



Exact results for corner contributions to the entanglement entropy and Rényi entropies of free bosons and fermions in 3d [☆]



Henriette Elvang^{**}, Marios Hadjiantonis^{*}

Randall Laboratory of Physics, Department of Physics, University of Michigan, Ann Arbor, MI 48109, USA

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ABSTRACT

In the presence of a sharp corner in the boundary of the entanglement region, the entanglement entropy (EE) and Rényi entropies for 3d CFTs have a logarithmic term whose coefficient, the corner function, is scheme-independent. In the limit where the corner becomes smooth, the corner function vanishes quadratically with coefficient σ for the EE and σ_n for the Rényi entropies. For a free real scalar and a free Dirac fermion, we evaluate analytically the integral expressions of Casini, Huerta, and Leitao to derive exact results for σ and σ_n for all $n = 2, 3, \dots$. The results for σ agree with a recent universality conjecture of Bueno, Myers, and Witczak-Krempa that $\sigma/C_T = \pi^2/24$ in all 3d CFTs, where C_T is the central charge. For the Rényi entropies, the ratios σ_n/C_T do not indicate similar universality. However, in the limit $n \rightarrow \infty$, the asymptotic values satisfy a simple relationship and equal $1/(4\pi^2)$ times the asymptotic values of the free energy of free scalars/fermions on the n -covered 3-sphere.

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1. Introduction and results

For a 3d conformal field theory (CFT) in the ground state, the entanglement entropy S for a region whose boundary has a sharp corner with angle θ can be written as

$$S = B \frac{L}{\epsilon} - a(\theta) \ln \left(\frac{L}{\epsilon} \right) + O(1). \quad (1)$$

Here L is a length scale associated with the size of the entangling region, ϵ is a short distance cutoff, and B is a non-universal constant. The *corner contribution* to the entanglement entropy is the scheme-independent positive function $a(\theta)$ of the opening angle θ [1–3]. Since the entanglement entropy of the region equals that of the complement region, the corner contribution satisfies $a(2\pi - \theta) = a(\theta)$. If the curve bounding the entangling region is smooth, the logarithmic term is absent, hence $a(\theta)$ must vanish in the limit $\theta \rightarrow \pi$ and it does so quadratically as

$$a(\theta) = \sigma (\theta - \pi)^2 + \dots \quad \text{for } \theta \rightarrow \pi. \quad (2)$$

The value of the *corner coefficient* σ depends on the theory.

For the theory of a free real scalar or a Dirac fermion, Casini, Huerta, and Leitao [4,5,2] derived expressions that give $a(\theta)$ implicitly in terms of some rather involved integrals. In the limit $\theta \rightarrow \pi$ one can extract double-integral expressions for the corner coefficient σ in (2). These integrals have been evaluated numerically [5,6] and the results indicate that the exact values are [6]

$$\sigma^{(B)} = \frac{1}{256} \quad \text{and} \quad \sigma^{(F)} = \frac{1}{128} \quad (3)$$

for the free boson and free fermion, respectively.

Bueno, Myers, and Witczak-Krempa [6] conjectured that the ratio of the coefficient σ in (2) to the central charge C_T is universal in 3d CFTs and that it takes the value

$$\text{conjecture [6]:} \quad \frac{\sigma}{C_T} = \frac{\pi^2}{24}. \quad (4)$$

The conjecture (4) has passed non-trivial holographic tests for gravity models with a family of higher derivative corrections [6,7]. The central charge C_T is defined as the coefficient of the vacuum 2-point function of the stress tensor (see eq. (3) in [6]). For free bosons and fermions, Osborn and Petkou [8] found that $C_T^{(B)} = 3/(32\pi^2)$ and $C_T^{(F)} = 3/(16\pi^2)$ in 3d. So with the values (3), the ratio σ/C_T is indeed $\pi^2/24$ for both free bosons and fermions.

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^{*} Primary corresponding author.

^{**} Corresponding author.

E-mail addresses: elvang@umich.edu (H. Elvang), mhadjian@umich.edu (M. Hadjiantonis).

Table 1
The first nine values of σ_n .

