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Which subnormal Toeplitz operators are either normal or analytic? ☆

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Abstract

We study subnormal Toeplitz operators on the vector-valued Hardy space of the unit circle, along with an appropriate reformulation of P.R. Halmos's Problem 5: Which subnormal block Toeplitz operators are either normal or analytic? We extend and prove Abrahamse's theorem to the case of matrix-valued symbols; that is, we show that every subnormal block Toeplitz operator with bounded type symbol (i.e., a quotient of two bounded analytic functions), whose analytic and co-analytic parts have the "left coprime factorization", is normal or analytic. We also prove that the left coprime factorization condition is essential. Finally, we examine a well-known conjecture, of whether every subnormal Toeplitz operator with finite rank self-commutator is normal or analytic.

Keywords: Block Toeplitz operators; Subnormal; Hyponormal; Bounded type functions

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1. Introduction

Toeplitz operators arise naturally in several fields of mathematics and in a variety of problems in physics (in particular, in the field of quantum mechanics). On the other hand, the theory of subnormal operators is an extensive and highly developed area, which has made important contributions to a number of problems in functional analysis, operator theory, and mathematical physics. Thus, it becomes of central significance to describe in detail subnormality for Toeplitz operators. This paper focuses on subnormality for *block* Toeplitz operators and more precisely, the case of block Toeplitz operators with bounded type symbols. Our main result is an appropriate generalization of Abrahamse's theorem to the case of matrix-valued symbols; that is, we show that every subnormal block Toeplitz operator with bounded type symbol (i.e., a quotient of two bounded analytic functions), whose analytic and co-analytic parts have the "left coprime factorization", is normal or analytic.

Naturally, this research is closely related to the study of subnormal operators with finite rank self-commutator, a class that has been extensively researched by many authors. However, until now a complete description of that class has proved elusive. Recently, D. Yakubovich [30] has shown that if S is a pure subnormal operator with finite rank self-commutator and admits a normal extension with no nonzero eigenvectors, then S is unitarily equivalent to a block Toeplitz operator with analytic rational normal matrix symbol. A corollary of our main result illustrates, in a certain sense, the case of subnormal Toeplitz operators with finite rank self-commutator.

To describe our results in more detail, we first need to review a few essential facts about (block) Toeplitz operators, and for that we will use [10,11,14,28]. Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators acting on $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *hyponormal* if its self-commutator $[T^*, T] := T^*T - TT^*$ is positive (semi-definite), and *subnormal* if there exists a normal operator N on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that $\mathcal{H}$ is invariant under N and $N|_{\mathcal{H}} = T$. Let $\mathbb{T} \equiv \partial\mathbb{D}$ be the unit circle in the complex plane. Let $L^2 \equiv L^2(\mathbb{T})$ be the set of all square-integrable measurable functions on $\mathbb{T}$ and let $H^2 \equiv H^2(\mathbb{T})$ be the corresponding Hardy space. Let $H^\infty \equiv H^\infty(\mathbb{T}) := L^\infty(\mathbb{T}) \cap H^2(\mathbb{T})$, that is, H^∞ is the set of bounded analytic functions on $\mathbb{D}$. Given $\phi \in L^\infty$, the Toeplitz operator T_ϕ and the Hankel operator H_ϕ are defined by

$$T_\phi g := P(\phi g) \quad \text{and} \quad H_\phi g := JP^\perp(\phi g) \quad \big(g \in H^2\big),$$

where P and $P^\perp$ denote the orthogonal projections that map from L^2 onto H^2 and $(H^2)^\perp$, respectively, and where J denotes the unitary operator on L^2 defined by $J(f)(z) = \bar{z}f(\bar{z})$.

In the early 1960s, normal Toeplitz operators were characterized by a property of their symbols by A. Brown and P.R. Halmos [3]. On the other hand, the exact nature of the relationship between the symbol $\phi \in L^\infty$ and the hyponormality of T_ϕ was understood much later, in 1988, via Cowen's theorem [6].

Cowen's theorem. *(See [6,27].) For each $\phi \in L^\infty$, let*

$$\mathcal{E}(\phi) \equiv \big\{k \in H^\infty\colon\ \|k\|_\infty \leqslant 1 \text{ and } \phi - k\bar{\phi} \in H^\infty\big\}.$$

Then T_ϕ is hyponormal if and only if $\mathcal{E}(\phi)$ is nonempty.

The elegant and useful theorem of C. Cowen has been used in the works [8,9,12,15–17,21–27, 31], which have been devoted to the study of hyponormality for Toeplitz operators on H^2. When one studies hyponormality (also, normality and subnormality) of the Toeplitz operator T_ϕ one may, without loss of generality, assume that $\phi(0)=0$; this is because hyponormality is invariant under translation by scalars. We now recall that a function $\phi \in L^\infty$ is said to be of *bounded type* (or in the Nevanlinna class) if there are analytic functions $\psi_1, \psi_2 \in H^\infty(\mathbb{D})$ such that

$$\phi(z) = \frac{\psi_1(z)}{\psi_2(z)} \quad \text{for almost all } z \in \mathbb{T}.$$

It is well known [1, Lemma 3] that if $\phi \notin H^\infty$ then

$$\phi \text{ is of bounded type} \quad \Leftrightarrow \quad \ker H_\phi \neq \{0\}. \tag{1}$$

If $\phi \in L^\infty$, we write

$$\phi_+ \equiv P\phi \in H^2 \quad \text{and} \quad \phi_- \equiv \overline{P^\perp \phi} \in zH^2.$$

Assume now that both ϕ and $\bar\phi$ are of bounded type. Since $T_{\bar z} H_\psi = H_\psi T_z$ for all $\psi \in L^\infty$, it follows from Beurling's theorem that $\ker H_{\overline{\phi_-}} = \theta_0 H^2$ and $\ker H_{\overline{\phi_+}} = \theta_+ H^2$ for some inner functions θ_0, θ_+. We thus have $b := \overline{\phi_-}\theta_0 \in H^2$, and hence we can write

$$\phi_- = \theta_0 \bar b \quad \text{and similarly} \quad \phi_+ = \theta_+ \bar a \quad \text{for some } a \in H^2.$$

In particular, if T_ϕ is hyponormal and $\phi \notin H^\infty$, and since

$$\left[T_\phi^*, T_\phi\right] = H_{\bar\phi}^* H_{\bar\phi} - H_\phi^* H_\phi = H_{\overline{\phi_+}}^* H_{\overline{\phi_+}} - H_{\overline{\phi_-}}^* H_{\overline{\phi_-}},$$

it follows that $\|H_{\overline{\phi_+}} f\| \geqslant \|H_{\overline{\phi_-}} f\|$ for all $f \in H^2$, and hence

$$\theta_+ H^2 = \ker H_{\overline{\phi_+}} \subseteq \ker H_{\overline{\phi_-}} = \theta_0 H^2,$$

which implies that θ_0 divides θ_+, i.e., $\theta_+ = \theta_0\theta_1$ for some inner function θ_1. We write, for an inner function θ,

$$\mathcal{H}_\theta := H^2 \ominus \theta H^2.$$

Note that if $f = \theta \bar a \in L^2$, then $f \in H^2$ if and only if $a \in \mathcal{H}_{z\theta}$; in particular, if $f(0)=0$ then $a \in \mathcal{H}_\theta$. Thus, if $\phi = \overline{\phi_-} + \phi_+ \in L^\infty$ is such that ϕ and $\bar\phi$ are of bounded type such that $\phi_+(0)=0$ and T_ϕ is hyponormal, then we can write

$$\phi_+ = \theta_0\theta_1 \bar a \quad \text{and} \quad \phi_- = \theta_0 \bar b, \quad \text{where } a \in \mathcal{H}_{\theta_0\theta_1} \text{ and } b \in \mathcal{H}_{\theta_0}.$$

By Kronecker's lemma [28, p. 183], if $f \in H^\infty$ then $\overline{f}$ is a rational function if and only if $\operatorname{rank} H_{\overline{f}} < \infty$, which implies that

$$\overline{f} \text{ is rational} \quad \Leftrightarrow \quad f = \theta \bar b \text{ with a finite Blaschke product } \theta. \tag{2}$$

On the other hand, M.B. Abrahamse [1, Lemma 6] also showed that if T_ϕ is hyponormal, if $\phi \notin H^\infty$, and if ϕ or $\overline{\phi}$ is of bounded type then both ϕ and $\overline{\phi}$ are of bounded type.

