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Exact sequences for the homology of the matching complex ☆

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ABSTRACT

Building on work by Bouc and by Shareshian and Wachs, we provide a toolbox of long exact sequences for the reduced simplicial homology of the matching complex M_n, which is the simplicial complex of matchings in the complete graph K_n. Combining these sequences in different ways, we prove several results about the 3-torsion part of the homology of M_n. First, we demonstrate that there is nonvanishing 3-torsion in $\tilde{H}_d(\mathsf{M}_n;\mathbb{Z})$ whenever $\nu_n \leqslant d \leqslant \lfloor \frac{n-6}{2} \rfloor$, where $\nu_n = \lceil \frac{n-4}{3} \rceil$. By results due to Bouc and to Shareshian and Wachs, $\tilde{H}_{\nu_n}(\mathsf{M}_n;\mathbb{Z})$ is a nontrivial elementary 3-group for almost all n and the bottom nonvanishing homology group of M_n for all $n \neq 2$. Second, we prove that $\tilde{H}_d(\mathsf{M}_n;\mathbb{Z})$ is a nontrivial 3-group whenever $\nu_n \leqslant d \leqslant \lfloor \frac{2n-9}{5} \rfloor$. Third, for each $k \geqslant 0$, we show that there is a polynomial $f_k(r)$ of degree $3k$ such that the dimension of $\tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r};\mathbb{Z}_3)$, viewed as a vector space over $\mathbb{Z}_3$, is at most $f_k(r)$ for all $r \geqslant k+2$.

1. Introduction

Given a family Δ of graphs on a fixed vertex set, we identify each member of Δ with its edge set. In particular, if Δ is closed under deletion of edges, then Δ is an abstract simplicial complex.

A *matching* in a simple graph G is a subset σ of the edge set of G such that no vertex appears in more than one edge in σ. Let $\mathsf{M}(G)$ be the family of matchings in G; $\mathsf{M}(G)$ is a simplicial complex. We write $\mathsf{M}_n = \mathsf{M}(K_n)$, where K_n is the complete graph on the vertex set $[n] = \{1, \ldots, n\}$.

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The topology of M_n and related complexes has been subject to analysis in a number of theses [1,6,9–11,13,15] and papers [2–5,7,8,14,16,17,20]; see Wachs [19] for an excellent survey and further references.

Despite the simplicity of the definition, the homology of the matching complex M_n turns out to have a complicated structure. The rational homology is well-understood and easy to describe thanks to a beautiful result due to Bouc [5], but very little is known about the integral homology and the homology over finite fields.

Over the integers, the bottom nonvanishing reduced homology group of M_n is known to appear in degree $\nu_n = \lceil \frac{n-4}{3} \rceil$ and is an elementary 3-group for almost all n. For $n \equiv 1 \pmod 3$, this result is due to Bouc [5], who proved that $\tilde{H}_{r-1}(\mathsf{M}_{3r+1}; \mathbb{Z}) \cong \mathbb{Z}_3$ for $r \geqslant 2$; see Section 4.1. Shareshian and Wachs [17] settled the general case, proving that $\tilde{H}_{\nu_n}(\mathsf{M}_n; \mathbb{Z}) \cong (\mathbb{Z}_3)^{e_n}$ for some $e_n \geqslant 1$ whenever $n \geqslant 15$ or $n \in \{7, 10, 12, 13\}$; see Section 4.2. Regarding the exact value of e_n when $n \not\equiv 1 \pmod 3$, the best previously known upper bound is superexponential in n [17]. In Section 5.4, we improve on this bound as follows:

Theorem 1. *We have that e_{3r+3} is bounded by a polynomial in r of degree three and that e_{3r+5} is bounded by a polynomial in r of degree six. More generally, for every $k \geqslant 0$, the dimension of the $\mathbb{Z}_3$-vector space $\tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r}; \mathbb{Z}_3)$ is bounded by a polynomial in r of degree $3k$.*

To establish Theorem 1, we construct a new long exact sequence for the matching complex, relating the homology of $\mathsf{M}_n \setminus e$ to that of $\mathsf{M}_{n-2} \setminus e$, M_{n-3}, and M_{n-5}, where e is an edge and $\mathsf{M}_n \setminus e$ is the complex obtained from M_n by removing the 0-cell corresponding to this edge. See Section 3.5 for details. Combining this sequence with the long exact sequence for the pair $(\mathsf{M}_n, \mathsf{M}_n \setminus e)$ (see Section 3.4) and using an induction argument, we derive bounds of the form

$$\hat{\beta}_{k,r} \leqslant \hat{\beta}_{k,r-1} + C_k r^{3k-1}\bigl(1 + O(1/r)\bigr),$$

where $\hat{\beta}_{k,r} = \dim_{\mathbb{Z}_3} \tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r}; \mathbb{Z}_3)$. Summing over r, we obtain the desired result.

As it turns out, for any fixed $k \geqslant 0$ and for sufficiently large r, we have that $\tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r}; \mathbb{Z})$ is a nontrivial 3-group. In fact, we prove the following result in Section 5.2.

Theorem 2. *$\tilde{H}_d(\mathsf{M}_n; \mathbb{Z})$ is a nontrivial 3-group whenever $\lceil \frac{n-4}{3} \rceil \leqslant d \leqslant \lfloor \frac{2n-9}{5} \rfloor$.*

The groups being finite in the given interval is a consequence of Bouc's formula for the rational homology [5]; see Section 1.2. To settle the nonexistence of p-torsion in $\tilde{H}_d(\mathsf{M}_n; \mathbb{Z})$ for $p \neq 3$, we use three long exact sequences. Bouc [5] introduced two of these sequences, one of which relates M_n to M_{n-1} and M_{n-2} and the other M_n to M_{n-3} and M_{n-4}; see Sections 3.1 and 3.2, respectively. The third sequence is new but based on the same idea and relates M_n to M_{n-3}, M_{n-5}, and M_{n-6}; see Section 3.3.

These three sequences are all special cases of a more general construction involving a filtration of M_n with respect to a given parameter $m \in [n]$,

$$\Delta_n^0 \subseteq \Delta_n^1 \subseteq \cdots \subseteq \Delta_n^{\min\{m,n-m\}} = \mathsf{M}_n.$$

We obtain Δ_n^i from M_n by removing all matchings containing at least $i+1$ edges $ab = \{a, b\}$ such that $a \in [m]$ and $b \in [m+1, n] = \{m+1, m+2, \ldots, n-1, n\}$. It is a straightforward exercise to show that the relative homology of $(\Delta_n^i, \Delta_n^{i-1})$ is isomorphic to a direct sum of homology groups of $\mathsf{M}_{m-i} * \mathsf{M}_{n-m-i}$, where $*$ denotes simplicial join. For $m \in \{1, 2\}$, the construction boils down to Bouc's two exact sequences, whereas the parameter choice $m = 3$ yields our new exact sequence. For larger m, one would need more than one exact sequence to fully describe the correlations between the different matching complexes involved. See Section 2 for basic properties of the filtration.

The group $\tilde{H}_d(\mathsf{M}_n; \mathbb{Z})$ being nontrivial when d falls within the bounds of Theorem 2 is a consequence of the following result, which we prove in Section 5.1:

Theorem 3. *For $n \geqslant 1$, there is nonvanishing 3-torsion in $\tilde{H}_d(\mathsf{M}_n; \mathbb{Z})$ whenever $\lceil \frac{n-4}{3} \rceil \leqslant d \leqslant \lfloor \frac{n-6}{2} \rfloor$. In particular, $\tilde{H}_d(\mathsf{M}_n; \mathbb{Z})$ is nonzero if and only if $\lceil \frac{n-4}{3} \rceil \leqslant d \leqslant \lfloor \frac{n-3}{2} \rfloor$.*

Table 1
The homology of M_n for $n \leqslant 14$. T_1 and T_2 are nontrivial finite groups of exponent a multiple of 3 and 15, respectively; see Proposition 5.5 and Theorem 4.5

$\tilde{H}_d(\mathsf{M}_n;\mathbb{Z})$	$d=0$	1	2	3	4	5
$n=3$	$\mathbb{Z}^2$	–	–	–	–	–
4	$\mathbb{Z}^2$	–	–	–	–	–
5	–	$\mathbb{Z}^6$	–	–	–	–
6	–	$\mathbb{Z}^{16}$	–	–	–	–
7	–	$\mathbb{Z}_3$	$\mathbb{Z}^{20}$	–	–	–
8	–	–	$\mathbb{Z}^{132}$	–	–	–
9	–	–	$\mathbb{Z}_3^8 \oplus \mathbb{Z}^{42}$	$\mathbb{Z}^{70}$	–	–
10	–	–	$\mathbb{Z}_3$	$\mathbb{Z}^{1216}$	–	–
11	–	–	–	$\mathbb{Z}_3^{45} \oplus \mathbb{Z}^{1188}$	$\mathbb{Z}^{252}$	–
12	–	–	–	$\mathbb{Z}_3^{56}$	$\mathbb{Z}^{12440}$	–
13	–	–	–	$\mathbb{Z}_3$	$T_1 \oplus \mathbb{Z}^{24596}$	$\mathbb{Z}^{924}$
14	–	–	–	–	T_2	$\mathbb{Z}^{138048}$

To prove the first statement in Theorem 3, we only need Bouc's original two sequences and the results of Bouc and of Shareshian and Wachs about the bottom nonvanishing homology. The second statement is a consequence of the first statement and Bouc's formula for the rational homology of M_n.

In Section 4.1, we find another application of the new long exact sequence introduced in Section 3.3 as we present a new proof of Bouc's result that $\tilde{H}_{r-1}(\mathsf{M}_{3r+1};\mathbb{Z}) \cong \mathbb{Z}_3$ for $r \geqslant 2$.

So far, all our results have been about the existence of 3-torsion and the nonexistence of other torsion. Almost nothing is known about p-torsion when $p \neq 3$, but in a previous paper [12], the author used a result due to Andersen [1] to prove that $\tilde{H}_4(\mathsf{M}_{14};\mathbb{Z})$ is a finite nontrivial group of exponent a multiple of 15. We have not been able to detect 5-torsion in any other homology group $\tilde{H}_d(\mathsf{M}_n;\mathbb{Z})$, but in Section 5.3, we show that the case $5d = 2n-8$ is crucial for the general behavior:

Theorem 4. *For $q \geqslant 3$, if $\tilde{H}_{2q}(\mathsf{M}_{5q+4};\mathbb{Z})$ contains nonvanishing 5-torsion, then so does $\tilde{H}_{2q+u}(\mathsf{M}_{5q+4+2u};\mathbb{Z})$ for each $u \geqslant 0$. In particular, if $\tilde{H}_{2q}(\mathsf{M}_{5q+4};\mathbb{Z})$ contains nonvanishing 5-torsion for each $q \geqslant 3$, then so does $\tilde{H}_d(\mathsf{M}_n;\mathbb{Z})$ whenever $\lceil \frac{2n-8}{5} \rceil \leqslant d \leqslant \lfloor \frac{n-7}{2} \rfloor$.*

See Table 1 for the homology of M_n for $n \leqslant 14$. Many values were obtained via computer calculations [3]; we have yet to find a computer-free method for calculating $\tilde{H}_d(\mathsf{M}_n;\mathbb{Z})$ in the case that the group is not free and not of size 3.

1.1. Notation

For a finite set S, we let M_S denote the matching complex on the complete graph with vertex set S. In particular, $\mathsf{M}_{[n]} = \mathsf{M}_n$, where $[n] = \{1,\dots,n\}$. For integers $a \leqslant b$, we write $[a,b] = \{a, a+1, \dots, b-1, b\}$.

The *join* of two families of sets Δ and Σ, assumed to be defined on disjoint ground sets, is the family $\Delta * \Sigma = \{\delta \cup \sigma\colon\ \delta \in \Delta,\ \sigma \in \Sigma\}$.

Whenever we discuss the homology of a simplicial complex or the relative homology of a pair of simplicial complexes, we mean reduced simplicial homology. For a simplicial complex Σ and a coefficient ring $\mathbb{F}$, we denote the generator of $\tilde{C}_d(\Sigma;\mathbb{F})$ corresponding to a set $\{e_0,\dots,e_d\} \in \Sigma$ as $e_0 \wedge \cdots \wedge e_d$. Given a cycle z in a chain group $\tilde{C}_d(\Sigma;\mathbb{F})$, whenever we talk about z as an element in the induced homology group $\tilde{H}_d(\Sigma;\mathbb{F})$, we really mean the homology class of z.