n	$\sigma_n^{(B)}$	$\sigma_n^{(F)}$
2	$\frac{1}{48\pi^2} \approx 0.00211086$	$\frac{1}{64\pi} \approx 0.00497359$
3	$\frac{1}{108\sqrt{3}\pi} \approx 0.00170163$	$\frac{5}{216\sqrt{3}\pi} \approx 0.00425408$
4	$\frac{8+3\pi}{1152\pi^2} \approx 0.00153255$	$\frac{1+6\sqrt{2}}{768\pi} \approx 0.00393133$
5	$\frac{\sqrt{25-2\sqrt{5}}}{1000\pi} \approx 0.00144219$	$\frac{\sqrt{425+58\sqrt{5}}}{2000\pi} \approx 0.00374840$
6	$\frac{81+34\sqrt{3}\pi}{19440\pi^2} \approx 0.00138643$	$\frac{261+20\sqrt{3}}{25920\pi} \approx 0.00363061$
7	$\frac{2 \cot(\frac{\pi}{14})+5 \cot(\frac{3\pi}{14})+5 \tan(\frac{\pi}{7})}{4116\pi} \approx 0.00134874$	$\frac{13 \cot(\frac{\pi}{14})+22 \cot(\frac{3\pi}{14})+15 \tan(\frac{\pi}{7})}{8232\pi} \approx 0.00354841$
8	$\frac{32+9\pi(1+2\sqrt{2})}{10752\pi^2} \approx 0.00132161$	$\frac{1+6\sqrt{2}+4\sqrt{274+17\sqrt{2}}}{7168\pi} \approx 0.00348777$
9	$\frac{27\sqrt{3}+10 \cot(\frac{\pi}{18})+28 \tan(\frac{\pi}{9})+35 \tan(\frac{2\pi}{9})}{34992\pi} \approx 0.00130116$	$\frac{135\sqrt{3}+68 \cot(\frac{\pi}{18})+77 \tan(\frac{\pi}{9})+130 \tan(\frac{2\pi}{9})}{69984\pi} \approx 0.00344118$
10	$\frac{125+6\pi\sqrt{565+142\sqrt{5}}}{54000\pi^2} \approx 0.00128522$	$\frac{5+300\sqrt{5}+4\sqrt{425+58\sqrt{5}}}{72000\pi} \approx 0.00340427$

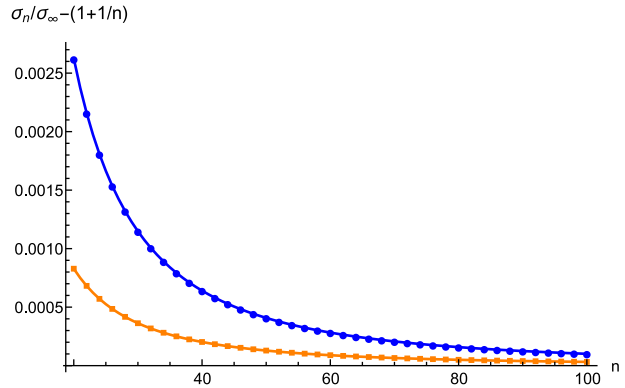
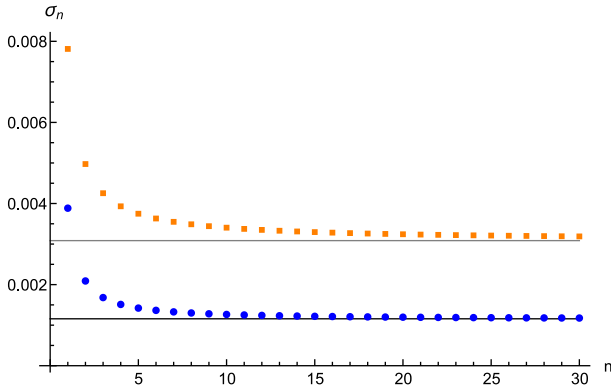


Fig. 1. Left: Plot showing that σ_n decreases monotonically from the entanglement entropy value included for $n = 1$ to the asymptotic value σ_∞ for free scalars (circles) and free Dirac fermions (squares). The asymptotic values $\sigma_\infty^{(B)} = \frac{3\zeta(3)}{32\pi^4} \approx 0.0011569$ (black) and $\sigma_\infty^{(F)} = \frac{\zeta(3)}{4\pi^4} \approx 0.00308507$ (gray) are indicated as horizontal lines. Right: The plot illustrates our numerical fit $\sigma_n = \sigma_\infty \left(1 + \frac{b_1}{n} + \frac{b_2}{n^2} + \frac{b_3}{n^3} + \dots \right)$, for which we find $b_1 = b_2 = b_3 = 1$ for the free scalar, and $b_1 = 1$ and $b_2 = b_3 = 1 - \frac{\pi^2}{12\zeta(3)} \approx 0.31578$ for the free fermion; the solid curves are $\frac{b_2}{n^2} + \frac{b_3}{n^3}$ for those respective values of b_2 and b_3 .

In this paper, we evaluate analytically the integral expressions [6] of Casini, Huerta, and Leita [4,5,2] for $\sigma^{(B)}$ and $\sigma^{(F)}$ and prove that their exact values are indeed those in (3). This verifies the universality conjecture (4) for the case of free bosons and fermions. One way of viewing the conjecture is simply as the statement that the corner coefficient σ in (2) does not contain independent information about the CFT, but is fixed in terms of the central charge C_T .