We now introduce the notion of block Toeplitz operators. For a Hilbert space $\mathcal{X}$, let $L^2_{\mathcal{X}} \equiv L^2_{\mathcal{X}}(\mathbb{T})$ be the Hilbert space of $\mathcal{X}$-valued norm square-integrable measurable functions on $\mathbb{T}$ and let $H^2_{\mathcal{X}} \equiv H^2_{\mathcal{X}}(\mathbb{T})$ be the corresponding Hardy space. We observe that $L^2_{\mathbb{C}^n} = L^2 \otimes \mathbb{C}^n$ and $H^2_{\mathbb{C}^n} = H^2 \otimes \mathbb{C}^n$. If Φ is a matrix-valued function in $L^\infty_{M_n} \equiv L^\infty_{M_n}(\mathbb{T})$ $(= L^\infty \otimes M_n)$ then $T_\Phi : H^2_{\mathbb{C}^n} \to H^2_{\mathbb{C}^n}$ denotes the block Toeplitz operator with symbol Φ defined by

$$T_\Phi F := P_n(\Phi F) \quad \text{for } F \in H^2_{\mathbb{C}^n},$$

where P_n is the orthogonal projection of $L^2_{\mathbb{C}^n}$ onto $H^2_{\mathbb{C}^n}$. A block Hankel operator with symbol $\Phi \in L^\infty_{M_n}$ is the operator $H_\Phi : H^2_{\mathbb{C}^n} \to H^2_{\mathbb{C}^n}$ defined by

$$H_\Phi F := J_n P_n^\perp(\Phi F) \quad \text{for } F \in H^2_{\mathbb{C}^n},$$

where J_n denotes the unitary operator from $(H^2_{\mathbb{C}^n})^\perp$ to $H^2_{\mathbb{C}^n}$ given by $J_n(F)(z) := \bar{z} I_n F(\bar{z})$ for $F \in H^2_{\mathbb{C}^n}$, and where I_n is the $n \times n$ identity matrix. If we set $H^2_{\mathbb{C}^n} := H^2 \oplus \cdots \oplus H^2$ then we see that

$$T_\Phi = \begin{bmatrix} T_{\phi_{11}} & \dots & T_{\phi_{1n}} \\ & \vdots & \\ T_{\phi_{n1}} & \dots & T_{\phi_{nn}} \end{bmatrix} \quad \text{and} \quad H_\Phi = \begin{bmatrix} H_{\phi_{11}} & \dots & H_{\phi_{1n}} \\ & \vdots & \\ H_{\phi_{n1}} & \dots & H_{\phi_{nn}} \end{bmatrix},$$

where

$$\Phi = \begin{bmatrix} \phi_{11} & \dots & \phi_{1n} \\ & \vdots & \\ \phi_{n1} & \dots & \phi_{nn} \end{bmatrix} \in L^\infty_{M_n}.$$

For $\Phi \in L^\infty_{M_n}$, write

$$\widetilde{\Phi}(z) := \Phi^*(\bar{z}).$$

A matrix-valued function $\Theta \in H^\infty_{M_{n\times m}}$ $(= H^\infty \otimes M_{n\times m})$ is called *inner* if $\Theta(z)^*\Theta(z) = I_m$ for almost all $z \in \mathbb{T}$. The following basic relations can be easily derived:

$$T^*_\Phi = T_{\Phi^*}, \qquad H^*_\Phi = H_{\widetilde{\Phi}} \quad \big(\Phi \in L^\infty_{M_n}\big);$$

$$T_{\Phi\Psi} - T_\Phi T_\Psi = H^*_{\Phi^*} H_\Psi \quad \big(\Phi, \Psi \in L^\infty_{M_n}\big); \tag{3}$$

$$H_\Phi T_\Psi = H_{\Phi\Psi}, \qquad H_{\Psi\Phi} = T^*_{\widetilde{\Psi}} H_\Phi \quad \big(\Phi \in L^\infty_{M_n},\ \Psi \in H^\infty_{M_n}\big); \tag{4}$$

$$H^*_\Phi H_\Phi - H^*_{\Theta\Phi} H_{\Theta\Phi} = H^*_\Phi H_{\Theta^*} H^*_{\Theta^*} H_\Phi \quad \big(\Theta \in H^\infty_{M_n} \text{ is inner},\ \Phi \in L^\infty_{M_n}\big).$$

For a matrix-valued function $\Phi = [\phi_{ij}] \in L^\infty_{M_n}$, we say that Φ is of *bounded type* if each entry ϕ_{ij} is of bounded type and that Φ is *rational* if each entry ϕ_{ij} is a rational function. The *shift* operator S on $H^2_{\mathbb{C}^n}$ is defined by

$$S := \sum_{j=1}^{n} \oplus T_z.$$

The following fundamental result known as the Beurling–Lax–Halmos theorem is useful in the sequel.

Beurling–Lax–Halmos theorem. *A nonzero subspace M of $H^2_{\mathbb{C}^n}$ is invariant under the shift operator S on $H^2_{\mathbb{C}^n}$ if and only if $M = \Theta H^2_{\mathbb{C}^m}$, where Θ is an inner matrix function in $H^\infty_{M_{n\times m}}$ $(m \leqslant n)$.*

In view of (4), the kernel of a block Hankel operator H_Φ is an invariant subspace of the shift operator on $H^2_{\mathbb{C}^n}$. Thus if $\ker H_\Phi \neq \{0\}$ then by the Beurling–Lax–Halmos theorem,

$$\ker H_\Phi = \Theta H^2_{\mathbb{C}^m}$$

for some inner matrix function Θ. But we don't guarantee that Θ is a square matrix. In fact, as we will refer in the sequel, Θ is square if and only if Φ is of bounded type. Recently, Gu, Hendricks and Rutherford [18] considered the hyponormality of block Toeplitz operators and characterized the hyponormality of block Toeplitz operators in terms of their symbols. In particular they showed that if T_Φ is a hyponormal block Toeplitz operator on $H^2_{\mathbb{C}^n}$, then Φ is normal, i.e., $\Phi^*\Phi = \Phi\Phi^*$. Their characterization for hyponormality of block Toeplitz operators resembles the Cowen's theorem except for an additional condition – the normality condition of the symbol.

Hyponormality of block Toeplitz operators. *(See Gu, Hendricks and Rutherford [18].) For each $\Phi \in L^\infty_{M_n}$, let*

$$\mathcal{E}(\Phi) := \left\{K \in H^\infty_{M_n}\colon\ \|K\|_\infty \leqslant 1 \text{ and } \Phi - K\Phi^* \in H^\infty_{M_n}\right\}.$$

Then T_Φ is hyponormal if and only if Φ is normal and $\mathcal{E}(\Phi)$ is nonempty.

For a matrix-valued function $\Phi \in H^2_{M_{n\times r}}$, we say that $\Delta \in H^2_{M_{n\times m}}$ is a *left inner divisor* of Φ if Δ is an inner matrix function such that $\Phi = \Delta A$ for some $A \in H^2_{M_{m\times r}}$ $(m \leqslant n)$. We also say that two matrix functions $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{n\times m}}$ are *left coprime* if the only common left inner divisor of both Φ and Ψ is a unitary constant and that $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{m\times r}}$ are *right coprime* if $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are left coprime. Two matrix functions Φ and Ψ in $H^2_{M_n}$ are said to be *coprime* if they are both left and right coprime. We remark that if $\Phi \in H^2_{M_n}$ is such that $\det\Phi$ is not identically zero then any left inner divisor Δ of Φ is square, i.e., $\Delta \in H^2_{M_n}$. If $\Phi \in H^2_{M_n}$ is such that $\det\Phi$ is not identically zero then we say that $\Delta \in H^2_{M_n}$ is a *right inner divisor* of Φ if $\widetilde{\Delta}$ is a left inner divisor of $\widetilde{\Phi}$.

The following lemma will be useful in the sequel.

Lemma 1.1. *(See [18].) For $\Phi \in L^\infty_{M_n}$, the following statements are equivalent*:

(i) *Φ is of bounded type*;
(ii) $\ker H_\Phi = \Theta H^2_{\mathbb{C}^n}$ *for some square inner matrix function Θ*;
(iii) *$\Phi = A\Theta^*$, where $A \in H^\infty_{M_n}$ and A and Θ are right coprime.*

For $\Phi \in L^\infty_{M_n}$ we write

$$\Phi_+ := P_n\Phi \in H^2_{M_n} \quad \text{and} \quad \Phi_- := \big(P_n^\perp \Phi\big)^* \in H^2_{M_n}.$$

Thus we can write $\Phi = \Phi_-^* + \Phi_+$. For an inner matrix function $\Theta \in H^\infty_{M_n}$, write

$$\mathcal{H}_\Theta := \big(\Theta H^2_{\mathbb{C}^n}\big)^\perp \equiv H^2_{\mathbb{C}^n} \ominus \Theta H^2_{\mathbb{C}^n}.$$

Suppose $\Phi = [\phi_{ij}] \in L^\infty_{M_n}$ is such that Φ^* is of bounded type. Then we may write $\phi_{ij} = \theta_{ij}\bar{b}_{ij}$, where θ_{ij} is an inner function and θ_{ij} and b_{ij} are coprime. Thus if θ is the least common multiple of θ_{ij}'s (i.e., the θ_{ij} divide θ and if they divide an inner function θ' then θ in turn divides θ'), then we can write

$$\Phi = [\phi_{ij}] = [\theta_{ij}\bar{b}_{ij}] = [\theta\bar{a}_{ij}] = \Theta A^* \quad \big(\Theta = \theta I_n,\ A \in H^2_{M_n}\big). \tag{5}$$

We note that the representation (5) is "minimal", in the sense that if ωI_n (ω is inner) is a common inner divisor of Θ and A, then ω is constant. Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type. Then in view of (5) we can write

$$\Phi_+ = \Theta_1 A^* \quad \text{and} \quad \Phi_- = \Theta_2 B^*,$$

where $\Theta_i = \theta_i I_n$ with an inner function θ_i for $i = 1, 2$ and $A, B \in H^2_{M_n}$. In particular, if $\Phi \in L^\infty_{M_n}$ is rational then the θ_i are chosen as finite Blaschke products as we observed in (2).