We will often consider pairs of complexes (Γ,Δ) such that $\Gamma \setminus \Delta$ is a union of families of the form

$$\Sigma = \{\sigma\} * \mathsf{M}_S,$$

where $\sigma = \{e_1, \dots, e_s\}$ is a set of pairwise disjoint edges and S is a subset of $[n]$ such that $S \cap e_i = \emptyset$ for each i. We may write the chain complex of Σ as

$$\tilde{C}_d(\Sigma; \mathbb{F}) = (e_1 \wedge \cdots \wedge e_s)\mathbb{F} \otimes_{\mathbb{F}} \tilde{C}_{d-s}(\mathsf{M}_S; \mathbb{F}),$$

defining the boundary operator as

$$\partial(e_1 \wedge \cdots \wedge e_s \otimes_{\mathbb{F}} c) = (-1)^s e_1 \wedge \cdots \wedge e_s \otimes_{\mathbb{F}} \partial(c).$$

For simplicity, we will often suppress $\mathbb{F}$ from notation. For example, by some abuse of notation, we will write

$$\langle e_1 \wedge \cdots \wedge e_s \rangle \otimes \tilde{C}_{d-s}(\mathsf{M}_S) = (e_1 \wedge \cdots \wedge e_s)\mathbb{F} \otimes_{\mathbb{F}} \tilde{C}_{d-s}(\mathsf{M}_S; \mathbb{F}).$$

We say that a cycle z in $\tilde{C}_{d-1}(\mathsf{M}_n; \mathbb{F})$ has *type* $\left[\begin{smallmatrix} n_1 \\ d_1 \end{smallmatrix}\right] \wedge \cdots \wedge \left[\begin{smallmatrix} n_s \\ d_s \end{smallmatrix}\right]$ if there is a partition $[n] = \bigcup_{i=1}^s S_i$ such that size of S_i is n_i and such that $z = z_1 \wedge \cdots \wedge z_s$, where z_i is a cycle in $\tilde{C}_{d_i-1}(\mathsf{M}_{S_i}; \mathbb{F})$ for each i. We define a *refinement* of a type in the natural manner; $\left[\begin{smallmatrix} n_1 \\ d_1 \end{smallmatrix}\right] \wedge \cdots \wedge \left[\begin{smallmatrix} n_{s-2} \\ d_{s-2} \end{smallmatrix}\right] \wedge \left[\begin{smallmatrix} n_{s-1} \\ d_{s-1} \end{smallmatrix}\right] \wedge \left[\begin{smallmatrix} n_s \\ d_s \end{smallmatrix}\right]$ is a refinement of $\left[\begin{smallmatrix} n_1 \\ d_1 \end{smallmatrix}\right] \wedge \cdots \wedge \left[\begin{smallmatrix} n_{s-2} \\ d_{s-2} \end{smallmatrix}\right] \wedge \left[\begin{smallmatrix} n_{s-1}+n_s \\ d_{s-1}+d_s \end{smallmatrix}\right]$ and so on. We write $T \prec T'$ to denote that the type T is a refinement of the type T'. If z is of type T and $T \prec T'$, then z is also of type T'. Finally, we write $\left[\begin{smallmatrix} n \\ d \end{smallmatrix}\right]^2 = \left[\begin{smallmatrix} n \\ d \end{smallmatrix}\right] \wedge \left[\begin{smallmatrix} n \\ d \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} n \\ d \end{smallmatrix}\right]^3 = \left[\begin{smallmatrix} n \\ d \end{smallmatrix}\right] \wedge \left[\begin{smallmatrix} n \\ d \end{smallmatrix}\right] \wedge \left[\begin{smallmatrix} n \\ d \end{smallmatrix}\right]$, and so on.

When dealing with the group $\tilde{H}_d(\mathsf{M}_n; \mathbb{Z})$, we will find the following transformation very useful:

$$\begin{cases} k = 3d - n + 4, \\ r = n - 2d - 3, \end{cases} \quad \Leftrightarrow \quad \begin{cases} n = 2k + 1 + 3r, \\ d = k - 1 + r. \end{cases} \tag{1}$$

In particular, we have the equivalences

$$\left\lceil \frac{n-4}{3} \right\rceil \leqslant d \leqslant \left\lfloor \frac{n-3}{2} \right\rfloor \quad \Leftrightarrow \quad 2d + 3 \leqslant n \leqslant 3d + 4 \quad \Leftrightarrow \quad \begin{cases} k \geqslant 0, \\ r \geqslant 0. \end{cases}$$

For $n \geqslant 1$, Theorem 3 yields that $\tilde{H}_d(\mathsf{M}_n; \mathbb{Z})$ is nonzero if and only if these inequalities are satisfied.

1.2. Two classical results

Before proceeding, we list two classical results pertaining to the topology of the matching complex.

Theorem 1.1. *(See Bouc [5].) For $n \geqslant 1$, the homology group $\tilde{H}_d(\mathsf{M}_n; \mathbb{Q}) = \tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r}; \mathbb{Q})$ is nonzero if and only if*

$$\left\lceil \frac{n - \lfloor \sqrt{n} \rfloor - 2}{2} \right\rceil \leqslant d \leqslant \left\lfloor \frac{n-3}{2} \right\rfloor \quad \Leftrightarrow \quad \begin{cases} k \geqslant \binom{r}{2}, \\ r \geqslant 0. \end{cases}$$

Theorem 1.1 is an immediate consequence of a concrete formula for the rational homology of M_n; see Bouc [5] for details and Wachs' survey [19] for an overview.

Theorem 1.2. *(See Björner et al. [4].) For $n \geqslant 1$, M_n is $(\nu_n - 1)$-connected, where $\nu_n = \lceil \frac{n-4}{3} \rceil$.*

Indeed, the ν_n-skeleton of M_n is shellable [17] and even vertex decomposable [2]. As already mentioned in the introduction, there is nonvanishing homology in degree ν_n for all $n \neq 2$; see Section 4 for details.

2. Filtration of M_n with respect to a fixed vertex set

The following general construction forms the basis of the three exact sequences presented in Sections 3.1–3.3. The first two sequences already appeared in the work of Bouc [5], whereas the third one is new.

Given a vertex set $S \subseteq [n]$, form a sequence

$$\Delta_n^0 \subseteq \Delta_n^1 \subseteq \cdots \subseteq \Delta_n^{\min\{\#S, n-\#S\}}$$

of simplicial complexes, where we obtain Δ_n^{i-1} from M_n by removing all matchings σ containing at least i edges ab such that $a \in S$ and $b \in [n] \setminus S$. We also define $\Delta_n^{-1} = \emptyset$. Assuming that $S = [m]$, one easily checks that

$$\Delta_n^i \setminus \Delta_n^{i-1} = \bigcup \bigl\{\{a_1b_1, \ldots, a_ib_i\}\bigr\} * \mathsf{M}_{[m]\setminus A} * \mathsf{M}_{[m+1,n]\setminus B}, \tag{2}$$

where the union is over all pairs of sequences $(a_1, \ldots, a_i)$ and $(b_1, \ldots, b_i)$ of distinct elements such that $1 \leqslant a_1 < \cdots < a_i \leqslant m$ and $b_1, \ldots, b_i \in [m+1, n]$; $A = \{a_1, \ldots, a_i\}$ and $B = \{b_1, \ldots, b_i\}$. The families in the union form an antichain under inclusion, meaning that if σ belongs to one of the families and τ to another, then $\sigma \not\subseteq \tau$ and $\tau \not\subseteq \sigma$. One readily verifies that this implies the following:

Lemma 2.1. *For $0 \leqslant i \leqslant \min\{m, n-m\}$ and all d, we have that*

$$\tilde{H}_d\bigl(\Delta_n^i, \Delta_n^{i-1}\bigr) \cong \bigoplus \langle a_1b_1 \wedge \cdots \wedge a_ib_i\rangle \otimes \tilde{H}_{d-i}(\mathsf{M}_{[m]\setminus A} * \mathsf{M}_{[m+1,n]\setminus B}),$$

where the direct sum is over all pairs of ordered sequences $(a_1, \ldots, a_i)$ and $(b_1, \ldots, b_i)$ with properties as above.

As a consequence, we have a long exact sequence of the form

$$\begin{aligned} \cdots &\longrightarrow \textstyle\bigoplus_t \tilde{H}_{d-i+1}(\mathsf{M}_{m-i} * \mathsf{M}_{n-m-i}) \\ &\longrightarrow \tilde{H}_d(\Delta_n^{i-1}) \longrightarrow \tilde{H}_d(\Delta_n^i) \longrightarrow \textstyle\bigoplus_t \tilde{H}_{d-i}(\mathsf{M}_{m-i} * \mathsf{M}_{n-m-i}) \\ &\longrightarrow \tilde{H}_{d-1}(\Delta_n^{i-1}) \longrightarrow \cdots, \end{aligned}$$

where $t = i!\binom{m}{i}\binom{n-m}{i}$. For $m - i \leqslant 3$, the situation is particularly simple, as M_{m-i} is then either the empty complex $\{\emptyset\}$, a single point, or three isolated points. We will exploit this fact in Sections 3.1–3.3.

3. Five long exact sequences

We present five long exact sequences relating different families of matching complexes. Throughout this section, we consider an arbitrary coefficient ring $\mathbb{F}$ with unit, which we suppress from notation for convenience.

3.1. Long exact sequence relating M_n, M_{n-1}, and M_{n-2}

The choice $m = 1$ yields the simplest special case of the construction in Section 2. Inserting $i = 0$ and $i = 1$ in (2), we obtain families involving complexes isomorphic to M_{n-1} and M_{n-2}, respectively. More exactly, we have the following result.

Theorem 3.1. *(See Bouc [5].) For each $n \geqslant 2$, we have a long exact sequence*

$$\begin{aligned} \cdots &\longrightarrow \textstyle\bigoplus_{s=2}^n \langle 1s\rangle \otimes \tilde{H}_d(\mathsf{M}_{[2,n]\setminus\{s\}}) \\ &\longrightarrow \tilde{H}_d(\mathsf{M}_{[2,n]}) \longrightarrow \tilde{H}_d(\mathsf{M}_n) \xrightarrow{\ \omega\ } \textstyle\bigoplus_{s=2}^n \langle 1s\rangle \otimes \tilde{H}_{d-1}(\mathsf{M}_{[2,n]\setminus\{s\}}) \\ &\longrightarrow \tilde{H}_{d-1}(\mathsf{M}_{[2,n]}) \longrightarrow \cdots, \end{aligned}$$

where ω is induced by the natural projection map.

We refer to this sequence as the *0-1-2 sequence*, thereby indicating that the sequence relates M_{n-0}, M_{n-1}, and M_{n-2}.

3.2. *Long exact sequence relating* M_n, M_{n-3}, *and* M_{n-4}

We proceed with the case $m = 2$ of the construction in Section 2. In this case, $i \in \{1,2\}$ inserted into (2) yields families involving complexes isomorphic to M_{n-3} and M_{n-4}, whereas $i = 0$ yields a family involving contractible complexes; M_2 is a point. This turns out to imply the following result.

Theorem 3.2. *(See Bouc [5].) Let $n \geqslant 4$ and define*

$$Q_d^{n-4} = \bigoplus_{s \neq t \in [3,n]} \langle 1s \wedge 2t \rangle \otimes \tilde{H}_d(\mathsf{M}_{[3,n]\setminus\{s,t\}}),$$

$$R_d^{n-3} = \bigoplus_{a=1}^{2} \bigoplus_{u=3}^{n} \langle au \rangle \otimes \tilde{H}_d(\mathsf{M}_{[3,n]\setminus\{u\}}).$$

Then we have a long exact sequence

$$\begin{array}{l} \qquad\qquad\qquad\qquad \cdots \longrightarrow Q_{d-1}^{n-4} \\ \xrightarrow{\psi^*} R_{d-1}^{n-3} \xrightarrow{\varphi^*} \tilde{H}_d(\mathsf{M}_n) \xrightarrow{\kappa^*} Q_{d-2}^{n-4} \\ \xrightarrow{\psi^*} R_{d-2}^{n-3} \longrightarrow \cdots, \end{array}$$

where ψ^* is induced by the map $\psi : 1s \wedge 2t \otimes x \mapsto 2t \otimes x - 1s \otimes x$, φ^* is induced by the map $\varphi : au \otimes x \mapsto (au - 12) \wedge x$, and κ^* is induced by the natural projection map.

We refer to this sequence as the *0-3-4 sequence*.

3.3. *Long exact sequence relating* M_n, M_{n-3}, M_{n-5}, *and* M_{n-6}

For our third application of the construction in Section 2, we consider $m = 3$. In this case, the relevant matching complexes are isomorphic to $\mathsf{M}_{n-3}, \mathsf{M}_{n-5}$, and M_{n-6}.

As in Section 2, we define Δ_n^i to be the complex of matchings σ such that at most i of the vertices in $\{1,2,3\}$ are matched in $\sigma \setminus \{12, 13, 23\}$.

Lemma 3.3. *Let $n \geqslant 5$. We have an isomorphism*

$$\varphi^* : P_{d-2}^{n-5} \oplus Q_{d-1}^{n-3} \to \tilde{H}_d(\Delta_n^2),$$

where

$$P_d^{n-5} = \bigoplus_{1 \leqslant a < b \leqslant 3} \bigoplus_{s \neq t \in [4,n]} \langle as \wedge bt \rangle \otimes \tilde{H}_d(\mathsf{M}_{[4,n]\setminus\{s,t\}})$$

and

$$Q_d^{n-3} = \bigoplus_{c=2}^{3} \langle 1c \rangle \otimes \tilde{H}_d(\mathsf{M}_{[4,n]}).$$

The isomorphism φ^* is induced by the map φ defined by $\varphi(1c \otimes x) = \varphi(1c) \wedge x$, where $\varphi(1c) = 1c - 23$, and $\varphi(as \wedge bt \otimes x) = \varphi(as \wedge bt) \wedge x$, where

$$\varphi(as \wedge bt) = as \wedge bt + ac \wedge (st - bt) + bc \wedge (as - st)$$

and $\{a,b,c\} = \{1,2,3\}$.