Turning to the Rényi entropies S_n , one can define a similar corner contribution $a_n(\theta)$ which in the smooth limit $\theta \rightarrow \pi$ goes to zero as $a_n(\theta) = \sigma_n (\theta - \pi)^2 + \dots$ for $n = 2, 3, 4, \dots$. (The $n \rightarrow 1$ limit of the Rényi entropy is the entanglement entropy.) It is not known if σ_n/C_T has any universal properties.

We calculate σ_n analytically for the free boson and free fermion using integral expressions for σ_n derived in [4,5,2].¹ For the free scalar we find

$$\sigma_n^{(B)} = \sum_{k=1}^{n-1} \frac{k(n-k)(n-2k) \tan\left(\frac{k\pi}{n}\right)}{24\pi n^3(n-1)}. \tag{5}$$

Note that when n is even, the contribution from $k = n/2$ must be evaluated carefully using $\lim_{k \rightarrow n/2} (n-2k) \tan\left(\frac{k\pi}{n}\right) = 2n/\pi$.

¹ We are grateful to Horacio Casini for sharing with us the integral expression for $\sigma_n^{(F)}$.

The result for the free fermion is

$$\sigma_n^{(F)} = \sum_{k=-(n-1)/2}^{(n-1)/2} \frac{k(n^2 - 4k^2) \tan\left(\frac{k\pi}{n}\right)}{24\pi n^3(n-1)}, \tag{6}$$

where sum is to be taken in integer steps from $-\frac{n-1}{2}$ to $\frac{n-1}{2}$.

For low values of n , the finite sums of the trigonometric functions in (5) and (6) simplify quite nicely. The results for first nine values of σ_n are shown in Table 1. In the case of the scalar, the exact $n = 2, 3$ results were guessed by the authors of [6] based on their high precision numerical evaluation of the integrals.

Since the ratios of the central charges of free fermions and bosons differ only by a factor of 2, universality of the ratio σ_n/C_T would require that $\sigma_n^{(B)}/\sigma_n^{(F)}$ obeys some simple, possibly n -dependent, relation. Based on our results above, there is no hint of such a simple relationship. Of course to fully exclude this, one would need values of σ_n for other 3d CFTs.

As a function of n , the Rényi corner coefficient σ_n decreases monotonically, as shown on the left in Fig. 1. When n is large, σ_n asymptotes to a constant value, which we calculate analytically:

$$\sigma_\infty^{(B)} = \frac{3\zeta(3)}{32\pi^4} \quad \text{and} \quad \sigma_\infty^{(F)} = \frac{\zeta(3)}{4\pi^4}. \tag{7}$$

The appearance of the Riemann zeta-function is intriguing since $\zeta(3) \approx 1.20206$ also shows up in the free energies and Rényi entropies for free scalars/fermions on a 3-sphere, as shown by Kle-

banov, Pufu, Sachdev, and Safdi [9]. Specifically, the free energy of a free real scalar or free fermion on an n -covered 3-sphere behaves as² $\mathcal{F}_n \rightarrow n\mathcal{F}_\infty$ for $n \rightarrow \infty$ with

$$\mathcal{F}_\infty^{(B)} = \frac{3\zeta(3)}{8\pi^2} \quad \text{and} \quad \mathcal{F}_\infty^{(F)} = \frac{\zeta(3)}{\pi^2}. \quad (8)$$

Thus, for both free scalars and fermions we have

$$\sigma_\infty^{(B/F)} = \frac{1}{4\pi^2} \mathcal{F}_\infty^{(B/F)}. \quad (9)$$

For finite n , there is no apparent relation between \mathcal{F}_n and σ_n , however there are some similarities in the subleading large- n behaviors, as we discuss in Section 4. The plot on the right in Fig. 1 shows the large- n behaviors of the Rényi corner coefficients σ_n . A priori it is not clear if there is any relation at large n between σ_n and \mathcal{F}_n , but it would be curious to test (9) in other examples.

The remainder of the paper details the derivations of the results summarized above. In Section 2, we derive the results (3) for the entanglement entropy corner coefficient σ . We then evaluate the Rényi entropy corner coefficients σ_n in Section 3. In Section 4, we discuss the asymptotic behavior at large n .

2. Evaluation of the EE integrals

In this section we describe the procedure for analytically evaluating the integrals for the coefficients $\sigma^{(B)}$ and $\sigma^{(F)}$ of the entanglement entropy. Our starting point is the integrals [4,5,2] presented in equations (B1)–(B3) of [6]. After a change of integration variable from m to $\mu = \sqrt{4m^2 - 1}$, the integrals take the form

$$\begin{aligned} \sigma^{(B)} &= -\frac{1}{2} \int_0^\infty d\mu \int_0^\infty db \mu^2 H a (1-a) \frac{\pi}{\cosh^2(\pi b)}, \\ \sigma^{(F)} &= -\int_0^\infty d\mu \int_0^\infty db \left[\mu^2 H a (1-a) - \frac{\mu F}{4\pi} \right] \frac{\pi}{\sinh^2(\pi b)}, \end{aligned} \quad (10)$$