We would remark that, in (5), by contrast with scalar-valued functions, Θ and A need not be (right) coprime: indeed, if $\Phi := \left[\begin{smallmatrix} z & z \\ z & z \end{smallmatrix}\right]$ then we can write

$$\Phi = \Theta A^* = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

but $\Theta := \left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right]$ and $A := \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ are not right coprime because $\frac{1}{\sqrt{2}}\left[\begin{smallmatrix} z & -z \\ 1 & 1 \end{smallmatrix}\right]$ is a common right inner factor, i.e.,

$$\Theta = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & z \\ -1 & z \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} z & -z \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A = \sqrt{2} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} z & -z \\ 1 & 1 \end{bmatrix}. \tag{6}$$

In this paper we consider the subnormality of block Toeplitz operators and in particular, the block version of Halmos's Problem 5: Which subnormal block Toeplitz operators are either normal or analytic? In 1976, M.B. Abrahamse showed that if $\phi = \bar{g} + f \in L^\infty$ ($f, g \in H^2$) is such that ϕ or $\bar{\phi}$ is of bounded type, if T_ϕ is hyponormal, and if $\ker[T_\phi^*, T_\phi]$ is invariant under T_ϕ then

T_ϕ is normal or analytic. The purpose of this paper is to establish an extension of Abrahamse's theorem for block Toeplitz operators. In Section 2 we make a brief sketch on Halmos's Problem 5 and Abrahamse's theorem. Section 3 is devoted to the proof of the main result. In Section 4 we consider the scalar Toeplitz operators with finite rank self-commutators.

2. Halmos's Problem 5 and Abrahamse's theorem

In 1970, P.R. Halmos posed the following problem, listed as Problem 5 in his lectures "Ten problems in Hilbert space" [19,20]:

Is every subnormal Toeplitz operator either normal or analytic?

A Toeplitz operator T_ϕ is called *analytic* if $\phi \in H^\infty$. Any analytic Toeplitz operator is easily seen to be subnormal: indeed, $T_\phi h = P(\phi h) = \phi h = M_\phi h$ for $h \in H^2$, where M_ϕ is the normal operator of multiplication by ϕ on L^2. The question is natural because the two classes, the normal and analytic Toeplitz operators, are fairly well understood and are subnormal. Halmos's Problem 5 has been partially answered in the affirmative by many authors (cf. [1,2,8,9,27], and etc.). In 1984, Halmos's Problem 5 was answered in the negative by C. Cowen and J. Long [7]: they found an analytic function ψ for which $T_{\psi+\alpha\overline{\psi}}$ $(0 < \alpha < 1)$ is subnormal – in fact, this Toeplitz operator is unitarily equivalent to a subnormal weighted shift W_β with weight sequence $\beta \equiv \{\beta_n\}$, where $\beta_n = (1-\alpha^{2n+2})^{\frac{1}{2}}$ for $n = 0, 1, 2, \ldots$. Unfortunately, Cowen and Long's construction does not provide an intrinsic connection between subnormality and the theory of Toeplitz operators. Until now researchers have been unable to characterize subnormal Toeplitz operators in terms of their symbols. On the other hand, surprisingly, as C. Cowen notes in [4,5], some analytic Toeplitz operators are unitarily equivalent to non-analytic Toeplitz operators; i.e., the analyticity of Toeplitz operators is not invariant under unitary equivalence. In this sense, we might ask whether Cowen and Long's non-analytic subnormal Toeplitz operator is unitarily equivalent to an analytic Toeplitz operator. To this end, we have:

Observation. *Cowen and Long's non-analytic subnormal Toeplitz operator T_ϕ is not unitarily equivalent to any analytic Toeplitz operator.*

Proof. Assume to the contrary that T_ϕ is unitarily equivalent to an analytic Toeplitz operator T_f. Then by the above remark, T_f is unitarily equivalent to the subnormal weighted shift W_β with weight sequence $\beta \equiv \{\beta_n\}$, where $\beta_n = (1-\alpha^{2n+2})^{\frac{1}{2}}$ $(0 < \alpha < 1)$ for $n = 0, 1, 2, \ldots$; i.e., there exists a unitary operator V such that

$$V^* T_f V = W_\beta.$$

Thus if $\{e_n\}$ is the canonical orthonormal basis for ℓ^2 then

$$V^* T_f V e_j = W_\beta e_j = \beta_j e_{j+1} \quad \text{for } j = 0, 1, 2, \ldots.$$

We thus have

$$\left(V^* T_{|f|^2} V\right) e_j = W_\beta^* W_\beta e_j = \beta_j^2 e_j,$$

and hence,

$$T_{|f|^2-\beta_j^2}(Ve_j)=0 \quad \text{for } j=0,1,2,\ldots.$$

Fix $j \geqslant 0$ and observe that $Ve_j \in \ker(T_{|f|^2-\beta_j^2})$. By Coburn's theorem, if $|f|^2-\beta_j^2$ is nonzero then either $T_{|f|^2-\beta_j^2}$ or $T^*_{|f|^2-\beta_j^2}$ is one-one. It follows that $|f|^2=\beta_j^2$ for $j=0,1,2,\ldots$. This readily implies that $\beta_0=\beta_1=\beta_2=\cdots$, a contradiction. □

Consequently, even if we interpret "*is*" in Halmos Problem 5 as "*is up to unitary equivalence*", the answer to Halmos Problem 5 is still negative.

We would like to reformulate Halmos's Problem 5 as follows:

Halmos's Problem 5 reformulated. *Which Toeplitz operators are subnormal*?

The most interesting partial answer to Halmos's Problem 5 was given by M.B. Abrahamse [1]. M.B. Abrahamse gave a general sufficient condition for the answer to Halmos's Problem 5 to be affirmative.

Abrahamse's theorem can be then stated as:

Abrahamse's theorem. *(See [1, Theorem].) Let $\phi=\overline{g}+f \in L^\infty$ $(f,g\in H^2)$ be such that ϕ or $\overline{\phi}$ is of bounded type. If T_ϕ is hyponormal and* $\ker[T_\phi^*, T_\phi]$ *is invariant under T_ϕ then T_ϕ is normal or analytic.*

Consequently, if $\phi=\overline{g}+f\in L^\infty$ $(f,g\in H^2)$ is such that ϕ or $\overline{\phi}$ is of bounded type, then every subnormal Toeplitz operator must be normal or analytic.

We say that a block Toeplitz operator T_Φ is *analytic* if $\Phi\in H^\infty_{M_n}$. Evidently, any analytic block Toeplitz operator with a normal symbol is subnormal because the multiplication operator M_Φ is a normal extension of T_Φ. As a first inquiry in the above reformulation of Halmos's Problem 5 the following question can be raised:

Is Abrahamse's theorem valid for block Toeplitz operators?

In this paper we answer this question in the affirmative (Theorem 3.5).

3. Abrahamse's theorem for matrix-valued symbols

Recall the representation (5), and for $\Psi\in L^\infty_{M_n}$ such that Ψ^* is of bounded type, write $\Psi=\Theta_2B^*=B^*\Theta_2$. Let Ω be the greatest common left inner divisor of B and Θ_2. Then $B=\Omega B_\ell$ and $\Theta_2=\Omega\Omega_2$ for some $B_\ell\in H^2_{M_n}$ and some inner matrix Ω_2. Therefore we can write

$$\Psi=B_\ell^*\Omega_2, \quad \text{where } B_\ell \text{ and } \Omega_2 \text{ are left coprime;}$$

in this case, $B_\ell^*\Omega_2$ is called a *left coprime factorization* of Ψ. Similarly,

$$\Psi = \Delta_2 B_r^*, \quad \text{where } B_r \text{ and } \Delta_2 \text{ are right coprime;}$$

in this case, $\Delta_2 B_r^*$ is called a *right coprime factorization* of Ψ.