Proof. First, we show that the sequence

$$0 \longrightarrow \tilde{H}_d(\Delta_n^1) \longrightarrow \tilde{H}_d(\Delta_n^2) \longrightarrow \tilde{H}_d(\Delta_n^2, \Delta_n^1) \longrightarrow 0 \tag{3}$$

is split exact for each d. To see this, first note that $\tilde{H}_d(\Delta_n^2, \Delta_n^1) \cong P_{d-2}^{n-5}$; apply Lemma 2.1. Next, define $\hat{\varphi}$ to be the restriction of φ to $\tilde{C}_d(\Delta_n^2, \Delta_n^1)$ and note that the projection of $\hat{\varphi}(as \wedge bt \otimes x)$ on $\tilde{C}_i(\Delta_n^2, \Delta_n^1)$ is again $as \wedge bt \otimes x$. Since $\hat{\varphi}$ clearly commutes with the boundary operator, the sequence (3) is split exact as desired. We conclude that we have an isomorphism

$$\tilde{H}_d(\Delta_n^2) \cong \tilde{H}_d(\Delta_n^1) \oplus P_{d-2}^{n-5}.$$

It remains to prove that the restriction of φ^* to Q_{d-1}^{n-3} defines an isomorphism from Q_{d-1}^{n-3} to $\tilde{H}_d(\Delta_n^1)$. By Lemma 2.1, that would be true if we replaced Δ_n^1 with Δ_n^0. Thus to conclude the proof, it suffices to prove that the relative homology of the pair (Δ_n^1, Δ_n^0) vanishes. By Lemma 2.1, we obtain that

$$\tilde{H}_d(\Delta_n^1, \Delta_n^0) \cong \bigoplus_{a=1}^{3} \bigoplus_{u=4}^{n} \langle au \rangle \otimes \tilde{H}_{d-1}(\mathsf{M}_{\{1,2,3\}\setminus\{a\}} * \mathsf{M}_{[4,n]\setminus\{u\}}) = 0;$$

the latter equality is a consequence of the fact that $\mathsf{M}_{\{1,2,3\}\setminus\{a\}} \cong \mathsf{M}_2$ is a point. □

Theorem 3.4. *Let $n \geqslant 6$. Define P_d^{n-5}, Q_d^{n-3}, and φ^* as in Lemma* 3.3 *and let*

$$R_d^{n-6} = \bigoplus_{(s,t,u)} \langle 1s \wedge 2t \wedge 3u \rangle \otimes \tilde{H}_d(\mathsf{M}_{[4,n]\setminus\{s,t,u\}}),$$

where the sum is over all triples of distinct integers (s,t,u) such that $s,t,u \in [4,n]$. Then we have a long exact sequence

$$\begin{aligned} &\cdots \longrightarrow R_{d-2}^{n-6} \\ &\xrightarrow{\psi^*} P_{d-2}^{n-5} \oplus Q_{d-1}^{n-3} \xrightarrow{\iota^* \circ \varphi^*} \tilde{H}_d(\mathsf{M}_n) \longrightarrow R_{d-3}^{n-6} \\ &\xrightarrow{\psi^*} P_{d-3}^{n-5} \oplus Q_{d-2}^{n-3} \longrightarrow \cdots, \end{aligned}$$

where ψ^* is induced by the map

$$\begin{aligned} \psi(1s \wedge 2t \wedge 3u \otimes x) &= 1s \wedge 2t \otimes x + 2t \wedge 3u \otimes x - 1s \wedge 3u \otimes x \\ &\quad + 12 \otimes (su - tu) \wedge x + 13 \otimes (tu - st) \wedge x \end{aligned}$$

and ι^* is induced by the natural inclusion map $\iota : \tilde{C}_d(\Delta_n^2) \to \tilde{C}_d(\mathsf{M}_n)$.

Proof. By Lemma 2.1, $\tilde{H}_d(\Delta_n^3, \Delta_n^2) \cong R_{d-3}^{n-6}$. Hence by Lemma 3.3, it remains to prove that ψ^* has properties as stated in the theorem. For this, note that the natural map

$$\hat{\psi} : \tilde{C}_d(\Delta_n^3, \Delta_n^2) \to \tilde{C}_d(\Delta_n^2)$$

is given by

$$\hat{\psi}(1s \wedge 2t \wedge 3u) = \partial(1s \wedge 2t \wedge 3u) = 1s \wedge 2t + 2t \wedge 3u - 1s \wedge 3u,$$

suppressing "$\otimes x$" from notation. Moreover, note that

$$\begin{aligned} \varphi\bigl(12 \otimes (su - tu) + 13 \otimes (tu - st)\bigr) &= (12 - 23) \wedge (su - tu) + (13 - 23) \wedge (tu - st) \\ &= 12 \wedge (su - tu) + 13 \wedge (tu - st) + 23 \wedge (st - su) \end{aligned}$$

and

$$\begin{aligned}
&\varphi(1s \wedge 2t + 2t \wedge 3u - 1s \wedge 3u) - \partial(1s \wedge 2t \wedge 3u) \\
&\quad = 13 \wedge (st - 2t) + 23 \wedge (1s - st) + 12 \wedge (tu - 3u) + 13 \wedge (2t - tu) \\
&\qquad - 12 \wedge (su - 3u) - 23 \wedge (1s - su) \\
&\quad = 12 \wedge (tu - su) + 13 \wedge (st - tu) + 23 \wedge (su - st).
\end{aligned}$$

Since ψ is given by $\varphi^{-1} \circ \hat{\psi}$, we are done. □

We refer to this sequence as the *0-3-5-6 sequence*.

Corollary 3.5. *For each $n \geqslant 6$, we have the exact sequence*

$$R^{n-6}_{\nu_n-2} \xrightarrow{\psi^*} P^{n-5}_{\nu_n-2} \oplus Q^{n-3}_{\nu_n-1} \xrightarrow{\iota^* \circ \varphi^*} \tilde{H}_{\nu_n}(\mathsf{M}_n) \longrightarrow 0,$$

where $\nu_n = \lceil \frac{n-4}{3} \rceil$. If $n \equiv 1 \pmod 3$, then $P^{n-5}_{\nu_n-2} = 0$.

Proof. This is immediate by Theorems 1.2 and 3.4. □

3.4. *Long exact sequence relating* M_n, $\mathsf{M}_n \setminus e$ *and* M_{n-2}

We proceed with the long exact sequence for the pair $(\mathsf{M}_n, \mathsf{M}_n \setminus e)$, where e is any edge and $\mathsf{M}_n \setminus e$ is the complex obtained by removing the 0-cell e.

Theorem 3.6. *For each $n \geqslant 2$ and each edge e in the complete graph K_n, we have a long exact sequence*

$$\begin{aligned}
&\cdots \longrightarrow \langle e \rangle \otimes \tilde{H}_d(\mathsf{M}_{[n]\setminus e}) \\
&\longrightarrow \tilde{H}_d(\mathsf{M}_n \setminus e) \longrightarrow \tilde{H}_d(\mathsf{M}_n) \xrightarrow{\omega} \langle e \rangle \otimes \tilde{H}_{d-1}(\mathsf{M}_{[n]\setminus e}) \\
&\longrightarrow \tilde{H}_{d-1}(\mathsf{M}_n \setminus e) \longrightarrow \cdots,
\end{aligned}$$

where ω is induced by the natural projection map.

Proof. Simply note that $\mathsf{M}_n \setminus (\mathsf{M}_n \setminus e) = \{\{e\}\} * \mathsf{M}_{[n]\setminus e}$. □

We refer to this sequence as the *0-e-2 sequence*. We will make use of this sequence when providing bounds on the homology in Section 5.4.

3.5. *Long exact sequence relating* $\mathsf{M}_n \setminus e$, $\mathsf{M}_{n-2} \setminus e$, M_{n-3}, *and* M_{n-5}

Using an approach similar to the one in Section 3.3, we construct a long exact sequence relating $\mathsf{M}_n \setminus e$, $\mathsf{M}_{n-2} \setminus e$, M_{n-3}, and M_{n-5}, where e is any edge. The main benefit of this sequence is that it provides good bounds on the homology when combined with the sequence in Section 3.4; see Section 5.4. Since we will not make use of the homomorphisms in this exact sequence, we do not define them explicitly; the interested reader will note that they are straightforward, though a bit cumbersome, to derive from the proof.

Theorem 3.7. *Let $n \geqslant 5$. Define*

$$P^{n-5}_d = \bigoplus_{s \neq t \in [4,n]} \langle 1s \wedge 2t \rangle \otimes \tilde{H}_d(\mathsf{M}_{[4,n]\setminus\{s,t\}})$$

and

$$Q_d^{n-2} = \bigoplus_{i=4}^{n} \langle 3u \rangle \otimes \tilde{H}_d(\mathsf{M}_{[n]\setminus\{3,u\}} \setminus 12).$$

Then we have a long exact sequence

$$\cdots \longrightarrow Q_d^{n-2}$$

$$\longrightarrow \langle 13 \rangle \otimes \tilde{H}_{d-1}(\mathsf{M}_{[4,n]}) \oplus P_{d-2}^{n-5} \longrightarrow \tilde{H}_d(\mathsf{M}_n \setminus 12) \longrightarrow Q_{d-1}^{n-2}$$

$$\longrightarrow \langle 13 \rangle \otimes \tilde{H}_{d-2}(\mathsf{M}_{[4,n]}) \oplus P_{d-3}^{n-5} \longrightarrow \cdots.$$

Proof. Consider the long exact sequence for the pair $(\mathsf{M}_n \setminus 12, \Delta_n^2)$, where Δ_n^2 is the complex obtained from $\mathsf{M}_n \setminus 12$ by removing the elements $34, \ldots, 3n$. Analogously to Lemma 2.1, we have that $\tilde{H}_d(\mathsf{M}_n \setminus 12, \Delta_n^2) \cong Q_{d-1}^{n-2}$.

To settle the theorem, it suffices to prove that

$$\tilde{H}_d(\Delta_n^2) \cong \langle 13 \rangle \otimes \tilde{H}_{d-1}(\mathsf{M}_{[4,n]}) \oplus P_{d-2}^{n-5}.$$

To achieve this goal, define Δ_n^1 to be the subcomplex of Δ_n^2 obtained by removing all faces containing $\{1s, 2t\}$ for some $s, t \in [4, n]$. Analogously to Lemma 2.1, we have that $\tilde{H}_d(\Delta_n^2, \Delta_n^1) \cong P_{d-2}^{n-5}$. A homomorphism φ^* from P_{d-2}^{n-5} to Δ_n^2 is given by mapping $1s \wedge 2t$ to the cycle

$$1s \wedge 2t + 2t \wedge 13 + 13 \wedge st + st \wedge 23 + 23 \wedge 1s.$$

It is clear that the natural map back to P_{d-2}^{n-5} has the property that $\varphi^*(1s \wedge 2t \otimes z)$ is mapped to $1s \wedge 2t \otimes z$; hence we have a split exact sequence just as in (3) in the proof of Lemma 3.3. This implies that

$$\tilde{H}_d(\Delta_n^2) \cong \tilde{H}_d(\Delta_n^1) \oplus P_{d-2}^{n-5},$$

again as in the proof of Lemma 3.3, except that the complexes and groups are different.

It remains to prove that $\tilde{H}_d(\Delta_n^1) \cong \langle 13 \rangle \otimes \tilde{H}_{d-1}(\mathsf{M}_{[4,n]})$. Let Δ_n^0 be the subcomplex of Δ_n^1 obtained by removing the elements $14, \ldots, 1n$ and $24, \ldots, 2n$. Since

$$\Delta_n^0 = \{\emptyset, 13, 23\} * \mathsf{M}_{[4,n]},$$

we obtain that $\tilde{H}_d(\Delta_n^0) \cong \langle 13 \rangle \otimes \tilde{H}_{d-1}(\mathsf{M}_{[4,n]})$. Thus the only thing remaining is to prove that $\tilde{H}_d(\Delta_n^1) \cong \tilde{H}_d(\Delta_n^0)$. Now,

$$\Delta_n^1 \setminus \Delta_n^0 = \bigcup_{a=1}^{2} \bigcup_{u=4}^{n} \{\{au\}\} * \mathsf{M}_{\{3-a,3\}} * \mathsf{M}_{[4,n]\setminus\{u\}},$$

which yields that

$$\tilde{H}_d(\Delta_n^1, \Delta_n^0) \cong \bigoplus_{a=1}^{2} \bigoplus_{u=4}^{n} \langle au \rangle \otimes \tilde{H}_{d-1}(\mathsf{M}_{\{3-a,3\}} * \mathsf{M}_{[4,n]\setminus\{u\}}) = 0;$$

the homology of a cone vanishes. $\square$

We refer to this sequence as the *0-2-3-5 sequence*.

4. Bottom nonvanishing homology

We consider the bottom nonvanishing homology group $\tilde{H}_{\nu_n}(\mathsf{M}_n;\mathbb{Z})$, starting with the case $n \equiv 1 \pmod 3$ in Section 4.1 and proceeding with the general case in Section 4.2.

Before examining the different cases, we present a nice result due to Shareshian and Wachs about the structure of the bottom nonvanishing homology group of M_n. Using the 0-3-5-6 sequence from Section 3.3, we may provide a more streamlined proof for the case $n \equiv 2 \pmod 3$.

Recall the concept of type introduced in Section 1.1.