where $a = 1/2 + ib$ for the scalar and $a = ib$ for the fermion. The functions H and F are defined as

$$H = -\frac{T}{2} \left(\frac{c}{h} X_1 + \frac{1}{c} X_2 \right) + \frac{1}{16\pi a (a-1)}, \quad F = -\frac{F_1}{F_2}, \quad (11)$$

with³

$$\begin{aligned} F_1 &= 4\pi c h H a (1-a) \left[(2a-1)^2 + \mu^2 \right] - \frac{1}{4} c h^2 (\mu^2 + 1), \\ F_2 &= \frac{c h \left[(2a-1)^2 + \mu^2 \right]^2}{2(2a-1)\mu}. \end{aligned} \quad (12)$$

The functions h , c , X_1 , X_2 , and T are defined as follows:

$$\begin{aligned} h &= \frac{2(\mu^2 + (2a-1)^2) \sin^2(\pi a)}{(\mu^2 + 1)(\cos(2\pi a) + \cosh(\pi\mu))}, \\ c &= \frac{2^{2a} \pi a (1-a) \sec\left(\pi a + \frac{i\pi\mu}{2}\right) \Gamma\left(\frac{3}{2} - a + \frac{i\mu}{2}\right)}{\sqrt{\mu^2 + 1} (\Gamma(2-a))^2 \Gamma\left(-\frac{1}{2} + a + \frac{i\mu}{2}\right)} \end{aligned} \quad (13)$$

and

$$\begin{aligned} X_1 &= -\frac{\Gamma(-a)}{2^{2a+1} \mu \Gamma(a+1)} \\ &\times \frac{\pi \sinh\left(\frac{\pi\mu}{2}\right) + i \cosh\left(\frac{\pi\mu}{2}\right) \left(\psi\left(\frac{1}{2} + a + \frac{i\mu}{2}\right) - \psi\left(\frac{1}{2} + a - \frac{i\mu}{2}\right) \right)}{\Gamma\left(\frac{1}{2} - a + \frac{i\mu}{2}\right) \Gamma\left(\frac{1}{2} - a - \frac{i\mu}{2}\right) (\cos(2\pi a) + \cosh(\pi\mu))}, \\ X_2 &= (X_1 \text{ with } a \text{ replaced by } (1-a)), \\ T &= \frac{1}{2} \sqrt{h \left[(h+1)(\mu^2 + 1) - 4a(1-a) \right]}. \end{aligned} \quad (14)$$

Here ψ denotes the digamma function, $\psi(z) = \frac{d}{dz} \ln \Gamma(z)$.

Our first line of attack involves calculating the quantities cX_1/h and X_2/c that appear in H in (11). Beyond the immediate cancellations that occur in these ratios, one can perform further simplifications using identities involving gamma functions. Namely, one can use the recurrence relation

$$\Gamma(1+z) = z\Gamma(z) \quad (15)$$

and the reflection relation

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}. \quad (16)$$

Surprisingly, all the gamma functions cancel after a series of such substitutions, giving

$$\begin{aligned} \frac{c}{h} X_1 &= \frac{\sqrt{\mu^2 + 1} \csc(\pi a)}{16\pi \mu (a-1) a} \left[\pi \sinh\left(\frac{\pi\mu}{2}\right) \right. \\ &\quad \left. + i \cosh\left(\frac{\pi\mu}{2}\right) \left(\psi\left(\frac{1}{2} + a + \frac{i\mu}{2}\right) - \psi\left(\frac{1}{2} + a - \frac{i\mu}{2}\right) \right) \right], \\ \frac{1}{c} X_2 &= \frac{\sqrt{\mu^2 + 1} \csc(\pi a)}{16\pi \mu (a-1) a} \left[\pi \sinh\left(\frac{\pi\mu}{2}\right) \right. \\ &\quad \left. + i \cosh\left(\frac{\pi\mu}{2}\right) \left(\psi\left(\frac{3}{2} - a + \frac{i\mu}{2}\right) - \psi\left(\frac{3}{2} - a - \frac{i\mu}{2}\right) \right) \right]. \end{aligned} \quad (17)$$

It is suggestive that the pre-factors and the form of these two results are the same. We then proceed by adding them together as in (11). The linear combination of digamma functions that appears in the result can be simplified using properties easily derived from (15) and (16). In the form that is useful for our purpose, these identities are

$$\psi\left(\frac{3}{2} - a \pm \frac{i\mu}{2}\right) = \psi\left(\frac{1}{2} - a \pm \frac{i\mu}{2}\right) + \frac{1}{\frac{1}{2} - a \pm \frac{i\mu}{2}}$$

and

$$\psi\left(\frac{1}{2} + a \pm \frac{i\mu}{2}\right) - \psi\left(\frac{1}{2} - a \mp \frac{i\mu}{2}\right) = \pi \tan\left(\pi a \pm i \frac{\pi\mu}{2}\right).$$