To prove our main result (Theorem 3.5), we need several auxiliary lemmas.

We begin with:

Lemma 3.1. *Suppose $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is such that Φ and Φ^* are of bounded type of the form*

$$\Phi_+ = A^*\Theta_1 \quad \textit{and} \quad \Phi_- = B^*\Theta_2,$$

where $\Theta_i := \theta_i I_n$ with an inner function θ_i $(i = 1, 2)$. If T_Φ is hyponormal, then Θ_2 is a right inner divisor of Θ_1.

Proof. Suppose T_Φ is hyponormal. Then there exists a matrix function $K \in H^\infty_{M_n}$ such that $\Phi_-^* - K\Phi_+^* \in H^2_{M_n}$. Thus $B\Theta_2^* - KA\Theta_1^* = F$ for some $F \in H^2_{M_n}$, which implies that $B\Theta_2^*\Theta_1 \in H^2_{M_n}$. Now we write $\Phi_- = [\, f_{ij}\,]_{n\times n}$. Since Φ is of bounded type we can write $f_{ij} = \theta_{ij}\bar{c}_{ij}$, where θ_{ij} is an inner function, c_{ij} is in H^2, and θ_{ij} and c_{ij} are coprime. Write $B = [\, b_{ij}\,]_{n\times n}$. We thus have

$$f_{ij} = \theta_{ij}\bar{c}_{ij} = \theta_2\bar{b}_{ji} \quad \text{for each } i, j = 1, \dots, n,$$

which implies that $b_{ji} = \bar{\theta}_{ij}\theta_2 c_{ij}$. But since $B\Theta_2^*\Theta_1 = [\theta_1\bar{\theta}_2 b_{ij}] \in H^2_{M_n}$, we have $\theta_1\bar{\theta}_{ji}c_{ji} \in H^2$. Since θ_{ji} and c_{ji} are coprime for each $i, j = 1, \dots, n$, it follows that $\bar{\theta}_{ji}\theta_1 \in H^2$, which implies that $\bar{\theta}_2\theta_1 \in H^2$ and therefore, Θ_2 divides Θ_1, i.e., $\Theta_1 = \Theta_0\Theta_2$ for some inner matrix function Θ_0. $\square$

In the sequel, when we consider the symbol $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$, which is such that Φ and Φ^* are of bounded type and for which T_Φ is hyponormal, we will, in view of Lemma 3.1, assume that

$$\Phi_+ = A^*\Omega_1\Omega_2 \quad \text{and} \quad \Phi_- = B_\ell^*\Omega_2 \quad \text{(left coprime factorization)},$$

where $\Omega_1\Omega_2 = \Theta = \theta I_n$. We also note that $\Omega_2\Omega_1 = \Theta$: indeed, if $\Omega_1\Omega_2 = \Theta = \theta I_n$, then $(\bar{\theta} I_n\Omega_1)\Omega_2 = I_n$, so that $\Omega_1(\bar{\theta} I_n\Omega_2) = I_n$, which implies that $(\bar{\theta} I_n\Omega_2)\Omega_1 = I_n$, and hence $\Omega_2\Omega_1 = \theta I_n = \Theta$.

Lemma 3.2. *Suppose $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is such that Φ and Φ^* are of bounded type of the form*

$$\Phi_+ = \Delta_1 A_r^* \quad (\textit{right coprime factorization})$$

and

$$\Phi_- = \Delta_2 B_r^* \quad (\textit{right coprime factorization}).$$

If T_Φ is hyponormal, then Δ_2 is a left inner divisor of Δ_1, i.e., $\Delta_1 = \Delta_2\Delta_0$ for some Δ_0.

Proof. Suppose T_Φ is hyponormal. Then there exists $K \in H^\infty_{M_n}$ such that $\Phi - K\Phi^* \in H^\infty_{M_n}$. Thus $H_\Phi = H_{K\Phi^*} = T^*_{\tilde{K}} H_{\Phi^*}$, which implies that $\ker H_{\Phi_+^*} \subseteq \ker H_{\Phi_-^*}$, so that by Lemma 1.1, $\Delta_1 H^2_{\mathbb{C}^n} \subseteq \Delta_2 H^2_{\mathbb{C}^n}$. It follows (cf. [13, Corollary IX.2.2]) that Δ_2 is a left inner divisor of Δ_1. $\square$

On the other hand, the condition "(left/right) coprime factorization" is not so easy to check in general. For example, consider a simple case: $\Phi_- := \left[\begin{smallmatrix} z & z \\ z & z \end{smallmatrix}\right]$. One is tempted to write

$$\Phi_- := \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^*.$$

But $\left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ are not right coprime as we have seen in the Introduction. On the other hand, observe that

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \equiv \Delta B^*,$$

where

$$\Delta := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & z \\ -1 & z \end{bmatrix} \text{ is inner} \quad \text{and} \quad B := \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 2z \\ 0 & 2z \end{bmatrix}.$$

Again, Δ and B are not right coprime because $\ker H_{\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]} = H^2_{\mathbb{C}^2}$. Thus we might choose

$$\Phi_- = (zI_2\Delta) \cdot B^* \quad \text{or} \quad \Phi_- = \Delta \cdot (\bar{z} I_2 B)^*.$$

A straightforward calculation shows that $\ker H_{\Phi_-^*} = \Delta H^2_{\mathbb{C}^2}$. Hence the latter of the above factorizations is the desired factorization: i.e., Δ and $\bar{z} I_2 B$ are right coprime.

However, if Θ is given in a form $\Theta = \theta I_n$ with a finite Blaschke product θ, then we can obtain a more tractable criterion on the coprime-ness of Θ and $B \in H^2_{M_n}$. To see this, recall that an $n \times n$ matrix-valued function D is called *a finite Blaschke–Potapov product* if D is of the form

$$D(z) = \nu \prod_{m=1}^{M} \big(b_m(z) P_m + (I - P_m)\big),$$

where ν is an $n \times n$ unitary constant matrix, b_m is a *Blaschke factor*, which is of the form

$$b_m(z) := \frac{z - \alpha_m}{1 - \bar{\alpha}_m z} \quad (\alpha_m \in \mathbb{D}),$$

and P_m is an orthogonal projection in $\mathbb{C}^n$. In particular, a scalar-valued function D reduces to a finite Blaschke product $D(z) = \nu \prod_{m=1}^M b_m(z)$, where $\nu = e^{i\omega}$. It was known [29] that an $n \times n$ matrix-valued function D is rational and inner if and only if it can be represented as a finite Blaschke–Potapov product.

We write $\mathcal{Z}(\theta)$ for the set of zeros of an inner function θ. We then have:

Lemma 3.3. *Let $B \in H^2_{M_n}$ and $\Theta := \theta I_n$ with a finite Blaschke product θ. Then the following statements are equivalent*:

(a) *$B(\alpha)$ is invertible for each $\alpha \in \mathcal{Z}(\theta)$;*
(b) *B and Θ are right coprime*;
(c) *B and Θ are left coprime.*

Proof. We first write

$$\theta(z) = e^{i\xi} \prod_{i=1}^{N} \left(\frac{z - \alpha_i}{1 - \overline{\alpha_i} z} \right)^{m_i} \quad \left(\sum_{i=1}^{N} m_i =: d \right).$$

(a) $\Leftrightarrow$ (b): Suppose $B(\alpha)$ is invertible for each $\alpha \in \mathcal{Z}(\theta)$. Assume to the contrary that B and Θ are not right coprime. Thus there exists a finite Blaschke–Potapov product D of the form

$$D(z) = \nu \prod_{m=1}^{M} \big(b_m(z) P_m + (I - P_m) \big)$$

satisfying that

$$B = B_1 D \quad \text{and} \quad \Theta = \Theta_0 D \quad \text{for some inner function } \Theta_0.$$

Thus if $\alpha \in \mathcal{Z}(b_{m_0})$ for some $1 \leqslant m_0 \leqslant M$, then $\Theta(\alpha) = \Theta_0(\alpha) D(\alpha)$ is not invertible. But since $\Theta = \theta I_n$, it follows that $\Theta(\alpha) = 0$ and hence $\alpha \in \mathcal{Z}(\theta)$. Moreover,

$$\det B(\alpha) = \det B_1(\alpha) \det D(\alpha) = \det(\nu) \det B_1(\alpha) \prod_{m=1}^{M} \det\big(b_m(\alpha) P_m + (I - P_m) \big) = 0,$$

giving a contradiction. Therefore B and Θ are right coprime.