Lemma 4.1. *(See Shareshian and Wachs [17].) For $k \in \{0,1,2\}$ and $r \geqslant 0$, the group $\tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r})$ is generated by cycles of type $\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]^r \wedge \left[\begin{smallmatrix}2k+1\\k\end{smallmatrix}\right]$.*

Proof. For $k \in \{0,1\}$, Shareshian and Wachs [17, Lemmas 2.3 and 2.5] provided a straightforward proof based on the tail end of the 0-3-4 sequence in Section 3.2. Assume that $k=2$ and write $n(k,r) = 2k+1+3r$ and $d(k,r) = k-1+r$; recall (1). The case $r=0$ is trivially true; hence assume that $r \geqslant 1$. The tail end in Corollary 3.5 becomes

$$P^{n(1,r-1)}_{d(1,r-1)} \oplus Q^{n(2,r-1)}_{d(2,r-1)} \xrightarrow{\iota^*\circ\varphi^*} \tilde{H}_{d(2,r)}(\mathsf{M}_{n(2,r)}) \longrightarrow 0.$$

By properties of $\iota^*\circ\varphi^*$, it follows that $\tilde{H}_{d(2,r)}(\mathsf{M}_{n(2,r)})$ is generated by cycles of type $\left[\begin{smallmatrix}5\\2\end{smallmatrix}\right] \wedge \left[\begin{smallmatrix}n(1,r-1)\\d(1,r-1)+1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right] \wedge \left[\begin{smallmatrix}n(2,r-1)\\d(2,r-1)+1\end{smallmatrix}\right]$. Now, a cycle of type $\left[\begin{smallmatrix}n(1,r-1)\\d(1,r-1)+1\end{smallmatrix}\right]$ is a sum of cycles of type $\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]^r$, whereas induction on r yields that a cycle of type $\left[\begin{smallmatrix}n(2,r-1)\\d(2,r-1)+1\end{smallmatrix}\right]$ is a sum of cycles of type $\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]^{r-1} \wedge \left[\begin{smallmatrix}5\\2\end{smallmatrix}\right]$. □

Lemma 4.1 does not generalize to arbitrary k. For example, for $(k,r) = (6,4)$, we obtain $\tilde{H}_9(\mathsf{M}_{25};\mathbb{Z})$, which is infinite by Theorem 1.1. In particular, this group cannot be generated by cycles of type $\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]^4 \wedge \left[\begin{smallmatrix}13\\6\end{smallmatrix}\right] \prec \left[\begin{smallmatrix}12\\4\end{smallmatrix}\right] \wedge \left[\begin{smallmatrix}13\\6\end{smallmatrix}\right]$, as these cycles all have finite exponent dividing three; $\tilde{H}_3(\mathsf{M}_{12};\mathbb{Z})$ is finite of exponent three.

4.1. The case $n \equiv 1 \pmod 3$

For $r \geqslant 0$, define

$$\gamma_{3r} = (12-23)\wedge(45-56)\wedge(78-89)\wedge\cdots\wedge\big((3r-2)(3r-1)-(3r-1)(3r)\big); \tag{4}$$

this is a cycle in both $\tilde{C}_{r-1}(\mathsf{M}_{3r};\mathbb{Z})$ and $\tilde{C}_{r-1}(\mathsf{M}_{3r+1};\mathbb{Z})$. By Lemma 4.1, $\tilde{H}_{r-1}(\mathsf{M}_{3r+1};\mathbb{Z})$ is generated by $\{\pi(\gamma_{3r})\colon \pi \in \mathfrak{S}_{3r+1}\}$, where the action of $\mathfrak{S}_{3r+1}$ on $\tilde{H}_{r-1}(\mathsf{M}_{3r+1};\mathbb{Z})$ is the one induced by the natural action on the underlying vertex set $[3r+1]$.

Using the long exact 0-3-5-6 sequence in Section 3.3, we give a new proof of a celebrated result due to Bouc about the bottom nonvanishing homology of M_n for $n \equiv 1 \pmod 3$.

Theorem 4.2. *(See Bouc [5].) For $r \geqslant 2$, we have that $\tilde{H}_{r-1}(\mathsf{M}_{3r+1};\mathbb{Z}) \cong \mathbb{Z}_3$.*

Proof. By Corollary 3.5, we have the exact sequence

$$R^{3r-5}_{r-3} \xrightarrow{\psi^*} Q^{3r-2}_{r-2} \xrightarrow{\iota^*\circ\varphi^*} \tilde{H}_{r-1}(\mathsf{M}_{3r+1}) \longrightarrow 0.$$

For $r=2$, this becomes

$$\bigoplus_{s,t,u}\langle 1s\wedge 2t\wedge 3u\rangle \xrightarrow{\psi^*} \bigoplus_{c=2}^{3}\langle 1c\rangle \otimes \tilde{H}_0(\mathsf{M}_{[4,7]}) \xrightarrow{\iota^*\circ\varphi^*} \tilde{H}_1(\mathsf{M}_7) \longrightarrow 0,$$

where the first direct sum ranges over all triples of distinct vertices $s, t, u \in [4,7]$. A basis for $\mathsf{M}_{[4,7]}$ is given by $\{45-56, 46-56\}$; hence a basis for Q_0^4 is given by $\{e_{25}, e_{26}, e_{35}, e_{36}\}$, where $e_{cd} = 1c \otimes (4d-56)$. Now,

$$\psi^*(1s \wedge 2t \wedge 3u) = 12 \otimes (su - tu) + 13 \otimes (tu - st);$$

apply Theorem 3.4 and Corollary 3.5. In particular, if $\{s,t,u\} = \{4,5,6\}$, then

$$\begin{aligned}\psi^*(1s \wedge 2t \wedge 37) &= 12 \otimes (s7 - t7) + 13 \otimes (t7 - st)\\ &= 12 \otimes (tu - su) + 13 \otimes (su - st)\\ &= \psi^*(1t \wedge 2s \wedge 3u).\end{aligned}$$

Similarly, $\psi^*(1s \wedge 27 \wedge 3u) = \psi^*(1u \wedge 2t \wedge 3s)$ and $\psi^*(17 \wedge 2t \wedge 3u) = \psi^*(1s \wedge 2u \wedge 3t)$. Moreover, one easily checks that

$$\psi^*(1s \wedge 2t \wedge 3u) + \psi^*(1t \wedge 2u \wedge 3s) + \psi^*(1u \wedge 2s \wedge 3t) = 0.$$

In particular, the image under ψ^* is generated by the four elements

$$\begin{aligned}\psi^*(14 \wedge 25 \wedge 36) &= 12 \otimes (46-56) + 13 \otimes (56-45) = e_{26} - e_{35};\\ \psi^*(14 \wedge 26 \wedge 35) &= 12 \otimes (45-56) + 13 \otimes (56-46) = e_{25} - e_{36};\\ \psi^*(15 \wedge 24 \wedge 36) &= 12 \otimes (56-46) + 13 \otimes (46-45) = -e_{26} - e_{35} + e_{36};\\ \psi^*(15 \wedge 26 \wedge 34) &= 12 \otimes (45-46) + 13 \otimes (46-56) = e_{25} - e_{26} + e_{36}.\end{aligned}$$

Since

$$\det\begin{pmatrix}0 & 1 & -1 & 0\\ 1 & 0 & 0 & -1\\ 0 & -1 & -1 & 1\\ 1 & -1 & 0 & 1\end{pmatrix} = 3,$$

it follows that $\tilde{H}_1(\mathsf{M}_7;\mathbb{Z}) \cong \mathbb{Z}_3$. Moreover, by Lemma 4.1 and symmetry, $\gamma_6 = (12-23) \wedge (45-56)$ must be a generator of $\tilde{H}_1(\mathsf{M}_7;\mathbb{Z})$.

For $r > 2$, assume by induction that $\tilde{H}_{r-2}(\mathsf{M}_{3r-2};\mathbb{Z})$ is a group of order three. Again by Lemma 4.1, this group is generated by any element of the form $\pi(\gamma_{3r-3})$, where γ_{3r-3} is defined as in (4) and $\pi \in \mathfrak{S}_{3r-2}$. Flipping $\pi(1)$ and $\pi(3)$ yields $-\pi(\gamma_{3r-3})$; hence the action of $\mathfrak{S}_{3r-2}$ on $\tilde{H}_{r-2}(\mathsf{M}_{3r-2};\mathbb{Z})$ is given by $\pi(z) = \mathrm{sgn}(\pi) \cdot z$.

By induction, we have the following exact sequence:

$$R_{r-3}^{3r-5} \xrightarrow{\psi^*} \langle 12\rangle \otimes \mathbb{Z}_3 \oplus \langle 13\rangle \otimes \mathbb{Z}_3 \xrightarrow{\iota^* \circ \varphi^*} \tilde{H}_{r-1}(\mathsf{M}_{3r+1}) \longrightarrow 0.$$

Another application of Theorem 3.4 and Corollary 3.5 yields that

$$\begin{aligned}\psi^*(1s \wedge 2t \wedge 3u \otimes z) &= 12 \otimes (su - tu) \wedge z + 13 \otimes (tu - st) \wedge z\\ &=: 12 \otimes \delta + 13 \otimes \delta'.\end{aligned}$$

Note that $\delta = (s,t,u)(\delta') = \delta'$, which implies that the image under ψ^* is contained in $(12+13) \otimes \mathbb{Z}_3$. Moreover,

$$\psi^*\big(14 \wedge 25 \wedge 36 \otimes \gamma_{3r-6}^{(6)}\big) = 12 \otimes (46-56) \wedge \gamma_{3r-6}^{(6)} + 13 \otimes (56-45) \wedge \gamma_{3r-6}^{(6)},$$

where $\gamma_{3r-6}^{(6)}$ is defined as in (4) but with all elements shifted six steps up. This is nonzero; hence the image under ψ^* is indeed equal to $(12+13) \otimes \mathbb{Z}_3$. We conclude that $\tilde{H}_{r-1}(\mathsf{M}_{3r+1};\mathbb{Z}) \cong \mathbb{Z}_3$. □

4.2. The general case

Bouc [5] proved that the exponent of $\tilde{H}_{\nu_n}(\mathsf{M}_n;\mathbb{Z})$ divides nine whenever $n = 3r + 3$ for some $r \geqslant 3$. Using the exact 0-3-4 sequence in Section 3.2, Shareshian and Wachs extended and improved this result:

Theorem 4.3. *(See Shareshian and Wachs [17].) For $n \in \{7, 10, 12, 13\}$ and for $n \geqslant 15$, $\tilde{H}_{\nu_n}(\mathsf{M}_n;\mathbb{Z})$ is of the form $(\mathbb{Z}_3)^{e_n}$ for some $e_n \geqslant 1$. The torsion subgroup of $\tilde{H}_{\nu_n}(\mathsf{M}_n;\mathbb{Z})$ is again an elementary 3-group for $n \in \{9, 11\}$ and zero for $n \in \{1, 2, 3, 4, 5, 6, 8\}$. For the remaining case $n = 14$, $\tilde{H}_{\nu_n}(\mathsf{M}_n;\mathbb{Z})$ is a finite group with nonvanishing 3-torsion.*

The only existing proofs for the cases $n \in \{9, 11, 12\}$ are computer-based. Our hope is that one may exploit properties of the exact sequences in this paper to find a proof without computer assistance.

By Theorem 4.2, $e_{3r+1} = 1$ whenever $r \geqslant 2$. In Section 5.1, we show that e_{3r+3} is bounded by a polynomial of degree 3 and that e_{3r+5} is bounded by a polynomial of degree 6.

Corollary 4.4. *For $n = 1$ and for $n \geqslant 3$, the group $\tilde{H}_{\nu_n}(\mathsf{M}_n;\mathbb{Z})$ is nonzero. In particular, the connectivity degree of M_n equals $\nu_n - 1$.*

For $n = 14$, the following is known:

Theorem 4.5. *(See Jonsson [12].) $\tilde{H}_4(\mathsf{M}_{14};\mathbb{Z})$ is a finite nontrivial group of exponent a multiple of 15.*

5. Higher-degree homology

In Section 5.1, we detect 3-torsion in higher-degree homology groups of M_n. In Section 5.2, we demonstrate that whenever the degree falls within a given interval, the whole homology group is a 3-group. We discuss the situation outside this interval in Section 5.3, providing some loose evidence for the existence of large intervals with 5-torsion. In Section 5.4, we proceed with upper bounds on the dimension of the homology over $\mathbb{Z}_3$.

5.1. 3-Torsion in higher-degree homology groups

This section builds on work previously published in the author's thesis [10]. First, let us state an elementary but useful result; the proof is straightforward.

Lemma 5.1. *Let $k \geqslant 1$ and let G be a graph on $2k$ vertices. Then $\mathsf{M}(G)$ admits a collapse to a complex of dimension at most $k - 2$.*

Let $k_0 \geqslant 0$ and let $\mathcal{G} = \{G_k\colon\ k \geqslant k_0\}$ be a family of graphs such that the following conditions hold:

- For each $k \geqslant k_0$, the vertex set of G_k is $[2k + 1]$.
- For each $k > k_0$ and for each vertex s such that $1s$ is an edge in G_k, the induced subgraph $G_k([2k + 1] \setminus \{1, s\})$ is isomorphic to G_{k-1}.

We say that such a family is *compatible*.