Then the combination of X_1 and X_2 that appears in H simplifies to

$$\begin{aligned} \frac{c}{h} X_1 + \frac{1}{c} X_2 &= \frac{\sqrt{\mu^2 + 1}}{4\pi a(1-a)} \left(\frac{\pi \sin(\pi a) \sinh\left(\frac{\pi\mu}{2}\right)}{\mu [\cos(2\pi a) + \cosh(\pi\mu)]} \right. \\ &\quad \left. - \frac{\csc(\pi a) \cosh\left(\frac{\pi\mu}{2}\right)}{(1-2a)^2 + \mu^2} \right). \end{aligned} \quad (18)$$

The last ingredient we need to construct H in (11) is T . Using (14), it is

² The authors of [9] work with a complex scalar, so the free energy there is twice that of a real scalar.

³ We simplified the expression for F_1 in [6] by writing it in terms of H .

$$T = \sqrt{\frac{((1 - 2a)^2 + \mu^2)^2 \sin^2(\pi a) \cosh^2\left(\frac{\pi\mu}{2}\right)}{(\mu^2 + 1)(\cos(2\pi a) + \cosh(\pi\mu))^2}}. \tag{19}$$

Further simplifications of H depend on the nature of variable a , as we will see when we specialize to the cases of the free scalar and the free fermion.

Free scalar. To proceed with the evaluation of the integral $\sigma^{(B)}$, we set $a = 1/2 + ib$ as prescribed for the free scalar. It is furthermore convenient to change integration variable $b \rightarrow b/2$. Using that both μ and b are positive, the integrand of $\sigma^{(B)}$ simplifies dramatically and becomes

$$\sigma^{(B)} = \frac{1}{64} \int_0^\infty d\mu \int_0^\infty db \left(\frac{\pi\mu(\mu^2 - b^2) \sinh(\pi\mu)}{[\cosh(\pi b) - \cosh(\pi\mu)]^2} + \frac{2\mu^2}{\cosh(\pi b) - \cosh(\pi\mu)} \right). \tag{20}$$

Next, we integrate by parts. Writing

$$\sigma^{(B)} = \frac{1}{64} \int_0^\infty d\mu \int_0^\infty db \left[\frac{\partial}{\partial\mu} \left(\frac{\mu(\mu^2 - b^2)}{\cosh(\pi b) - \cosh(\pi\mu)} \right) + \frac{b^2 - \mu^2}{\cosh(\pi b) - \cosh(\pi\mu)} \right], \tag{21}$$

we see that the boundary term vanishes and we get

$$\sigma^{(B)} = \frac{1}{256} \int_{-\infty}^{+\infty} d\mu \int_{-\infty}^{+\infty} db \frac{b^2 - \mu^2}{\cosh(\pi b) - \cosh(\pi\mu)}. \tag{22}$$

We have extended the limits of integration to facilitate the change of integration variables

$$\mu = x - y \quad \text{and} \quad b = x + y. \tag{23}$$

This separates the two integrations and reduces the expression to

$$\sigma^{(B)} = \left(\frac{1}{8} \int_{-\infty}^{+\infty} dx \frac{x}{\sinh(\pi x)} \right)^2 = \frac{1}{256}. \tag{24}$$

This completes the derivation of the result (3) for the free scalar.

Free fermion. With F_1 given in terms of H as in (12), we have already done most of the leg-work needed to compute $\sigma^{(F)}$. For the free fermion, we have to take $a = ib$ and it is again convenient to change integration variable $b \rightarrow b/2$. After putting everything together, we have

$$\sigma^{(F)} = -\frac{1}{32} \int_0^\infty d\mu \int_0^\infty db \left(\frac{\pi\mu(\mu^2 - b^2 - 1) \sinh(\pi\mu)}{[\cosh(\pi b) + \cosh(\pi\mu)]^2} - \frac{2\mu^2}{\cosh(\pi b) + \cosh(\pi\mu)} \right). \tag{25}$$

We can express the integrand as a total derivative plus remaining terms as

$$\sigma^{(F)} = \frac{1}{32} \int_0^\infty d\mu \int_0^\infty db \left[\frac{\partial}{\partial\mu} \left(\frac{\mu(\mu^2 - b^2 - 1)}{\cosh(\pi b) + \cosh(\pi\mu)} \right) + \frac{1 - \mu^2 + b^2}{\cosh(\pi b) + \cosh(\pi\mu)} \right]. \tag{26}$$

As before, the boundary term vanishes and we are left with the expression (after extending the limits of integration)

$$\begin{aligned} \sigma^{(F)} &= \frac{1}{128} \int_{-\infty}^{+\infty} d\mu db \frac{1 - \mu^2 + b^2}{\cosh(\pi b) + \cosh(\pi\mu)} \\ &= \frac{1}{128} \int_{-\infty}^{+\infty} dx dy \frac{1 + 4xy}{\cosh(\pi x) \cosh(\pi y)}. \end{aligned} \tag{27}$$

In the last step, we changed integration variables using (23). Since $x/\cosh(\pi x)$ is odd, that part of the integral vanishes and the result is therefore simply

$$\sigma^{(F)} = \frac{1}{128} \left(\int_{-\infty}^{+\infty} dx \frac{1}{\cosh(\pi x)} \right)^2 = \frac{1}{128}. \tag{28}$$

Thus we have derived the result (3) for the free fermion.