For the converse we assume that $B(\alpha_{i_0})$ is not invertible for some i_0. Then the following matrix is not invertible:

$$\mathcal{B} := \begin{bmatrix} B_0 & 0 & 0 & 0 & \cdots & 0 \\ B_1 & B_0 & 0 & 0 & \cdots & 0 \\ B_2 & B_1 & B_0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ B_{m_{i_0}-2} & B_{m_{i_0}-3} & \ddots & \ddots & B_0 & 0 \\ B_{m_{i_0}-1} & B_{m_{i_0}-2} & \ldots & B_2 & B_1 & B_0 \end{bmatrix} \quad \left(B_j := \frac{B^{(j)}(\alpha_{i_0})}{j!} \right).$$

Thus there exists a nonzero $n \times m_{i_0}$ matrix $\mathcal{G} = (\mathcal{G}_0 \mathcal{G}_1 \cdots \mathcal{G}_{m_{i_0}-1})^t$ such that $\mathcal{B}\mathcal{G} = 0$. We now want to show that there exists $\mathfrak{h} = (h_1 h_2 \cdots h_n)^t \in H^2_{\mathbb{C}^n}$ satisfying the following property:

$$\frac{\mathfrak{h}^{(j)}(\alpha_i)}{j!} = \begin{cases} \mathcal{G}_j & (i = i_0), \\ 0 & (i \neq i_0). \end{cases} \tag{7}$$

This is exactly the classical Hermite–Fejér interpolation problem (cf. [13]), so that we use an argument of a solution for the interpolation of this type. Thus we can construct a function (in fact, a polynomial) $\mathfrak{h}(z) \equiv P(z)$ satisfying (7) (see [13, p. 299]). Then $P(z)$ belongs to $\ker H_{B\Theta^*}$. Since

$$\mathcal{G} = [\,\mathcal{G}_0 \mathcal{G}_1 \cdots \mathcal{G}_{m_{i_0}-1}\,]^t \neq 0,$$

it follows that $P(z) \notin \Theta H^2_{\mathbb{C}^n}$. Therefore we have $\ker H_{B\Theta^*} \neq \Theta H^2_{\mathbb{C}^n}$, which implies that B and Θ are not right coprime.

(b) $\Leftrightarrow$ (c): Suppose B and Θ are right coprime. If B and Θ are not left coprime, there exists a nonconstant inner matrix $\Delta \in H^2_{M_n}$ such that $B = \Delta B_1$ and $\Theta = \Delta\Omega$. We thus have that for each $i = 1, 2, \ldots, N$

$$\begin{bmatrix} \Delta_{i,0} & 0 & 0 & 0 & \cdots & 0 \\ \Delta_{i,1} & \Delta_{i,0} & 0 & 0 & \cdots & 0 \\ \Delta_{i,2} & \Delta_{i,1} & \Delta_{i,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \Delta_{i,m_i-2} & \Delta_{i,m_i-3} & \ddots & \ddots & \Delta_{i,0} & 0 \\ \Delta_{i,m_i-1} & \Delta_{i,m_i-2} & \cdots & \Delta_{i,2} & \Delta_{i,1} & \Delta_{i,0} \end{bmatrix} \begin{bmatrix} \Omega_{i,0} \\ \Omega_{i,1} \\ \Omega_{i,2} \\ \vdots \\ \Omega_{i,m_i-2} \\ \Omega_{i,m_i-1} \end{bmatrix} = 0,$$

where

$$\Delta_{i,j} := \frac{\Delta^{(j)}(\alpha_i)}{j!} \quad \text{and} \quad \Omega_{i,j} := \frac{\Omega^{(j)}(\alpha_i)}{j!}.$$

But since $B(\alpha)$ is invertible for each $\alpha \in \mathcal{Z}(\theta)$, we have that $\Delta_{i,0}$ is invertible for each $i = 1, 2, \ldots, N$. Thus

$$\Omega_{i,j} = 0 \quad (i = 1, 2, \ldots, N,\ j = 0, 1, 2, \ldots, m_i - 1),$$

which implies that $\Omega = \Theta\Omega_1$ for some $\Omega_1 \in H^2_{M_n}$. Thus $\Theta = \Delta\Omega = \Delta\Theta\Omega_1$, so that $I = \Delta\Omega_1$ and hence $\Delta^* = \Omega_1$, which implies that Δ is a constant matrix, a contradiction. Thus B and Θ are left coprime. The converse follows from the same argument. This completes the proof. $\square$

Lemma 3.4. *Let θ_0 be a nonconstant inner function. Then $\mathcal{H}_{\theta_0}$ contains an outer function that is invertible in H^∞.*

Proof. If θ_0 has at least one Blaschke factor, say $\frac{z-\alpha}{1-\bar{\alpha}z}$ ($|\alpha| < 1$), then $\frac{1}{1-\bar{\alpha}z}$ is an outer function and $\frac{1}{1-\bar{\alpha}z} \in \mathcal{H}_{\theta_0}$ because $\frac{1}{1-\bar{\alpha}z}$ is the reproducing kernel for α, so that for any $f \in H^2$,

$$\left\langle \theta_0 f, \frac{1}{1-\bar{\alpha}z} \right\rangle = \theta_0(\alpha) f(\alpha) = 0.$$

Now suppose θ_0 is a nonconstant singular inner function of the form

$$\theta_0(z) := \exp\left(-\int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(\theta) \right),$$

where μ is a finite positive Borel measure on $\mathbb{T}$ which is singular with respect to Lebesgue measure. We put

$$\omega(z) := \exp\left(-\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\frac{\mu}{2}(\theta)\right).$$

Then $\omega^2 = \theta_0$. If $\alpha := \overline{\omega}(0)$ then evidently, $0 < |\alpha| < 1$ since ω is not constant. Note that $\bar{\theta}_0(\omega - \frac{1}{\alpha}) = \overline{\omega} - \frac{1}{\alpha}\bar{\theta}_0 \in (H^2)^\perp$, since $(\overline{\omega} - \frac{1}{\alpha}\bar{\theta}_0)(0) = \alpha - \frac{1}{\alpha}\alpha^2 = 0$. We thus have $\omega - \frac{1}{\alpha} \in \mathcal{H}_{\theta_0}$. Also a straightforward calculation shows that $\frac{1}{\omega - \frac{1}{\alpha}}$ is bounded and analytic in $\mathbb{D}$, which says that $\omega - \frac{1}{\alpha}$ is invertible in H^∞. Hence $\omega - \frac{1}{\alpha}$ is an outer function in $\mathcal{H}_{\theta_0}$. This completes the proof. □

Before proving the main result, we recall the inner–outer factorization of vector-valued functions. If D and E are Hilbert spaces and if F is a function with values in $\mathcal{B}(E, D)$ such that $F(\cdot)e \in H^2_D(\mathbb{T})$ for each $e \in E$, then F is called a strong H^2-function. The strong H^2-function F is called an *inner function* if $F(\cdot)$ is an isometric operator from D into E. Write $\mathcal{P}_E$ for the set of all polynomials with values in E, i.e., $p(\zeta) = \sum_{k=0}^n \widehat{p}(k)\zeta^k$, $\widehat{p}(k) \in E$. Then the function $Fp = \sum_{k=0}^n F\widehat{p}(k)z^k$ belongs to $H^2_D(\mathbb{T})$. The strong H^2-function F is called *outer* if

$$\mathrm{cl}\, F \cdot \mathcal{P}_E = H^2_D(\mathbb{T}).$$

Note that if $\dim D = \dim E = n < \infty$, then evidently, every $F \in H^2_{M_n}$ is a strong H^2-function. We then have an analogue of the scalar inner–outer factorization theorem.

Inner–outer factorization. *(Cf. [28].) Every strong H^2-function F with values in $\mathcal{B}(E, D)$ can be expressed in the form*

$$F = F^i F^e,$$

where F^e is an outer function with values in $\mathcal{B}(E, D')$ and F^i is an inner function with values in $\mathcal{B}(D', D)$ for some Hilbert space D'.

We are now ready to prove the main result of this paper.

Theorem 3.5 *(Abrahamse's theorem for matrix-valued symbols). Suppose $\Phi := \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is such that Φ and Φ^* are of bounded type. In view of Lemma 3.1, we may write*

$$\Phi_+ = A^*\Theta_0\Theta_2 \quad \textit{and} \quad \Phi_- = B^*\Theta_2,$$

where $\Theta_i = \theta_i I_n$ with an inner function θ_i $(i = 0, 2)$ and $A, B \in H^2_{M_n}$. Assume that A, B and Θ_2 are left coprime. If

(i) T_Φ *is hyponormal*; *and*
(ii) $\ker[T_\Phi^*, T_\Phi]$ *is invariant under* T_Φ;

then T_Φ is normal or analytic. Hence, in particular, if T_Φ is subnormal then it is normal or analytic.

Remark 3.6. We note that if $n = 1$ (i.e., T_Φ is a scalar Toeplitz operator) then $\Phi_+ = \bar{a}\theta_0\theta_2$ and $\Phi_- = \bar{b}\theta_2$ with $a, b \in H^2$. Thus, we can always arrange that a, b and θ are coprime. Consequently, if $n = 1$ then our matrix version reduces to the original Abrahamse's theorem.