Proposition 5.2. *In each of the following three cases, $\mathcal{G} = \{G_k\colon\ k \geqslant k_0\}$ is a compatible family:*

(1) *$G_k = K_{2k+1}$ for all k.*
(2) *$G_k = K_{k+1,k}$ for all k, where $K_{k+1,k}$ is the complete bipartite graph with blocks $[k + 1]$ and $[k + 2, 2k + 1]$.*
(3) *$G_k = K_{2k+1} \setminus \{23, 45, 67, \ldots, 2k(2k + 1)\}$ for all k.*

Proof. It suffices to prove that $G_k([2k+1] \setminus \{1, s\})$ is isomorphic to G_{k-1} whenever $1s$ is an edge in G_k and $k > k_0$. This is immediate in all three cases. □

Now, fix $k_0, n, d \geqslant 0$. Let $\mathcal{G} = \{G_k\colon\ k \geqslant k_0\}$ be a family of compatible graphs and let γ be an element in $\tilde{H}_{d-1}(\mathsf{M}_n; \mathbb{Z})$, hence a cycle of type $\left[\begin{smallmatrix} n \\ d \end{smallmatrix}\right]$. For each $k \geqslant k_0$, define a map

$$\begin{cases} \theta_k : \tilde{H}_{k-1}\big(\mathsf{M}(G_k); \mathbb{Z}\big) \to \tilde{H}_{k-1+d}(\mathsf{M}_{2k+1+n}; \mathbb{Z}), \\ \theta_k(z) = z \wedge \gamma^{(2k+1)}, \end{cases}$$

where we obtain $\gamma^{(2k+1)}$ from γ by replacing each occurrence of the vertex i with $i + 2k + 1$ for every $i \in [n]$. Note that $\tilde{H}_{k-1}(\mathsf{M}(G_k); \mathbb{Z})$ is the top homology group of $\mathsf{M}(G_k)$ (provided G_k contains matchings of size k). For any prime p, we have that θ_k induces a homomorphism

$$\theta_k \otimes_{\mathbb{Z}} \iota_p : \tilde{H}_{k-1}\big(\mathsf{M}(G_k); \mathbb{Z}\big) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \tilde{H}_{k-1+d}(\mathsf{M}_{2k+1+n}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_p,$$

where $\iota_p : \mathbb{Z}_p \to \mathbb{Z}_p$ is the identity.

Theorem 5.3. *With notation and assumptions as above, if $\theta_{k_0} \otimes_{\mathbb{Z}} \iota_p$ is a monomorphism, then $\theta_k \otimes_{\mathbb{Z}} \iota_p$ is a monomorphism for each $k \geqslant k_0$. If, in addition, the exponent of γ in $\tilde{H}_{d-1}(\mathsf{M}_n; \mathbb{Z})$ is p, then we have a monomorphism*

$$\begin{cases} \hat{\theta}_k : \tilde{H}_{k-1}\big(\mathsf{M}(G_k); \mathbb{Z}\big) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \tilde{H}_{k-1+d}(\mathsf{M}_{2k+1+n}; \mathbb{Z}), \\ \hat{\theta}_k(z \otimes_{\mathbb{Z}} \lambda) = \theta_k(\lambda z) = \lambda z \wedge \gamma^{(2k+1)}, \end{cases}$$

for each $k \geqslant k_0$. In particular, the group $\tilde{H}_{k-1+d}(\mathsf{M}_{2k+1+n}; \mathbb{Z})$ contains p-torsion of rank at least the rank of $\tilde{H}_{k-1}(\mathsf{M}(G_k); \mathbb{Z})$.

Proof. To prove the first part of the theorem, we use induction on k; the base case $k = k_0$ is true by assumption. Assume that $k > k_0$ and consider the head end of the long exact sequence for the pair $(\mathsf{M}(G_k), \mathsf{M}(G_k \setminus \{1\}))$, where $G_k \setminus \{1\} = G_k([2k+1] \setminus \{1\})$:

$$0 \longrightarrow \tilde{H}_{k-1}(\mathsf{M}(G_k \setminus \{1\}); \mathbb{Z})$$

$$\longrightarrow \tilde{H}_{k-1}(\mathsf{M}(G_k); \mathbb{Z}) \xrightarrow{\hat{\omega}} P_{k-2}(G_k) \longrightarrow \tilde{H}_{k-2}(\mathsf{M}(G_k \setminus \{1\}); \mathbb{Z}).$$

Here,

$$P_{k-2}(G_k) = \bigoplus_{s:\ 1s \in G_k} \langle 1s \rangle \otimes \tilde{H}_{k-2}\big(\mathsf{M}\big(G_k \setminus \{1, s\}\big); \mathbb{Z}\big)$$

and $\hat{\omega}$ is defined in the natural manner.

Now, the group $\tilde{H}_{k-1}(\mathsf{M}(G_k \setminus \{1\}); \mathbb{Z})$ is zero by Lemma 5.1. As a consequence, $\hat{\omega}$ is a monomorphism. Moreover, all groups in the second row of the above sequence are torsion-free. Namely, the dimensions of $\mathsf{M}(G_k)$ and $\mathsf{M}(G_k \setminus \{1, s\})$ are at most $k-1$ and $k-2$, respectively, and Lemma 5.1 yields that $\mathsf{M}(G_k \setminus \{1\})$ is homotopy equivalent to a complex of dimension at most $k-2$. It follows that the induced homomorphism

$$\hat{\omega} \otimes \iota_p : \tilde{H}_{k-1}\big(\mathsf{M}(G_k); \mathbb{Z}\big) \otimes \mathbb{Z}_p \to P_{k-2}(G_k) \otimes \mathbb{Z}_p$$

remains a monomorphism.

Now, consider the following diagram:

$$\begin{array}{ccc} \tilde{H}_{k-1}(\mathsf{M}(G_k); \mathbb{Z}) \otimes \mathbb{Z}_p & \xrightarrow{\hat{\omega} \otimes \iota_p} & P_{k-2}(G_k) \otimes \mathbb{Z}_p \\ \Big\downarrow{\scriptstyle \theta_k \otimes \iota_p} & & \Big\downarrow{\scriptstyle \theta_{k-1}^{\oplus} \otimes \iota_p} \\ \tilde{H}_{k-1+d}(\mathsf{M}_{2k+1+n}; \mathbb{Z}) \otimes \mathbb{Z}_p & \xrightarrow{\omega \otimes \iota_p} & P_{k-2+d}^{2k-1+n} \otimes \mathbb{Z}_p. \end{array}$$

Here,

$$P_{k-2+d}^{2k-1+n} = \bigoplus_{s=2}^{2k+1+n} \langle 1s \rangle \otimes \tilde{H}_{k-2+d}(\mathsf{M}_{[2,2k+1+n]\setminus\{s\}}; \mathbb{Z}),$$

ω is defined as in Theorem 3.1, and $\theta_{k-1}^{\oplus}$ is defined by

$$\theta_{k-1}^{\oplus}(1s \otimes z) = 1s \otimes z \wedge \gamma^{(2k+1)}.$$

One easily checks that the diagram commutes; going to the right and then down or going down and then to the right both give the same map

$$\left(c_1 + \sum_{s:\ 1s \in G_k} 1s \wedge z_{1s}\right) \otimes 1 \mapsto \sum_{s:\ 1s \in G_k} \left(1s \otimes z_{1s} \wedge \gamma^{(2k+1)}\right) \otimes 1,$$

where c_1 is a sum of oriented simplices from $\mathsf{M}(G_k \setminus \{1\})$ and each z_{1s} is a sum of oriented simplices from $\mathsf{M}(G_k \setminus \{1, s\})$ satisfying $\partial(z_{1s}) = 0$ and $\partial(c_1) + \sum_s z_{1s} = 0$. Moreover, $\theta_{k-1}^{\oplus} \otimes \iota_p$ is a monomorphism, because the restriction to each summand is a monomorphism by induction on k. Namely, since $\mathcal{G}$ is compatible, $G_k \setminus \{1, s\}$ is isomorphic to G_{k-1} for each s such that $1s \in G_k$. As a consequence, $(\theta_{k-1}^{\oplus} \circ \hat{\omega}) \otimes \iota_p$ is a monomorphism, which implies that $\theta_k \otimes \iota_p$ is a monomorphism.

For the very last statement, it suffices to prove that $\hat{\theta}_k$ is a well-defined homomorphism, which is true if and only if $\theta_k(pz) = 0$ for each $z \in \tilde{H}_{k-1}(\mathsf{M}(G_k); \mathbb{Z})$. Now, let $c \in \tilde{C}_d(\mathsf{M}_n; \mathbb{Z})$ be such that $\partial(c) = p\gamma$; such a c exists by assumption. We obtain that

$$\partial\left(z \wedge c^{(2k+1)}\right) = \pm z \wedge \left(p\gamma^{(2k+1)}\right) = \pm(pz) \wedge \gamma^{(2k+1)};$$

hence $\theta_k(pz) = 0$ as desired. □

One may generalize Theorem 5.3 by allowing a whole family $\mathcal{G}_k$ of graphs for each k rather than just one single graph G_k. The condition for compatibility would then be that for any $G \in \mathcal{G}_k$ and for any s such that $1s \in G$, the induced subgraph $G([2k+1] \setminus \{1, s\})$ is isomorphic to some graph in $\mathcal{G}_{k-1}$. We do not need this generalization in this paper.

Theorem 5.4. *For $k \geqslant 0$ and $r \geqslant 4$, there is 3-torsion of rank at least $\binom{2k}{k}$ in $\tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r}; \mathbb{Z})$. Moreover, for $k \geqslant 0$, there is 3-torsion of rank at least $\binom{k+1}{\lfloor (k+1)/2 \rfloor}$ in $\tilde{H}_{k+2}(\mathsf{M}_{2k+10}; \mathbb{Z})$. To summarize, $\tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r}; \mathbb{Z})$ contains nonvanishing 3-torsion whenever $k \geqslant 0$ and $r \geqslant 3$.*

Proof. For the first statement, consider the compatible family $\{K_{2k+1}\colon\ k \geqslant 0\}$ and the cycle $\gamma_{3r} \in \tilde{H}_{r-1}(\mathsf{M}_{3r}; \mathbb{Z})$ defined as in (4). By Theorem 4.2 and Lemma 4.1,

$$\theta_0 \otimes \iota_3 : \tilde{H}_{-1}(\mathsf{M}_1; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_3 \cong \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_3 \to \tilde{H}_{r-1}(\mathsf{M}_{3r+1}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_3$$

defines an isomorphism, where $\theta_0(\lambda) = \lambda\gamma_{3r}^{(1)}$. By Lemma 4.1 and Theorem 4.3, γ_{3r} has exponent 3 in $\tilde{H}_{r-1}(\mathsf{M}_{3r}; \mathbb{Z})$; hence Theorem 5.3 yields that the group $\tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r}; \mathbb{Z})$ contains 3-torsion of rank at least the rank of the group $\tilde{H}_{k-1}(\mathsf{M}_{2k+1}; \mathbb{Z})$. By a result due to Bouc [5], this rank equals $\binom{2k}{k}$.

For the second statement, consider the compatible family $\{G_k = K_{2k+1} \setminus \{23, 45, 67, \ldots, 2k(2k+1)\}\colon k \geqslant 1\}$ and the cycle $\gamma_6 = (12 - 23) \wedge (45 - 56) \in \tilde{H}_1(\mathsf{M}_7; \mathbb{Z})$. For $k = 1$, we obtain that G_1 is the graph P_3 on three vertices with edge set $\{12, 13\}$; clearly, $\tilde{H}_0(\mathsf{M}(P_3); \mathbb{Z}) \cong \mathbb{Z}$. As a consequence,

$$\theta_1 \otimes \iota_3 : \tilde{H}_0\left(\mathsf{M}(P_3); \mathbb{Z}\right) \otimes_{\mathbb{Z}} \mathbb{Z}_3 \to \tilde{H}_2(\mathsf{M}_{10}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_3$$

is an isomorphism; apply Theorem 4.2. Proceeding as in the first case and using the fact that γ_6 has exponent 3 in $\tilde{H}_1(\mathsf{M}_7; \mathbb{Z})$, we conclude that $\tilde{H}_{k+1}(\mathsf{M}_{2k+8}; \mathbb{Z})$ contains 3-torsion of rank at least the rank of $\tilde{H}_{k-1}(\mathsf{M}(G_k); \mathbb{Z})$ for each $k \geqslant 1$.

It remains to show that the rank of $\tilde{H}_{k-1}(\mathsf{M}(G_k);\mathbb{Z})$ is at least $\binom{k}{\lfloor k/2\rfloor}$. Let A be any subset of the removed edge set

$$E = \{23, 45, \ldots, 2k(2k+1)\}$$

such that the size of A is $\lfloor k/2\rfloor$; write $B = E \setminus A$. Consider the complete bipartite graph G_k^A with one block equal to $\{1\} \cup \bigcup_{e\in A} e$ and the other block equal to $\bigcup_{e\in B} e$. For even k, the size of the "A" block is $k+1$; for odd k, the size of the "A" block is k. It is clear that G_k^A is a subgraph of G_k.