3. Rényi entropies

We now proceed to calculate the corner coefficients σ_n for the Rényi entropies.

Free scalar. For the scalar field, the Rényi corner coefficient is given by the integral (B7) in [6]. We change the integration variable m to $\mu = \sqrt{4m^2 - 1}$ to write it as

$$\sigma_n^{(B)} = -\sum_{k=1}^{n-1} \frac{k(n-k)}{2n^2(n-1)} \int_0^\infty d\mu \mu^2 H_{k/n}, \tag{29}$$

where $H_{k/n}$ is H in (11) with a replaced by k/n . With the simplified expression for H from Section 2, we get

$$\begin{aligned} \sigma_n^{(B)} &= \sum_{k=1}^{n-1} \frac{\sin^2\left(\frac{\pi k}{n}\right)}{32\pi n^2(n-1)} \int_0^\infty d\mu \left(-\frac{2\mu^2 n^2}{\cos\left(\frac{2\pi k}{n}\right) + \cosh(\pi\mu)} \right. \\ &\quad \left. + \frac{\mu[(n-2k)^2 + \mu^2 n^2] \pi \sinh(\pi\mu)}{\left[\cos\left(\frac{2\pi k}{n}\right) + \cosh(\pi\mu)\right]^2} \right). \end{aligned} \tag{30}$$

As before, we write the integrand as a total derivative plus the remaining terms:

$$\begin{aligned} \sigma_n^{(B)} &= -\sum_{k=1}^{n-1} \frac{\sin^2\left(\frac{\pi k}{n}\right)}{32\pi n^2(n-1)} \int_0^\infty d\mu \left[-\frac{(n-2k)^2 + \mu^2 n^2}{\cos\left(\frac{2\pi k}{n}\right) + \cosh(\pi\mu)} \right. \\ &\quad \left. + \frac{\partial}{\partial\mu} \left(\frac{\mu[(n-2k)^2 + \mu^2 n^2]}{\cos\left(\frac{2\pi k}{n}\right) + \cosh(\pi\mu)} \right) \right]. \end{aligned} \tag{31}$$

The boundary term vanishes and the expression simplifies to

$$\sigma_n^{(B)} = \sum_{k=1}^{n-1} \frac{\sin^2\left(\frac{\pi k}{n}\right)}{32\pi n^2(n-1)} \int_0^\infty d\mu \frac{(n-2k)^2 + \mu^2 n^2}{\cos\left(\frac{2\pi k}{n}\right) + \cosh(\pi\mu)}. \tag{32}$$

The contribution of $k = n/2$ is easy to calculate and is equal to

$$\frac{1}{64\pi(n-1)} \int_0^\infty d\mu \frac{\mu^2}{\sinh^2\left(\frac{\pi\mu}{2}\right)} = \frac{1}{48\pi^2(n-1)}. \quad (33)$$

For $k \neq n/2$, there are contributions from two integrals:

$$I_{n;k}^{(1)} = \int_0^\infty \frac{d\mu}{\cos\left(\frac{2\pi k}{n}\right) + \cosh(\pi\mu)} = \frac{2 \tan^{-1}\left(\tan\left(\frac{\pi k}{n}\right)\right)}{\pi \sin\left(\frac{2k\pi}{n}\right)} \\ = \frac{2}{\sin\left(\frac{2k\pi}{n}\right)} \times \begin{cases} \frac{k}{n}, & k < n/2 \\ \frac{k}{n} - 1, & k > n/2 \end{cases} \quad (34)$$

and

$$I_{n;k}^{(2)} = \int_0^\infty \frac{\mu^2 d\mu}{\cos\left(\frac{2\pi k}{n}\right) + \cosh(\pi\mu)} \\ = \frac{2i \left[\text{Li}_3\left(-e^{\frac{2ik\pi}{n}}\right) - \text{Li}_3\left(-e^{-\frac{2ik\pi}{n}}\right) \right]}{\pi^3 \sin\left(\frac{2k\pi}{n}\right)} \\ = -\frac{i \log\left(e^{\frac{2ik\pi}{n}}\right) \left[\pi^2 + \log^2\left(e^{\frac{2ik\pi}{n}}\right) \right]}{3\pi^3 \sin\left(\frac{2k\pi}{n}\right)} \\ = \frac{2}{\sin\left(\frac{2k\pi}{n}\right)} \times \begin{cases} \frac{k(n^2-4k^2)}{3n^3}, & k < n/2 \\ -\frac{(n-k)(n-2k)(3n-2k)}{3n^3}, & k > n/2. \end{cases} \quad (35)$$

Above, we manipulated the tri-logarithm Li_3 using the polylog identity

$$\text{Li}_3(z) - \text{Li}_3(z^{-1}) = -\frac{1}{6} \log^3(-z) - \frac{\pi^2}{6} \log(-z), \quad (36)$$

which holds for $z \notin]0, 1[$.