Proof of Theorem 3.5. If Θ_2 is constant then $\Phi_- = 0$, so that T_Φ is analytic. Suppose that Θ_2 is nonconstant.

We split the proof into three steps.

STEP 1: We first claim that

$$\Theta_0 H^2_{\mathbb{C}^n} \subseteq \ker\left[T_\Phi^*, T_\Phi\right]. \tag{8}$$

To see this, we observe that

$$\left[T_\Phi^*, T_\Phi\right] = H^*_{\Phi_+^*} H_{\Phi_+^*} - H^*_{\Phi_-^*} H_{\Phi_-^*} = H^*_{A\Theta_2^*\Theta_0^*} H_{A\Theta_2^*\Theta_0^*} - H^*_{\Theta_2^* B} H_{\Theta_2^* B}, \tag{9}$$

which implies that

$$\Theta_0\Theta_2 H^2_{\mathbb{C}^n} \subseteq \ker\left[T_\Phi^*, T_\Phi\right]. \tag{10}$$

On the other hand, since $\Theta_0\Theta_2$ is diagonal, we have that for all $g \in \mathcal{P}_{\mathbb{C}^n}$,

$$\begin{aligned} T_\Phi(\Theta_0\Theta_2 g) &= P_n\left(\Theta_2^* B\Theta_0\Theta_2 g + \Phi_+\Theta_0\Theta_2 g\right) \\ &= \Theta_0 B g + \Theta_0\Theta_2\Phi_+ g \\ &= P_{\mathcal{H}_{\Theta_0\Theta_2}}(\Theta_0 B g) + P_{\Theta_0\Theta_2 H^2_{\mathbb{C}^n}}(\Theta_0 B g) + \Theta_0\Theta_2\Phi_+ g. \end{aligned}$$

Since $\mathcal{H}_{\Theta_0\Theta_2} = \mathcal{H}_{\Theta_0} \oplus \Theta_0\mathcal{H}_{\Theta_2}$, it follows that

$$P_{\mathcal{H}_{\Theta_0\Theta_2}}(\Theta_0 B g) = P_{\Theta_0\mathcal{H}_{\Theta_2}}(\Theta_0 B g).$$

We thus have

$$T_\Phi(\Theta_0\Theta_2 g) = P_{\Theta_0\mathcal{H}_{\Theta_2}}(\Theta_0 B g) + P_{\Theta_0\Theta_2 H^2_{\mathbb{C}^n}}(\Theta_0 B g) + \Theta_0\Theta_2\Phi_+ g. \tag{11}$$

We claim that

$$\mathcal{H}_{\Theta_2} = \mathrm{cl}\left\{P_{\mathcal{H}_{\Theta_2}}(Bg):\ g \in \mathcal{P}_{\mathbb{C}^n}\right\}. \tag{12}$$

In view of the above mentioned inner–outer factorization, let $B = B^i B^e$ be the inner–outer factorization of B (as a strong H^2-function), where $B^i \in H^\infty_{M_{n\times r}}$ and $B^e \in H^2_{M_{r\times n}}$. Since B and Θ_2 are left coprime, B^i and Θ_2 are left coprime. Thus it follows from the Beurling–Lax–Halmos theorem that

$$\Theta_2 H^2_{\mathbb{C}^n} \vee \mathrm{cl}\, B\mathcal{P}_{\mathbb{C}^n} = \Theta_2 H^2_{\mathbb{C}^n} \vee B^i \left(\mathrm{cl}\, B^e \mathcal{P}_{\mathbb{C}^n}\right) = \Theta_2 H^2_{\mathbb{C}^n} \vee B^i H^2_{\mathbb{C}^r} = H^2_{\mathbb{C}^n},$$

giving (12). Thus we have

$$\Theta_0 \mathcal{H}_{\Theta_2} = \mathrm{cl}\, \Theta_0 \left\{ P_{\mathcal{H}_{\Theta_2}}(Bg)\colon\ g \in \mathcal{P}_{\mathbb{C}^n} \right\} = \mathrm{cl} \left\{ P_{\Theta_0 \mathcal{H}_{\Theta_2}}(\Theta_0 Bg)\colon\ g \in \mathcal{P}_{\mathbb{C}^n} \right\}. \tag{13}$$

If $\ker[T^*_\Phi, T_\Phi]$ is invariant under T_Φ then since $\ker[T^*_\Phi, T_\Phi]$ is a closed subspace it follows from (10)–(13) that

$$\Theta_0 \mathcal{H}_{\Theta_2} \subseteq \ker\left[T^*_\Phi, T_\Phi\right].$$

We thus have

$$\Theta_0 H^2_{\mathbb{C}^n} = \Theta_0 \mathcal{H}_{\Theta_2} \oplus \Theta_0 \Theta_2 H^2_{\mathbb{C}^n} \subseteq \ker\left[T^*_\Phi, T_\Phi\right],$$

which proves (8).

STEP 2: We next claim that

$$\mathcal{E}(\Phi) \text{ contains an inner function } K. \tag{14}$$

To see this, we first observe that if $K \in \mathcal{E}(\Phi)$ then by (4),

$$\left[T^*_\Phi, T_\Phi\right] = H^*_{\Phi^*_+} H_{\Phi^*_+} - H^*_{K\Phi^*_+} H_{K\Phi^*_+} = H^*_{\Phi^*_+}\left(I - T_{\widetilde{K}} T^*_{\widetilde{K}}\right) H_{\Phi^*_+}, \tag{15}$$

so that

$$\ker\left[T^*_\Phi, T_\Phi\right] = \ker\left(I - T_{\widetilde{K}} T^*_{\widetilde{K}}\right) H_{\Phi^*_+}.$$

Thus by (8),

$$\{0\} = \left(I - T_{\widetilde{K}} T^*_{\widetilde{K}}\right) H_{A\Theta_2^*\Theta_0^*}\left(\Theta_0 H^2_{\mathbb{C}^n}\right) = \left(I - T_{\widetilde{K}} T^*_{\widetilde{K}}\right) H_{A\Theta_2^*}\left(H^2_{\mathbb{C}^n}\right),$$

which implies

$$\mathrm{cl}\,\mathrm{ran}\, H_{A\Theta_2^*} \subseteq \ker\left(I - T_{\widetilde{K}} T^*_{\widetilde{K}}\right). \tag{16}$$

Since by assumption, A and Θ_2 are left coprime, and hence $\widetilde{A}$ and $\widetilde{\Theta}_2$ are right coprime, it follows from Lemma 1.1 that

$$\mathrm{cl}\,\mathrm{ran}\, H_{A\Theta_2^*} = (\ker H_{\widetilde{A}\widetilde{\Theta}_2^*})^\perp = \left(\widetilde{\Theta}_2 H^2_{\mathbb{C}^n}\right)^\perp = \mathcal{H}_{\widetilde{\Theta}_2}, \tag{17}$$

which together with (16) implies

$$\mathcal{H}_{\widetilde{\Theta}_2} \subseteq \ker\left(I - T_{\widetilde{K}} T^*_{\widetilde{K}}\right). \tag{18}$$

We thus have

$$F = T_{\widetilde{K}} T^*_{\widetilde{K}} F \quad \text{for each } F \in \mathcal{H}_{\widetilde{\Theta}_2}. \tag{19}$$

But since $\|\widetilde{K}\|_\infty = \|K\|_\infty \leqslant 1$, we have

$$\|P_n(\widetilde{K}^* F)\|_2 \leqslant \|\widetilde{K}^* F\|_2 \leqslant \|F\|_2 = \|T_{\widetilde{K}} T^*_{\widetilde{K}} F\|_2 = \|\widetilde{K} P_n(\widetilde{K}^* F)\|_2 \leqslant \|P_n(\widetilde{K}^* F)\|_2,$$

which gives

$$\|P_n(\widetilde{K}^* F)\|_2 = \|\widetilde{K}^* F\|_2,$$

which implies $\widetilde{K}^* F \in H^2_{\mathbb{C}^n}$. Therefore by (19), we have

$$F = \widetilde{K}\widetilde{K}^* F \quad \text{for each } F \in \mathcal{H}_{\widetilde{\Theta}_2}.$$

In view of Lemma 3.4, we can choose an outer function $f \in \mathcal{H}_{\widetilde{\theta}_2}$, which is invertible in H^∞. For each $j = 1, 2, \dots, n$, we define

$$F_j := (0, \dots, 0, f, 0, \dots, 0)^t \quad (\text{where } f \text{ is the } j\text{-th component}).$$

Then $F_j \in \mathcal{H}_{\widetilde{\Theta}_2}$ for each $j = 1, 2, \dots, n$, so that $(I - \widetilde{K}\widetilde{K}^*)F_j = 0$ for each $j = 1, 2, \dots, n$. If we write $I - \widetilde{K}\widetilde{K}^* \equiv [q_{ij}]_{1 \leqslant i,j \leqslant n} \in L^\infty_{M_n}$, then $q_{ij} f = 0$ for each $i, j = 1, 2, \dots, n$, so that $q_{ij} = 0$ for each $i, j = 1, 2, \dots, n$ because f is invertible. Therefore we have $K^*K = \widetilde{I} = I$, which implies that K is an inner function. This proves (14).