Label the vertices in $[2, 2k+1]$ as $s_1, t_1, s_2, t_2, \ldots, s_k, t_k$ such that $s_i t_i \in A$ for even i and $s_i t_i \in B$ for odd i. Consider the matching

$$\sigma_A = \{1s_1, t_1 s_2, t_2 s_3, \ldots, t_{k-1} s_k\}.$$

One easily checks that $\sigma_A \in \mathsf{M}(G_k^{A'})$ if and only if $A = A'$. Now, as observed by Shareshian and Wachs [17, (6.2)], $\mathsf{M}(G_k^A)$ is an orientable pseudomanifold. Defining z_A to be the fundamental cycle of $\mathsf{M}(G_k^A)$, we obtain that $\{z_A\colon\ A \subset E,\ \#A = \lfloor k/2\rfloor\}$ forms an independent set in $\tilde{H}_{k-1}(\mathsf{M}(G_k);\mathbb{Z})$, which concludes the proof. □

Let $G_k = K_{2k+1} \setminus \{23, 45, 67, \ldots, 2k(2k+1)\}$ be the graph in the above proof. Based on computer calculations for $k \leqslant 5$, we conjecture that the rank r_k of $\tilde{H}_{k-1}(\mathsf{M}(G_k);\mathbb{Z})$ equals the coefficient of x^k in $(1+x+x^2)^k$; this is sequence A002426 in Sloane's Encyclopedia [18]. Equivalently,

$$\sum_{k\geqslant 0} r_k x^k = \frac{1}{\sqrt{1-2x-3x^2}}.$$

Proposition 5.5. *(See Jonsson [12].) We have that $\tilde{H}_4(\mathsf{M}_{13};\mathbb{Z}) \cong T \oplus \mathbb{Z}^{24596}$, where T is a finite group containing $\mathbb{Z}_3^{10}$ as a subgroup.*

Corollary 5.6. *For $n \geqslant 1$, there is nonvanishing 3-torsion in the homology group $\tilde{H}_d(\mathsf{M}_n;\mathbb{Z}) = \tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r};\mathbb{Z})$ whenever*

$$\left\lceil \frac{n-4}{3} \right\rceil \leqslant d \leqslant \left\lfloor \frac{n-6}{2} \right\rfloor \quad \Leftrightarrow \quad \begin{cases} k \geqslant 0, \\ r \geqslant 3, \end{cases}$$

or $r = 2$ and $k \in \{0, 1, 2, 3\}$. Moreover, $\tilde{H}_d(\mathsf{M}_n;\mathbb{Z})$ is nonzero if and only if

$$\left\lceil \frac{n-4}{3} \right\rceil \leqslant d \leqslant \left\lfloor \frac{n-3}{2} \right\rfloor \quad \Leftrightarrow \quad \begin{cases} k \geqslant 0, \\ r \geqslant 0. \end{cases}$$

Proof. The first statement is a consequence of Theorem 5.4, Proposition 5.5, and Table 1. For the second statement, Theorem 1.1 yields that the group $\tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r};\mathbb{Z})$ is infinite if and only if $r \geqslant 0$ and $k \geqslant \binom{r}{2}$. In particular, the group is infinite for all $k \geqslant 0$ and $0 \leqslant r \leqslant 2$ except $(k, r) = (0, 2)$. Since $\tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r};\mathbb{Z}) \cong \mathbb{Z}_3$ when $k = 0$ and $r = 2$, we are done by Theorem 1.2 and Lemma 5.1. □

Corollary 5.6 suggests the following conjecture:

Conjecture 5.7. *The group $\tilde{H}_d(\mathsf{M}_n;\mathbb{Z}) = \tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r};\mathbb{Z})$ contains 3-torsion if and only if*

$$\left\lceil \frac{n-4}{3} \right\rceil \leqslant d \leqslant \left\lfloor \frac{n-5}{2} \right\rfloor \quad \Leftrightarrow \quad \begin{cases} k \geqslant 0, \\ r \geqslant 2. \end{cases}$$

By Corollary 5.6, the conjecture remains unsettled if and only if $r = 2$ and $k \geqslant 4$; for the cases $r = 0$ and $r = 1$, one easily checks that the homology is free. The conjecture would follow if we were able to settle Conjecture 6.2 in Section 6.

Table 2
$\mu_n = \lceil \frac{2n-8}{5} \rceil$ for different values of n

n	μ_n	n	μ_n
$5q-5$	$2q-3$	$5q$	$2q-1$
$5q-4$	$2q-3$	$5q+1$	$2q-1$
$5q-3$	$2q-2$	$5q+2$	$2q$
$5q-2$	$2q-2$	$5q+3$	$2q$
$5q-1$	$2q-2$	$5q+4$	$2q$

5.2. Intervals with vanishing homology over $\mathbb{Z}_p$ for $p \neq 3$

Throughout this section, let p be a prime different from 3. Using the exact sequences in Sections 3.1–3.2, and 3.3, we provide bounds on d and n such that $\tilde{H}_d(\mathsf{M}_n; \mathbb{Z}_p)$ is zero.

Theorem 5.8. *The group $\tilde{H}_d(\mathsf{M}_n; \mathbb{Z}_p) = \tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r}; \mathbb{Z}_p)$ is zero unless $2n-8 \leqslant 5d \Leftrightarrow r \leqslant k+1$. Moreover, for each $q \geqslant 0$, the following hold (notation as in Section 1.1):*

- *$\tilde{H}_{2q-1}(\mathsf{M}_{5q})$ is generated by cycles of type $\left[\begin{smallmatrix}5\\2\end{smallmatrix}\right]^q$ and $\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right] \wedge \left[\begin{smallmatrix}5q-3\\2q-1\end{smallmatrix}\right]$.*
- *$\tilde{H}_{2q-1}(\mathsf{M}_{5q+1})$ is generated by cycles of type $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \wedge \left[\begin{smallmatrix}5\\2\end{smallmatrix}\right]^q$.*
- *$\tilde{H}_{2q}(\mathsf{M}_{5q+3})$ is generated by cycles of type $\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right] \wedge \left[\begin{smallmatrix}5\\2\end{smallmatrix}\right]^q$.*
- *$\tilde{H}_{2q}(\mathsf{M}_{5q+4})$ is generated by cycles of type $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \wedge \left[\begin{smallmatrix}3\\1\end{smallmatrix}\right] \wedge \left[\begin{smallmatrix}5\\2\end{smallmatrix}\right]^q$.*

Proof. Writing $\mu_n = \lceil \frac{2n-8}{5} \rceil$, we obtain Table 2, which might be of some help when reading this proof.

One easily checks the theorem for $n \leqslant 5$; thus assume that $n \geqslant 6$. Assume inductively that the theorem is true for all $m \leqslant n-1$. We have five cases for n:

• $n = 5q$. The first case is perhaps the hardest. By the long exact 0-3-4 sequence in Section 3.2, we have an exact sequence of the form

$$\bigoplus \tilde{H}_{d-1}(\mathsf{M}_{5q-3}) \longrightarrow \tilde{H}_d(\mathsf{M}_{5q}) \longrightarrow \bigoplus \tilde{H}_{d-2}(\mathsf{M}_{5q-4}).$$

By induction, the groups on the left and right are zero whenever $d < 2q-1$, which implies that the same is true for the group in the middle.

It remains to prove that $\tilde{H}_{2q-1}(\mathsf{M}_{5q})$ is generated by cycles of type $\left[\begin{smallmatrix}5\\2\end{smallmatrix}\right]^q$ and $\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right] \wedge \left[\begin{smallmatrix}5q-3\\2q-1\end{smallmatrix}\right]$. For this, consider the tail end of the long exact 0-3-4 sequence:

$$\bigoplus_{a,u} \langle au \rangle \otimes \tilde{H}_{2q-2}(\mathsf{M}_{[3,5q]\setminus\{u\}}) \xrightarrow{\varphi^*} \tilde{H}_{2q-1}(\mathsf{M}_{5q})$$

$$\xrightarrow{\kappa^*} \bigoplus_{s,t} \langle 1s \wedge 2t \rangle \otimes \tilde{H}_{2q-3}(\mathsf{M}_{[3,5q]\setminus\{s,t\}}) \longrightarrow 0;$$

see Theorem 3.2. To generate $\tilde{H}_{2q-1}(\mathsf{M}_{5q})$, we will combine two sets of cycles:

(1) The first set consists of the image under φ^* of an appropriate set of generators of the first group in the exact sequence.
(2) The second set consists of an appropriate set of cycles in $\tilde{H}_{2q-1}(\mathsf{M}_{5q})$ such that the image under κ^* of this set generates the third group in the sequence.

(1) By properties of φ^*, the image of any cycle in the leftmost group has type $\left[{3 \atop 1}\right] \wedge \left[{5q-3 \atop 2q-1}\right]$.

(2) Induction yields that $\tilde{H}_{2q-3}(\mathsf{M}_{[3,5q]\setminus\{s,t\}}) \cong \tilde{H}_{2q-3}(\mathsf{M}_{5q-4})$ is generated by cycles of type $\left[{1 \atop 0}\right] \wedge \left[{5 \atop 2}\right]^{q-1}$. Now, consider a cycle $z \in \tilde{H}_{2q-3}(\mathsf{M}_{[3,5q]\setminus\{s,t\}})$ of type $\left[{1 \atop 0}\right] \wedge \left[{5 \atop 2}\right]^{q-1}$; let x be the unused element in z corresponding to the empty cycle of type $\left[{1 \atop 0}\right]$. Define

$$\gamma = 1s \wedge 2t + 2t \wedge sx + sx \wedge 12 + 12 \wedge tx + tx \wedge 1s.$$

It is clear that κ^* maps $\gamma \wedge z$ to $1s \wedge 2t \otimes z$ and that $\gamma \wedge z$ has type $\left[{5 \atop 2}\right]^q$. Thus we are done.

- $n = 5q + 1$. Again using the long exact 0-3-4 sequence in Section 3.2, we deduce that $\tilde{H}_d(\mathsf{M}_{5q+1})$ is zero whenever $\tilde{H}_{d-1}(\mathsf{M}_{5q-2})$ and $\tilde{H}_{d-2}(\mathsf{M}_{5q-3})$ are zero, which is true for $d < 2q - 1$. For $d = 2q - 1$, we obtain the exact sequence

$$\bigoplus_{a,u} \langle au\rangle \otimes \tilde{H}_{2q-2}(\mathsf{M}_{[3,5q+1]\setminus\{u\}}) \xrightarrow{\varphi^*} \tilde{H}_{2q-1}(\mathsf{M}_{5q+1}) \longrightarrow 0.$$

By induction, $\tilde{H}_{2q-2}(\mathsf{M}_{[3,5q+1]\setminus\{u\}}) \cong \tilde{H}_{2q-2}(\mathsf{M}_{5q-2})$ is generated by cycles of type $\left[{3 \atop 1}\right] \wedge \left[{5 \atop 2}\right]^{q-1}$. Hence properties of φ^* yield that $\tilde{H}_{2q-1}(\mathsf{M}_{5q+1})$ is generated by cycles of type $\left[{3 \atop 1}\right] \wedge \left[{3 \atop 1}\right] \wedge \left[{5 \atop 2}\right]^{q-1}$. By the exact sequence for the pair $(\mathsf{M}_7, \mathsf{M}_6)$ in Section 3.1 and the fact that $\tilde{H}_1(\mathsf{M}_7) = 0$, we have that $\tilde{H}_1(\mathsf{M}_6; \mathbb{Z})$ is generated by cycles of type $\left[{1 \atop 0}\right] \wedge \left[{5 \atop 2}\right]$; use Theorem 3.1. As a consequence, any cycle of type $\left[{3 \atop 1}\right] \wedge \left[{3 \atop 1}\right] \wedge \left[{5 \atop 2}\right]^{q-1}$ can be written as a sum of cycles of type $\left[{1 \atop 0}\right] \wedge \left[{5 \atop 2}\right] \wedge \left[{5 \atop 2}\right]^{q-1} = \left[{1 \atop 0}\right] \wedge \left[{5 \atop 2}\right]^q$.

- $n = 5q + 2$. Using the long exact 0-1-2 sequence in Section 3.1, we conclude that $\tilde{H}_d(\mathsf{M}_{5q+2})$ is zero whenever $\tilde{H}_d(\mathsf{M}_{5q+1})$ and $\tilde{H}_{d-1}(\mathsf{M}_{5q})$ are zero, which is true for $d < 2q - 1$. For $d = 2q - 1$, we have the exact sequence

$$\tilde{H}_{2q-1}(\mathsf{M}_{[2,5q+2]}) \xrightarrow{\iota^*} \tilde{H}_{2q-1}(\mathsf{M}_{5q+2}) \longrightarrow 0,$$

where ι^* is induced by the inclusion map. By induction, $\tilde{H}_{2q-1}(\mathsf{M}_{[2,5q+2]}) \cong \tilde{H}_{2q-1}(\mathsf{M}_{5q+1})$ is generated by cycles of type $\left[{1 \atop 0}\right] \wedge \left[{5 \atop 2}\right]^q$. It follows that the group $\tilde{H}_{2q-1}(\mathsf{M}_{5q+2})$ is generated by cycles of type $\left[{2 \atop 0}\right] \wedge \left[{5 \atop 2}\right]^q$, which means that $\tilde{H}_{2q-1}(\mathsf{M}_{5q+2}) = 0$.