Combining the results (34) and (35), we find that the result is the same for $1 < k < n/2$ and $n/2 < k < n$, namely

$$\int_0^\infty d\mu \frac{(n-2k)^2 + \mu^2 n^2}{\cos\left(\frac{2\pi k}{n}\right) + \cosh(\pi\mu)} = (n-2k)^2 I_{n;k}^{(1)} + n^2 I_{n;k}^{(2)} \\ = \frac{8k(n-k)(n-2k)}{3n \sin\left(\frac{2k\pi}{n}\right)}. \quad (37)$$

Thus, having evaluated the integral in (32), we can write $\sigma_n^{(B)}$ as the finite sum

$$\sigma_n^{(B)} = \frac{1}{24\pi n^3(n-1)} \sum_{k=1}^{n-1} k(n-k)(n-2k) \tan\left(\frac{\pi k}{n}\right). \quad (38)$$

Note that taking the limit $k \rightarrow n/2$ as described below (5), the summand evaluates precisely to the special case (33). The expression (38) is the result for the Rényi corner coefficient presented in (5), so this completes our evaluation for the free scalar.

Free fermion. For the fermion field, the Rényi corner coefficient is given by the integral

$$\sigma_n^{(F)} = -\frac{2}{n-1} \sum_{k>0} \int_0^\infty d\mu \left[a(1-a)\mu^2 H - \frac{\mu F}{4\pi} \right]_{a=k/n}, \quad (39)$$

where the sum is over k from $1/2$ (n even) or 1 (n odd) in integer steps to $\frac{1}{2}(n-1)$. Substituting the expressions for H and F obtained earlier gives

$$\sigma_n^{(F)} = \sum_{k>0}^{\frac{1}{2}(n-1)} \frac{\sin^2\left(\frac{\pi k}{n}\right)}{8\pi(n-1)} \int_0^\infty d\mu \left(\frac{2\mu^2}{\cos\left(\frac{2\pi k}{n}\right) + \cosh(\pi\mu)} - \frac{\mu \left(\frac{4k^2}{n^2} + \mu^2 - 1 \right) \pi \sinh(\pi\mu)}{\left[\cos\left(\frac{2\pi k}{n}\right) + \cosh(\pi\mu) \right]^2} \right). \quad (40)$$

We then use integration by parts to simplify the integral

$$\sigma_n^{(F)} = \sum_{k>0}^{\frac{1}{2}(n-1)} \frac{\sin^2\left(\frac{\pi k}{n}\right)}{8\pi(n-1)} \int_0^\infty d\mu \left[\frac{\partial}{\partial \mu} \left(\frac{\mu \left(\frac{4k^2}{n^2} + \mu^2 - 1 \right)}{\cos\left(\frac{2\pi k}{n}\right) + \cosh(\pi\mu)} \right) - \frac{\frac{4k^2}{n^2} + \mu^2 - 1}{\cos\left(\frac{2\pi k}{n}\right) + \cosh(\pi\mu)} \right]. \quad (41)$$

The boundary term integrates to zero and the expression simplifies to

$$\sigma_n^{(F)} = \sum_{k>0}^{\frac{1}{2}(n-1)} \frac{\sin^2\left(\frac{\pi k}{n}\right)}{8\pi(n-1)} \int_0^\infty d\mu \frac{1 - \mu^2 - \frac{4k^2}{n^2}}{\cos\left(\frac{2\pi k}{n}\right) + \cosh(\pi\mu)}. \quad (42)$$

The result of the integral again involves a difference of two tri-logarithms and it can be simplified using equation (36). The result is even in $k \rightarrow -k$, so we can write the final answer as

$$\sigma_n^{(F)} = \frac{1}{24\pi n^3(n-1)} \sum_{k=-\frac{1}{2}(n-1)}^{\frac{1}{2}(n-1)} k(n^2 - 4k^2) \tan\left(\frac{\pi k}{n}\right). \quad (43)$$

This is the formula we presented in (6). Values for low n were tabulated in Section 1 for both $\sigma_n^{(B)}$ and $\sigma_n^{(F)}$.