STEP 3: Now since K is inner it follows from (3) that

$$I - T_{\widetilde{K}} T^*_{\widetilde{K}} = T_{\widetilde{K}\widetilde{K}^*} - T_{\widetilde{K}} T^*_{\widetilde{K}} = H^*_{\widetilde{K}^*} H_{\widetilde{K}^*}.$$

Thus by (18), we have

$$\mathcal{H}_{\widetilde{\Theta}_2} \subseteq \ker H^*_{\widetilde{K}^*} H_{\widetilde{K}^*} = \ker H_{\widetilde{K}^*}. \tag{20}$$

Write $K := [k_{ij}]_{1 \leqslant i,j \leqslant n} \in H^\infty_{M_n}$. Since, by Lemma 3.4, $\mathcal{H}_{\widetilde{\theta}_2}$ contains an outer function h that is invertible in H^∞, it follows from (20) that

$$k_{ij}(\bar{z}) h \in H^2 \quad \text{for each } i, j = 1, 2, \dots, n,$$

so that $k_{i,j}(\bar{z}) \in \frac{1}{h} H^2 \subseteq H^2$ for each $i, j = 1, 2, \dots, n$. Therefore each k_{ij} is constant and hence, K is constant. Therefore by (15), $[T^*_\Phi, T_\Phi] = 0$, i.e., T_Φ is normal. This completes the proof. $\square$

Remark 3.7. Theorem 3.5 may fail if the condition "B and Θ_2 are left coprime" is dropped even though A and Θ_2 are left coprime. To see this, let θ be a nonconstant finite Blaschke product. Consider the matrix-valued function

$$\Phi = \begin{bmatrix} 2\theta + \bar\theta & \bar\theta \\ \bar\theta & 2\theta + \bar\theta \end{bmatrix}.$$

Write

$$\Theta := \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix}.$$

Then

$$\Phi_+ = 2\Theta \quad \text{and} \quad \Phi_- = \begin{bmatrix} \theta & \theta \\ \theta & \theta \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^* \Theta.$$

A direct calculation shows that Φ is normal. Put $K := \frac{1}{2}\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. Remember that for a matrix-valued function A, we define $\|A\|_\infty := \sup_{t\in\mathbb{T}} \|A(t)\|$ (where $\|\cdot\|$ means the operator norm). Then $\|K\|_\infty = 1$ and $\Phi - K\Phi^* \in H^\infty_{M_2}$, so that T_Φ is hyponormal. Observe that

$$\begin{aligned}
[T_{\Phi^*}, T_\Phi] &= H^*_{\Phi^*_+} H_{\Phi^*_+} - H^*_{\Phi^*_-} H_{\Phi^*_-} \\
&= 4\begin{bmatrix} H^*_{\bar\theta} H_{\bar\theta} & 0 \\ 0 & H^*_{\bar\theta} H_{\bar\theta} \end{bmatrix} - 2\begin{bmatrix} H^*_{\bar\theta} H_{\bar\theta} & H^*_{\bar\theta} H_{\bar\theta} \\ H^*_{\bar\theta} H_{\bar\theta} & H^*_{\bar\theta} H_{\bar\theta} \end{bmatrix} \\
&= 2\begin{bmatrix} H^*_{\bar\theta} H_{\bar\theta} & -H^*_{\bar\theta} H_{\bar\theta} \\ -H^*_{\bar\theta} H_{\bar\theta} & H^*_{\bar\theta} H_{\bar\theta} \end{bmatrix} \\
&= 2\begin{bmatrix} P_{\mathcal{H}_\theta} & -P_{\mathcal{H}_\theta} \\ -P_{\mathcal{H}_\theta} & P_{\mathcal{H}_\theta} \end{bmatrix},
\end{aligned}$$

which gives

$$\begin{aligned}
\ker[T_{\Phi^*}, T_\Phi] &= \ker\begin{bmatrix} P_{\mathcal{H}_\theta} & -P_{\mathcal{H}_\theta} \\ -P_{\mathcal{H}_\theta} & P_{\mathcal{H}_\theta} \end{bmatrix} = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} :\ P_{\mathcal{H}_\theta} f = P_{\mathcal{H}_\theta} g \right\} \\
&= \Theta H^2_{\mathbb{C}^2} \oplus \{ f \oplus f:\ f \in \mathcal{H}_\theta \}.
\end{aligned}$$

We now claim that $\ker[T^*_\Phi, T_\Phi]$ is invariant under T_Φ. To show this we suppose

$$F = \begin{bmatrix} f \\ g \end{bmatrix} \in \ker\bigl[T^*_\Phi, T_\Phi\bigr].$$

Then

$$T_\Phi F = \begin{bmatrix} 2T_\theta + T_{\bar\theta} & T_{\bar\theta} \\ T_{\bar\theta} & 2T_\theta + T_{\bar\theta} \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} 2T_\theta f + T_{\bar\theta}(f+g) \\ 2T_\theta g + T_{\bar\theta}(f+g) \end{bmatrix}.$$

Observe that

$$P_{\mathcal{H}_\theta}\big(2T_\theta f + T_{\bar\theta}(f+g)\big) = P_{\mathcal{H}_\theta} T_{\bar\theta}(f+g) = P_{\mathcal{H}_\theta}\big(2T_\theta g + T_{\bar\theta}(f+g)\big),$$

which implies that $\ker[T_\Phi^*, T_\Phi]$ is invariant under T_Φ. But since

$$\{f \oplus f\colon f \in \mathcal{H}_\theta\} \subsetneq \mathcal{H}_\Theta, \quad \text{and hence,} \quad \ker\big[T_\Phi^*, T_\Phi\big] \neq H^2_{\mathbb{C}^2},$$

we can see that T_Φ is not normal. Note that by Lemma 3.3, $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and Θ are not left coprime.

If, in the left coprime factorization $\Phi_- = B^*\Theta_2$ ($\Theta_2 = \theta_2 I_n$) of Theorem 3.5, θ_2 has a Blaschke factor, then the assumption of the "left coprime factorization" for the analytic part Φ_+ of Φ can be dropped in Theorem 3.5.

Corollary 3.8. *Suppose $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is such that Φ and Φ^* are of bounded type. In view of* (5)*, we may write*

$$\Phi_- = B^*\Theta,$$

where $\Theta := \theta I_n$ with an inner function θ. Assume that B and Θ are left coprime. Assume also that θ contains a Blaschke factor. If

(i) T_Φ *is hyponormal*; *and*
(ii) $\ker[T_\Phi^*, T_\Phi]$ *is invariant under* T_Φ;

then T_Φ is normal or analytic. Hence, in particular, if T_Φ is subnormal then it is normal or analytic.

Proof. For notational convenience, we let $\Theta_2 := \Theta$ and $\theta_2 := \theta$. Now suppose θ_2 has a Blaschke factor b_α and write $B_\alpha := b_\alpha I_n$. By assumption, B and B_α are left coprime, so that by Lemma 3.3, B and B_α are right coprime. Thus, in view of Lemmas 3.1 and 3.2, we can write

$$\Phi_+ = A^*\Theta_0\Theta_2 = B_\alpha \Delta_1 A_r^* \quad \text{and} \quad \Phi_- = B^*\Theta_2,$$

where $\Theta_i = \theta_i I_n$ with an inner function θ_i ($i = 0, 2$), A_r and $B_\alpha\Delta_1$ are right coprime, and B and Θ_2 are left coprime. In particular, we note that A and B_α are right coprime, so that again by Lemma 3.3, A and B_α are left coprime. On the other hand, an analysis for the proof of STEP 1 of Theorem 3.5 shows that

$$\Theta_0 H^2_{\mathbb{C}^n} \subseteq \ker\big[T_\Phi^*, T_\Phi\big]. \tag{21}$$

(Note that we didn't employ the assumption "A and Θ_2 are left coprime" to get (21) in the proof of STEP 1 of Theorem 3.5.) Thus if $K \in \mathcal{E}(\Phi)$ then by the same argument as (16), we have

$$\operatorname{cl}\operatorname{ran} H_{A\Theta_2^*} \subseteq \ker\big(I - T_{\widetilde K} T_{\widetilde K}^*\big). \tag{22}$$