- $n = 5q + 3$. This time, we use the long exact 0-3-5-6 sequence from Section 3.3. By properties of this sequence, the group $\tilde{H}_d(\mathsf{M}_{5q+3})$ is zero whenever $\tilde{H}_{d-1}(\mathsf{M}_{5q})$, $\tilde{H}_{d-2}(\mathsf{M}_{5q-2})$, and $\tilde{H}_{d-3}(\mathsf{M}_{5q-3})$ are zero, which is true for $d < 2q$. For $d = 2q$, we have a surjection

$$\begin{array}{c} \bigoplus \langle as \wedge bt\rangle \otimes \tilde{H}_{2q-2}(\mathsf{M}_{[4,5q+3]\setminus\{s,t\}}) \oplus \bigoplus \langle 1c\rangle \otimes \tilde{H}_{2q-1}(\mathsf{M}_{[4,5q+3]}) \\ \Big\downarrow \varphi^* \\ \tilde{H}_{2q}(\mathsf{M}_{5q+3}) \end{array}$$

defined as in Lemma 3.3. To establish that $\tilde{H}_{2q}(\mathsf{M}_{5q+3})$ is generated by cycles of type $\left[{3 \atop 1}\right] \wedge \left[{5 \atop 2}\right]^q$, it suffices to prove that $\tilde{H}_{2q}(\mathsf{M}_{5q+3})$ is generated by cycles of type $\left[{5 \atop 2}\right] \wedge \left[{5q-2 \atop 2q-1}\right]$. Namely, by induction, $\tilde{H}_{2q-2}(\mathsf{M}_{5q-2})$ is generated by cycles of type $\left[{3 \atop 1}\right] \wedge \left[{5 \atop 2}\right]^{q-1}$.

Induction yields that $\tilde{H}_{2q-2}(\mathsf{M}_{[4,5q+3]\setminus\{s,t\}}) \cong \tilde{H}_{2q-2}(\mathsf{M}_{5q-2})$ is generated by cycles of type $\left[{3 \atop 1}\right] \wedge \left[{5q-5 \atop 2q-2}\right]^{q-1}$ and that $\tilde{H}_{2q-1}(\mathsf{M}_{[4,5q+3]}) \cong \tilde{H}_{2q-1}(\mathsf{M}_{5q})$ is generated by cycles of type $\left[{5 \atop 2}\right] \wedge \left[{5q-5 \atop 2q-2}\right]$ and $\left[{3 \atop 1}\right] \wedge \left[{5q-3 \atop 2q-1}\right]$. By properties of φ^*, it follows that $\tilde{H}_{2q}(\mathsf{M}_{5q+3})$ is generated by cycles of the following types:

- $\left[{5 \atop 2}\right] \wedge \left[{3 \atop 1}\right] \wedge \left[{5q-5 \atop 2q-2}\right] \prec \left[{5 \atop 2}\right] \wedge \left[{5q-2 \atop 2q-1}\right]$;
- $\left[{3 \atop 1}\right] \wedge \left[{5 \atop 2}\right] \wedge \left[{5q-5 \atop 2q-2}\right] \prec \left[{5 \atop 2}\right] \wedge \left[{5q-2 \atop 2q-1}\right]$;
- $\left[{3 \atop 1}\right] \wedge \left[{3 \atop 1}\right] \wedge \left[{5q-3 \atop 2q-1}\right]$.

By the discussion at the end of the case $n = 5q + 1$, cycles of the very last type can be written as a sum of cycles of type $\begin{bmatrix}1\\0\end{bmatrix} \wedge \begin{bmatrix}5\\2\end{bmatrix} \wedge \begin{bmatrix}5q-3\\2q-1\end{bmatrix}$. As a consequence, $\tilde{H}_{2q}(\mathsf{M}_{5q+3})$ is generated by cycles of type $\begin{bmatrix}5\\2\end{bmatrix} \wedge \begin{bmatrix}5q\\2q\end{bmatrix}$.

• $n = 5q + 4$. For the final case, we again consider the long exact 0-1-2 sequence from Section 3.1. We obtain that $\tilde{H}_d(\mathsf{M}_{5q+4})$ is zero whenever $\tilde{H}_d(\mathsf{M}_{5q+3})$ and $\tilde{H}_{d-1}(\mathsf{M}_{5q+2})$ are zero, which is true for $d < 2q$.

To conclude the proof, it remains to show that $\tilde{H}_{2q}(\mathsf{M}_{5q+4})$ is generated by cycles of type $\begin{bmatrix}1\\0\end{bmatrix} \wedge \begin{bmatrix}3\\1\end{bmatrix} \wedge \begin{bmatrix}5\\2\end{bmatrix}^q$. Now, induction yields that $\tilde{H}_{2q}(\mathsf{M}_{5q+3})$ is generated by cycles of type $\begin{bmatrix}3\\1\end{bmatrix} \wedge \begin{bmatrix}5\\2\end{bmatrix}^q$. Hence $\tilde{H}_{2q}(\mathsf{M}_{5q+4})$ is generated by cycles of type $\begin{bmatrix}1\\0\end{bmatrix} \wedge \begin{bmatrix}3\\1\end{bmatrix} \wedge \begin{bmatrix}5\\2\end{bmatrix}^q$ as desired. □

Corollary 5.9. *If*

$$\left\lceil \frac{n-4}{3} \right\rceil \leqslant d \leqslant \left\lfloor \frac{2n-9}{5} \right\rfloor \quad \Leftrightarrow \quad 0 \leqslant k \leqslant r-2,$$

then $\tilde{H}_d(\mathsf{M}_n;\mathbb{Z}) = \tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r};\mathbb{Z})$ *is a nontrivial* 3*-group.*

Proof. This is an immediate consequence of Corollary 5.6, Theorem 5.8, and the universal coefficient theorem. □

While the bottom nonvanishing groups are *elementary* 3-groups by Theorem 4.3, we do not know whether this is true in general for the groups under consideration.

The smallest n for which Corollary 5.9 implies something previously unknown is $n = 22$, in which case we may conclude that $\tilde{H}_7(\mathsf{M}_{22};\mathbb{Z})$ is a 3-group; note that $\nu_{22} = 6$.

5.3. On the existence of further 5*-torsion*

One may ask whether the upper bound $\frac{2n-9}{5}$ in Corollary 5.9 is best possible, meaning that there is p-torsion for some $p \neq 3$, most likely $p = 5$, in degree $\lceil \frac{2n-8}{5} \rceil$ of the homology of M_n whenever the group under consideration is finite. Our hope is that this is indeed the case. While we do not have much evidence to support this hope, we can provide the following potentially useful result.

Theorem 5.10. *For each* $q \geqslant 3$*, there is nonvanishing* 5*-torsion in the group* $\tilde{H}_{2q}(\mathsf{M}_{5q+4};\mathbb{Z})$ *if and only if there is a cycle* $\gamma \in \tilde{H}_{2q}(\mathsf{M}_{5q+4};\mathbb{Z})$ *of type* $\begin{bmatrix}1\\0\end{bmatrix} \wedge \begin{bmatrix}3\\1\end{bmatrix} \wedge \begin{bmatrix}5\\2\end{bmatrix}^q$ *such that* γ *has exponent* 5*. If this is the case, then there is nonvanishing* 5*-torsion in* $\tilde{H}_{2q+u}(\mathsf{M}_{5q+4+2u};\mathbb{Z})$ *for each* $u \geqslant 0$*.*

Proof. The first statement is an immediate consequence of Theorem 5.8.

For the second statement, assume that γ is a cycle with properties as in the theorem. Write $\gamma = \gamma_5 \wedge \gamma'$, where γ_5 is of type $\begin{bmatrix}5\\2\end{bmatrix}$ and γ' is of type $\begin{bmatrix}1\\0\end{bmatrix} \wedge \begin{bmatrix}3\\1\end{bmatrix} \wedge \begin{bmatrix}5\\2\end{bmatrix}^{q-1}$. It is clear that the exponent of γ' in $\tilde{H}_{2q-2}(\mathsf{M}_{5q-1};\mathbb{Z})$ is a finite multiple of 5. Namely, γ' is of type $\begin{bmatrix}14\\5\end{bmatrix} \wedge \begin{bmatrix}5q-15\\2q-6\end{bmatrix}$ and γ has exponent 5.

Now, consider the compatible family $\mathcal{G} = \{K_{k+1,k}\colon\ k \geqslant 2\}$; recall Proposition 5.2. We claim that every element $z \in \tilde{H}_1(\mathsf{M}_5;\mathbb{Z})$ has the property that $2z$ is a sum of cycles, each having the form

$$ac \wedge bd + bd \wedge ae + ae \wedge bc + bc \wedge ad + ad \wedge be + be \wedge ac,$$

where $\{a,b,c,d,e\} = [5]$; this is the fundamental cycle of $\mathsf{M}_{G_{a,b}}$, where $G_{a,b}$ is the complete bipartite graph with blocks $\{a,b\}$ and $\{c,d,e\}$. To prove the claim, let T be the subgroup of $\tilde{H}_1(\mathsf{M}_5;\mathbb{Z})$ generated by the fundamental cycles of $G_{1,2}$, $G_{2,3}$, $G_{3,4}$, $G_{4,5}$,$G_{5,1}$, and $G_{1,3}$. One easily checks that the matrix of the natural projection from T to the group generated by $51 \wedge 23$, $12 \wedge 34$, $23 \wedge 45$, $34 \wedge 51$, $45 \wedge 12$, and $13 \wedge 24$ has determinant ± 2. Since $\tilde{H}_1(\mathsf{M}_5;\mathbb{Z}) \cong \mathbb{Z}^6$, the claim is settled.

As a consequence, we may assume that γ_5 is the fundamental cycle of $\mathsf{M}(K_{3,2})$. In particular, the map

$$\theta_2 \otimes \iota_5 : \tilde{H}_1\big(\mathsf{M}(K_{3,2}); \mathbb{Z}\big) \otimes_{\mathbb{Z}} \mathbb{Z}_5 \to \tilde{H}_{2q}(\mathsf{M}_{5q+4}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_5$$

defined by $\theta_2(z) = z \wedge \gamma'$ is a monomorphism; $\gamma = \gamma_5 \wedge \gamma'$. Now, applying Theorem 5.3, we deduce that we have a monomorphism

$$\theta_{2+u} \otimes \iota_5 : \tilde{H}_{1+u}\big(\mathsf{M}(K_{3+u,2+u}); \mathbb{Z}\big) \otimes_{\mathbb{Z}} \mathbb{Z}_5 \to \tilde{H}_{2q+u}(\mathsf{M}_{5q+4+2u}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_5$$

defined by $\theta_{2+u}(z) = z \wedge (\gamma')^{(2u)}$ for each $u \geqslant 0$; notation is as in Section 5.1. Since the exponent of γ' in $\tilde{H}_{2q-2}(\mathsf{M}_{5q-1}; \mathbb{Z})$ is a finite multiple of 5, there is indeed nonvanishing 5-torsion in $\tilde{H}_{2q+u}(\mathsf{M}_{5q+4+2u}; \mathbb{Z})$. □

Corollary 5.11. *If there is nonvanishing* 5*-torsion in the group* $\tilde{H}_{2q}(\mathsf{M}_{5q+4}; \mathbb{Z})$ *for each* $q \geqslant 3$*, then* $\tilde{H}_d(\mathsf{M}_n; \mathbb{Z}) = \tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r}; \mathbb{Z})$ *contains nonvanishing* 5*-torsion whenever*

$$\left\lceil \frac{2n-8}{5} \right\rceil \leqslant d \leqslant \left\lfloor \frac{n-7}{2} \right\rfloor \quad \Leftrightarrow \quad 4 \leqslant r \leqslant k+1.$$

5.4. Bounds on the homology over $\mathbb{Z}_3$

The goal of this section is to provide nontrivial upper bounds on the dimension of $\tilde{H}_d(\mathsf{M}_n; \mathbb{Z}_3)$ when n and d satisfy the conditions in Corollary 5.9. To achieve this, we use the long exact 0-e-2 sequence from Section 3.4 and the long exact 0-2-3-5 sequence from Section 3.5.

Define

$$\begin{cases} \beta_d^n = \dim_{\mathbb{Z}_3} \tilde{H}_d(\mathsf{M}_n; \mathbb{Z}_3), \\ \alpha_d^n = \dim_{\mathbb{Z}_3} \tilde{H}_d(\mathsf{M}_n \setminus 12; \mathbb{Z}_3). \end{cases}$$

Lemma 5.12. *For all* $n \geqslant 2$ *and all* d*, we have that*

$$\beta_d^n \leqslant \alpha_d^n + \beta_{d-1}^{n-2}.$$

For $n \geqslant 5$ *and all* d*, we have that*

$$\alpha_d^n \leqslant \beta_{d-1}^{n-3} + 2\binom{n-3}{2}\beta_{d-2}^{n-5} + (n-3)\alpha_{d-1}^{n-2}.$$

Proof. The inequalities are immediate consequences of Theorems 3.6 and 3.7. □

Define $\hat{\beta}_{k,r} = \beta_d^n$ and $\hat{\alpha}_{k,r} = \alpha_d^n$, where k and r are defined as in (1).

Corollary 5.13. *For* $k \geqslant 0$, $r \geqslant 0$*, and* $k + r \geqslant 1$*, we have that*

$$\hat{\beta}_{k,r} \leqslant \hat{\alpha}_{k,r} + \hat{\beta}_{k-1,r};$$

$$\hat{\alpha}_{k,r} \leqslant \hat{\beta}_{k,r-1} + 2\binom{2k+3r-2}{2}\hat{\beta}_{k-1,r-1} + (2k+3r-2)\hat{\alpha}_{k-1,r}.$$

Theorem 5.14. *For each* $k \geqslant 0$*, there are polynomials* $f_k(r)$ *and* $g_k(r)$ *of degree* $3k$ *with dominating term* $\frac{3^k}{k!} r^{3k}$ *such that*

$$\begin{cases} \hat{\beta}_{k,r} \leqslant f_k(r), \\ \hat{\alpha}_{k,r} \leqslant g_k(r), \end{cases}$$

for all $r \geqslant k+2$. Equivalently,

$$\begin{cases} \beta_d^n \leqslant f_{3d-n+4}(n-2d-3), \\ \alpha_d^n \leqslant g_{3d-n+4}(n-2d-3), \end{cases}$$

for all $n \geqslant 7$ and $\lceil \frac{n-4}{3} \rceil \leqslant d \leqslant \lfloor \frac{2n-9}{5} \rfloor$.