4. Asymptotic behavior of the Rényi's

Let us now study the large n behavior of the Rényi entropy corner coefficients σ_n . In particular, we evaluate analytically the value for the coefficients σ_n in the limit where $n \rightarrow \infty$. This is done by introducing a new variable $x = k/n$ and multiplying by $n\Delta x = 1$. Then in the $n \rightarrow \infty$ limit, the sum becomes an integral and we have

$$\sigma_\infty^{(B)} = \frac{1}{24\pi} \int_0^1 dx x(x-1)(2x-1) \tan(\pi x) = \frac{3\zeta(3)}{32\pi^4}, \\ \sigma_\infty^{(F)} = \frac{1}{24\pi} \int_{-1/2}^{1/2} dx x(1-4x^2) \tan(\pi x) = \frac{\zeta(3)}{4\pi^4}. \quad (44)$$

These values turn out to be proportional to the asymptotic values of the $\mathcal{F}_n \rightarrow n\mathcal{F}_\infty$ calculated on the n -covered 3-sphere [9]; as noted in Equation (9) we have $\sigma_\infty^{(B/F)} = \frac{1}{4\pi^2} \mathcal{F}_\infty^{(B/F)}$.

On the right in Fig. 1, we illustrated the asymptotic behavior of the corner coefficient which we find to be

$$\sigma_n = \sigma_\infty \left(1 + \frac{b_1}{n} + \frac{b_2}{n^2} + \frac{b_3}{n^3} + \dots \right). \quad (45)$$

Numerical fits show that b_1, b_2 , and b_3 are 1 for the free boson while b_1 is 1 and $b_2 = b_3 \approx 0.31578$ in (45) for the free fermion. In fact, fitting up to $O(1/n^{16})$, we find numerical evidence that $b_{2k} = b_{2k+1}$ for both the scalar and fermion. This indicates that

a factor of $(n+1)/n$ can be factored out of the function in (45), so that

$$\sigma_n = \sigma_\infty \frac{n+1}{n} \left(1 + \frac{b_2}{n^2} + \frac{b_4}{n^4} + \frac{b_6}{n^6} + \dots \right). \quad (46)$$

It is also interesting to study the ratios of the Rényi corner coefficients at large n : based on numerical fits in the range $n = 100$ to 2000 we find

$$\frac{\sigma_n^{(B)}}{\sigma_n^{(F)}} = \frac{3}{8} \left[1 + \frac{\pi^2}{12\zeta(3)} \frac{1}{n^2} - 0.93871149 \frac{1}{n^4} + O\left(\frac{1}{n^5}\right) \right]. \quad (47)$$

The value of the $1/n^2$ -coefficient is inferred from the numerics. Specifically, we fit to the function

$$\frac{3}{8} \left(1 + \frac{d_1}{n} + \frac{d_2}{n^2} + \frac{d_3}{n^3} + \dots \right), \quad (48)$$

and find that $d_1 < 10^{-26}$, $|d_2 - \frac{\pi^2}{12\zeta(3)}| < 10^{-23}$, $d_3 < 10^{-19}$, $d_4 = -0.93871149\dots$, $d_5 < 10^{-13}$ etc. The vanishing of the odd powers in (48) is consistent with (46). Note also that we can now identify the number $b_2 = b_3 \approx 0.31578$ from the fit (45) of the free fermion Rényi entropy corner coefficient at large n as $1 - \frac{\pi^2}{12\zeta(3)}$; this is the value given in the caption of Fig. 1.

Taking the Hurwitz zeta-function expressions for $\mathcal{F}_n^{(B/F)}$ from [9] and using (48) to perform a similar fit at large n in the range 30 to 300, we find

$$\frac{\mathcal{F}_n^{(B)}}{\mathcal{F}_n^{(F)}} = \frac{3}{8} \left[1 - \frac{\pi^2}{12\zeta(3)} \frac{1}{n^2} + 0.937106586 \frac{1}{n^4} + O\left(\frac{1}{n^5}\right) \right]. \quad (49)$$

Again, the value of the $1/n^2$ -coefficient is inferred from the numerics which give $d_1 < 10^{-20}$, $|d_2 + \frac{\pi^2}{12\zeta(3)}| < 10^{-17}$, $d_3 < 10^{-14}$, $d_4 = 0.937106586\dots$, $d_5 < 10^{-12}$ etc. The behaviors of $\mathcal{F}_n^{(B/F)}$ individually is, however, very different that of the Rényi corner coefficients. We find that $\mathcal{F}_n^{(B)} \sim n\mathcal{F}_\infty^{(B)} (1 + O(\frac{1}{n^4}))$ while $\mathcal{F}_n^{(F)} \sim n\mathcal{F}_\infty^{(F)} (1 + \frac{\pi^2}{12\zeta(3)} \frac{1}{n^2} + O(\frac{1}{n^4}))$.

It is not clear whether the similarities observed at large n between σ_n and \mathcal{F}_n have any significance or if it is a coincidence. Perhaps future investigations will clarify this.

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