Since A and B_α are left coprime it follows that $\widetilde A$ and $\widetilde B_\alpha$ are right coprime. Thus we can write

$$\widetilde A \widetilde\Theta_2^* = \widetilde A_1 \widetilde\Omega^* \widetilde B_\alpha^* \quad \text{(right coprime factorization)}$$

for some inner function Ω and $A_1 \in H^2_{M_n}$. It thus follows from Lemma 1.1 that

$$\operatorname{cl}\operatorname{ran} H_{A\Theta_2^*} = (\ker H_{\widetilde{A}\widetilde{\Theta}_2^*})^\perp = \big(\widetilde{B}_\alpha \widetilde{\Omega} H^2_{\mathbb{C}^n}\big)^\perp \supseteq \big(\widetilde{B}_\alpha H^2_{\mathbb{C}^n}\big)^\perp = \mathcal{H}_{\widetilde{B}_\alpha}.$$

Thus by (22),

$$\mathcal{H}_{\widetilde{B}_\alpha} \subseteq \ker\big(I - T_{\widetilde{K}} T^*_{\widetilde{K}}\big).$$

Then the exactly same argument as the argument from (18) to the end of the proof of Theorem 3.5 with B_α in place of Θ_2 shows that T_Φ is normal. (We again note that we didn't employ the assumption "A and Θ_2 are left coprime" there.) This completes the proof. $\square$

We thus have:

Corollary 3.9. *Suppose $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is a matrix-valued rational function. In view of* (5) *and* (2)*, we may write*

$$\Phi_- = B^* \Theta,$$

where $\Theta := \theta I_n$ with a finite Blaschke product θ. Assume that $B(\alpha)$ is invertible for each $\alpha \in \mathcal{Z}(\theta)$. If T_Φ is subnormal then T_Φ is normal or analytic.

Proof. This follows at once from Corollary 3.8 together with Lemma 3.3. $\square$

4. Scalar Toeplitz operators with finite rank self-commutators

If Φ is normal and analytic then $[T^*_\Phi, T_\Phi] = H^*_{\Phi^*} H_{\Phi^*}$, so that by the Kronecker's lemma, T_Φ has a finite rank self-commutator if and only if Φ is rational. Therefore Corollary 3.9 illustrates the case of subnormal Toeplitz operators with finite rank self-commutators. But it is still open whether subnormal (even scalar-valued) Toeplitz operators with finite rank self-commutators are either normal or analytic. We would like to state:

Conjecture 4.1. *If T_ϕ is a subnormal Toeplitz operator with finite rank self-commutator, then T_ϕ is normal or analytic.*

We need not expect that if T_ϕ is a hyponormal Toeplitz operator with finite rank self-commutator then ϕ is of bounded type. Indeed, if $\psi \in H^\infty$ is such that $\bar{\psi}$ is not of bounded type and $\phi = \overline{\psi} + z\psi$ (and hence ϕ is not of bounded type) then a straightforward calculation shows that T_ϕ is hyponormal and $\operatorname{rank}[T^*_\phi, T_\phi] = 1$.

We would like to take this opportunity to give a positive evidence for Conjecture 4.1. First of all, we recall a theorem of Nakazi and Takahashi [27, Theorem 10] which states that if T_ϕ is hyponormal then $[T^*_\phi, T_\phi]$ is of finite rank if and only if there exists a finite Blaschke product b in $\mathcal{E}(\phi)$ such that the degree of b equals the rank of $[T^*_\phi, T_\phi]$. In what follows we let $b\mathcal{M} := \{bf\colon f \in \mathcal{M}\}$.

Theorem 4.2. *Suppose T_ϕ is a hyponormal Toeplitz operator with finite rank self-commutator. If* $\ker[T_\phi^*, T_\phi]$ *and* $b\ker[T_\phi^*, T_\phi]$ *(some $b \in \mathcal{E}(\phi)$) are invariant under T_ϕ, then T_ϕ is normal or analytic.*

Proof. Write $K := \ker[T_\phi^*, T_\phi]$ and $R := \operatorname{ran}[T_\phi^*, T_\phi]$. If ϕ or $\overline{\phi}$ is of bounded type then by Abrahamse's theorem, T_ϕ is either normal or analytic. Suppose both ϕ and $\overline{\phi}$ are not of bounded type. We first claim that

$$\operatorname{cl} H_\phi\big(\ker\big[T_\phi^*, T_\phi\big]\big) = H^2. \tag{23}$$

To see this we observe that by the Nakazi–Takahashi theorem, there exists a finite Blaschke product $b \in \mathcal{E}(\phi)$ such that $\deg(b) = \dim R$. Since

$$T_\phi^* T_\phi - T_\phi T_\phi^* = H_{\bar\phi}^* H_{\bar\phi} - H_\phi^* H_\phi = H_{\bar\phi}^* H_{\bar\phi} - H_{b\bar\phi}^* H_{b\bar\phi} = H_{\bar\phi}^* H_{\bar b} H_{\bar b}^* H_{\bar\phi},$$

we have

$$\ker\big[T_\phi^*, T_\phi\big] = \ker H_{\bar b}^* H_{\bar\phi} = \ker(T_{\bar\phi} T_b - T_b T_{\bar\phi}),$$

which shows that $H_{\tilde{\bar b}} H_{\bar\phi}(K) = 0$, and hence $H_{\bar\phi}(K) \subseteq \tilde{b} H^2$, so that $\operatorname{cl} H_{\bar\phi} K \subseteq \tilde{b} H^2$. But since $\dim R < \infty$ and by (1), $H_{\bar\phi}$ is one-one and has dense range, we have

$$H^2 = \operatorname{cl} H_{\bar\phi}(K + R) = \operatorname{cl}(H_{\bar\phi} K + H_{\bar\phi} R) = \operatorname{cl} H_{\bar\phi} K + H_{\bar\phi} R.$$

We therefore have $\operatorname{cl} H_{\bar\phi} K = \tilde{b} H^2$ since $\dim H_{\bar\phi} R = \deg(b)$. Hence

$$\operatorname{cl} H_\phi K = \operatorname{cl} H_{b\bar\phi} K = \operatorname{cl} T_{\tilde{\bar b}}^* H_{\bar\phi} K = T_{\tilde{\bar b}} \tilde{b} H^2 = H^2,$$

which proves (23). On the other hand, we note that $\mathcal{E}(\phi)$ is a singleton set: otherwise, ϕ is of bounded type. Thus $\mathcal{E}(\phi)$ consists of only a finite Blaschke product b. We next argue that if $T_\phi(bK) \subseteq bH^2$ then

$$T_\phi(bk) = bT_\phi k \quad \text{for each } k \in K. \tag{24}$$

To see this, let $k \in K$ and write $k_1 := T_\phi k$. Thus $\phi k = k_1 + \bar{k}_2$ for some $k_2 \in H_0^2 = zH^2$. Then

$$T_\phi(bk) = P(b\phi k) = P(b\bar{k}_2 + bk_1) = P(b\bar{k}_2) + bk_1 = P(b\bar{k}_2) + bT_\phi k.$$

Since, by assumption, $T_\phi(bK) \subseteq bH^2$, it follows that $P(b\bar{k}_2) \in bH^2$. But since $P(b\bar{k}_2) \in (bH^2)^\perp$, we have $P(b\bar{k}_2) = 0$, which proves (24). Since $T_\phi T_b - T_b T_\phi = H_{\tilde{\bar b}} H_\phi$ it follows from (23) and (24) that

$$H_{\tilde{\bar b}} H^2 = H_{\tilde{\bar b}}(\operatorname{cl} H_\phi K) = \operatorname{cl} H_{\tilde{\bar b}} H_\phi K = \operatorname{cl}(T_\phi T_b - T_b T_\phi)K = 0,$$

which implies that $\bar{\bar{b}}H^2 \subseteq H^2$, so that $b = e^{i\theta}$ for some $\theta \in [0, 2\pi)$. Therefore ϕ is of the form $\phi = \overline{f} + e^{i\theta} f$ for some $f \in H^\infty$ and $\theta \in [0, 2\pi)$ which implies that T_ϕ is normal. □

We thus have:

Corollary 4.3. *Suppose T_ϕ is a subnormal Toeplitz operator with finite rank self-commutator. If $b \ker[T_\phi^*, T_\phi]$ is invariant under T_ϕ (some $b \in \mathcal{E}(\phi)$), then T_ϕ is normal or analytic.*

Proof. Since $\ker[T^*, T]$ is invariant under T for every subnormal operator T, the result follows at once from Theorem 4.2. □

We were not unable to decide whether the condition "$b \ker[T_\phi^*, T_\phi]$ (some $b \in \mathcal{E}(\phi)$) is invariant under T_ϕ" can be dropped from Corollary 4.3: in other words, if T_ϕ is a subnormal operator with finite rank self-commutator and $b \in \mathcal{E}(\phi)$, is $b \ker[T_\phi^*, T_\phi]$ invariant under T_ϕ? If the answer to this question is affirmative we can conclude that Conjecture 4.1 is true.

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