Proof. For $k=0$, we have that $\hat{\beta}_{0,r} = 1$ for all $r \geqslant 2$; use Theorem 4.2. It is known that $\hat{\alpha}_{0,2} \leqslant 1$ [10, Theorem 11.20]; indeed, it is not hard to prove that $\tilde{H}_1(\mathsf{M}_7 \setminus e; \mathbb{Z}) \cong \tilde{H}_1(\mathsf{M}_7 \setminus e; \mathbb{Z}_3) \cong \mathbb{Z}_3$. Moreover, Lemma 5.12 implies that $1 = \hat{\beta}_{0,r} \leqslant \hat{\alpha}_{0,r} \leqslant \hat{\beta}_{0,r-1} = 1$ for $r \geqslant 3$.

Assume that $k \geqslant 1$ and $r \geqslant k+3$. By Corollary 5.13 and induction on k, we obtain that

$$\hat{\alpha}_{k,r} \leqslant \hat{\beta}_{k,r-1} + 2\binom{2k+3r-2}{2} f_{k-1}(r-1) + (2k+3r-2) g_{k-1}(r),$$
$$\hat{\beta}_{k,r} \leqslant \hat{\alpha}_{k,r} + f_{k-1}(r),$$

where f_{k-1} and $g_{k-1}(r)$ are polynomials with properties as in the theorem. As a consequence,

$$\hat{\beta}_{k,r} - \hat{\beta}_{k,r-1} \leqslant 2\binom{2k+3r-2}{2} f_{k-1}(r-1) + (2k+3r-2) g_{k-1}(r) + f_{k-1}(r).$$

Now, the right-hand side is of the form

$$h_k(r) = (3r)^2 \cdot \frac{3^{k-1} r^{3k-3}}{(k-1)!} + \rho_k(r) = \frac{3^{k+1} r^{3k-1}}{(k-1)!} + \rho_k(r),$$

where $\rho_k(r)$ is a polynomial of degree at most $3k-2$. Summing over r, we obtain that

$$\hat{\beta}_{k,r} \leqslant \hat{\beta}_{k,k+2} + \sum_{i=k+3}^{r} h_k(r).$$

The right-hand side is a polynomial $f_k(r)$ in r with dominating term

$$\frac{3^{k+1}}{(k-1)!} \cdot \frac{r^{3k}}{3k} = \frac{3^k r^{3k}}{k!}.$$

Defining

$$g_k(r) = f_k(r-1) + 2\binom{2k+3r-2}{2} f_{k-1}(r-1) + (2k+3r-2) g_{k-1}(r),$$

we obtain a bound on $\hat{\alpha}_{k,r}$ with similar properties, which concludes the proof. □

For $k \geqslant 1$, one may extend the theorem to all $r \geqslant 0$ by adding a sufficiently large constant to each of $f_k(r)$ and $g_k(r)$.

Let us provide a more precise bound for the case $k=1$.

Theorem 5.15. *We have that $\beta_0^3 = 2$, $\beta_1^6 = 16$, $\beta_2^9 = 50$, $\beta_3^{12} = 56$, and*

$$\beta_r^{3r+3} \leqslant \frac{6r^3 + 9r^2 + 5r}{2} - 73$$

for $r \geqslant 4$.

Proof. With notation as in the proof of Theorem 5.14, Lemma 5.12 implies that

$$\hat{\beta}_{1,r} \leqslant \hat{\beta}_{1,r-1} + 2\binom{3r}{2} + 3r + 1 = \hat{\beta}_{1,r-1} + 9r^2 + 1.$$

Table 1 and a straightforward computation yield the theorem. □

The first few values on the bound in Theorem 5.15, starting with $r = 4$, are 201, 427, 752, 1194, and 1771.

The set of pairs (n, d) corresponding to a given k in Theorem 5.14 is of the form $\{v + rw\colon\ r \geqslant k+2\}$, where $v = (2k+1, k-1)$ and $w = (3, 1)$. Choosing other vectors v and w, we obtain other sequences of Betti numbers. In this more general situation, it might be of interest to study other fields than $\mathbb{Z}_3$. For $w = (2, 1)$ and any field, the growth is at least exponential as soon as $v = (n_0, d_0)$ for some n_0 and d_0 satisfying $n_0 \geqslant 2d_0 + 3$. Namely, over $\mathbb{Q}$, $\beta^{n_0+2q}_{d_0+q}$ is known to equal the number of self-conjugate standard Young tableaux of size $n_0 + 2q$ with a Durfee square of size $n_0 - 2d_0 - 2$ [5]. One easily checks that the number of such tableaux grows at least exponentially when q tends to infinity. Yet, if we were to pick a vector $w = (a, b)$ such that $a/b > 2$, then the rational homology would disappear for sufficiently large q; apply Theorem 1.1.

By Theorem 5.4, there is 3-torsion of rank at least $\binom{2k}{k}$ in $\tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r}; \mathbb{Z})$ for $k \geqslant 0$ and $r \geqslant 4$. As a consequence, for $\mathbb{Z}_3$, the growth is at least exponential for every (a, b) satisfying $2 \leqslant a/b < 3$. Namely, writing $k_0 = 3d_0 - n_0 + 4$ and $\delta = 3b - a$ and assuming that $2 < a/b < 3$, we have that

$$\beta^{n_0+aq}_{d_0+bq} = \hat{\beta}_{k_0+q\delta, n_0-2d_0+q(a-2b)-3} \geqslant \binom{2(k_0+q\delta)}{k_0+q\delta}$$

as soon as $n_0 - 2d_0 + q(a - 2b) \geqslant 7$.

Finally, let us consider $\mathbb{Z}_p$ for $p \neq 3$. By Theorem 5.8, whenever $a/b > 5/2$, we have that $\beta^{n_0+aq}_{d_0+bq}$ is zero over $\mathbb{Z}_p$ for sufficiently large q. The situation remains unclear for $2 < a/b \leqslant 5/2$.

6. Concluding remarks and open problems

From our viewpoint, the most important open problem regarding the homology of M_n is whether there exists other torsion than 3-torsion for $n \neq 14$. In light of the discussion in Section 5.3, we are tempted to conjecture the following:

Conjecture 6.1. $\tilde{H}_d(\mathsf{M}_n; \mathbb{Z}) = \tilde{H}_{k-1+r}(\mathsf{M}_{2k+3r+1}; \mathbb{Z})$ *contains nonvanishing* 5-*torsion whenever*

$$\left\lceil \frac{2n-8}{5} \right\rceil \leqslant d \leqslant \left\lfloor \frac{n-6}{2} \right\rfloor \quad \Leftrightarrow \quad 3 \leqslant r \leqslant k+1.$$

The bounds are exactly the same as in Corollary 5.11, except that the upper bound in the corollary is $\lfloor \frac{n-7}{2} \rfloor$ rather than $\lfloor \frac{n-6}{2} \rfloor$. In fact, the conjecture would be true for $d = \frac{n-6}{2}$ and all even $n \geqslant 14$ if the following conjecture were true.

Conjecture 6.2. *The sequence*

$$0 \longrightarrow \tilde{H}_d(\mathsf{M}_n \setminus e; \mathbb{Z}) \longrightarrow \tilde{H}_d(\mathsf{M}_n; \mathbb{Z}) \longrightarrow \langle e \rangle \otimes \tilde{H}_{d-1}(\mathsf{M}_{[n]\setminus e}; \mathbb{Z}) \longrightarrow 0,$$

cut from the long exact 0-*e*-2 *sequence in Section* 3.4*, is split exact for every* $n \geqslant 3$ *and every* d.

We have checked the conjecture up to $n = 11$ using computer; see Table 3 and compare to Table 1. If Conjecture 6.2 were true for all n, then we would have p-torsion in $\tilde{H}_{d+k}(\mathsf{M}_{n+2k}; \mathbb{Z})$ for all $k \geqslant 0$ whenever $\tilde{H}_d(\mathsf{M}_n; \mathbb{Z})$ contains p-torsion.

Define $\hat{\beta}_{k,r} = \dim_{\mathbb{Z}_3} \tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r}; \mathbb{Z}_3)$. Conjecture 6.2 being true for the coefficient ring $\mathbb{Z}_3$ would imply that $\hat{\beta}_{k-1,r} \leqslant \hat{\beta}_{k,r}$. Combined with a quite modest conjecture about the behavior of $\{\hat{\beta}_{k,r}\colon\ r \geqslant 1\}$ for each fixed k, this would yield nontrivial lower bounds on $\hat{\beta}_{k,r}$ for every $k, r \geqslant 0$.

Proposition 6.3. *Suppose that* $\hat{\beta}_{k-1,r} \leqslant \hat{\beta}_{k,r}$ *for all* $k \geqslant 1$ *and* $r \geqslant 0$*. Suppose further that there are positive numbers* $\{C_k\colon\ k \geqslant 0\}$ *such that* $C_k\hat{\beta}_{k,r} \geqslant \hat{\beta}_{k,r-1}$ *for all* $k \geqslant 0$ *and* $r \geqslant 1$*. Then* $\hat{\beta}_{k,r}$ *is bounded from below by a polynomial of degree* k*.*

Table 3
The homology of $\mathsf{M}_n \setminus e$ for $n \leqslant 11$

$\tilde{H}_i(\mathsf{M}_n \setminus e; \mathbb{Z})$	$i = -1$	0	1	2	3	4
$n = 2$	$\mathbb{Z}$	–	–	–	–	–
3	–	$\mathbb{Z}$	–	–	–	–
4	–	$\mathbb{Z}^2$	–	–	–	–
5	–	–	$\mathbb{Z}^4$	–	–	–
6	–	–	$\mathbb{Z}^{14}$	–	–	–
7	–	–	$\mathbb{Z}_3$	$\mathbb{Z}^{14}$	–	–
8	–	–	–	$\mathbb{Z}^{116}$	–	–
9	–	–	–	$\mathbb{Z}_3^7 \oplus \mathbb{Z}^{42}$	$\mathbb{Z}^{50}$ –	–
10	–	–	–	$\mathbb{Z}_3$	$\mathbb{Z}^{1084}$ –	–
11	–	–	–	–	$\mathbb{Z}_3^{37} \oplus \mathbb{Z}^{1146}$	$\mathbb{Z}^{182}$

Table 4
List of all possible infinite and prime power exponents of elements in $\tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r}; \mathbb{Z})$ for $k \leqslant 5$ and $r \leqslant 7$. Legend: $\infty =$ infinite exponent; $p^* =$ exponent an unknown positive power of p; $e? =$ possibly other prime power exponents than those listed

Exponents	$k = 0$	1	2	3	4	5
$r = 0$	∞	∞	∞	∞	∞	∞
1	∞	∞	∞	∞	∞	∞
2	3	$\infty, 3$	$\infty, 3$	$\infty, 3^*, e?$	$\infty, e?$	$\infty, e?$
3	3	3	$3^*, 5^*, e?$	$\infty, 3^*, e?$	$\infty, 3^*, e?$	$\infty, 3^*, e?$
4	3	3	3	$3^*, e?$	$3^*, e?$	$3^*, e?$
5	3	3	3	3^*	$3^*, e?$	$3^*, e?$
6	3	3	3	3^*	3^*	$3^*, e?$
7	3	3	3	3^*	3^*	3^*

Proof. By the long exact 0-1-2 sequence, we have that

$$(2k + 3r)\hat{\beta}_{k-1,r-1} \leqslant \hat{\beta}_{k,r-1} + \hat{\beta}_{k-2,r}$$

for $r \geqslant 1$. Applying our assumptions, we obtain that

$$(2k + 3r)\hat{\beta}_{k-1,r-1} \leqslant C_k\hat{\beta}_{k,r} + \hat{\beta}_{k,r} = (C_k + 1)\hat{\beta}_{k,r},$$

which yields that $\hat{\beta}_{k,r} \geqslant (C_k + 1)^{-1}(2k + 3r)\hat{\beta}_{k-1,r-1}$. $\square$

For $k \leqslant 2$, $\hat{\beta}_{k,r}$ is indeed bounded from below by a polynomial of degree k [17].

We conclude with Table 4, which provides a list of possible exponents in $\tilde{H}_{k-1+r}(\mathsf{M}_{2k+1+3r}; \mathbb{Z})$ for small k and r; apply Theorems 1.1, 4.3, 4.5, 5.4, and 5.8 and Proposition 5.5. Note that $(k,r) = (0,2)$ yields the first occurrence of 3-torsion and that $(k,r) = (2,3)$ yields the only known occurrence of 5-torsion. These two pairs share the property that k is maximal for the given r such that the group at (k,r) is finite. Speculating wildly, one may ask whether there is further torsion to discover at other pairs (k,r) with this property, that is, $k = \binom{r}{2} - 1$; use Theorem 1.1.

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