On Generalized Inverses of a Block in a Partitioned Matrix

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ABSTRACT

Given symmetric generalized inverses of $\begin{pmatrix} A & x \\ x' & c \end{pmatrix}$, where x is a vector with n components, we obtain formulae for the corresponding generalized inverses (symmetric) of A. An application of such formulae in linear models is suggested.

1. INTRODUCTION

In an earlier paper (Mitra and Bhimasankaram [4]) we obtained various types of generalized inverses of A from the corresponding generalized inverses of $(A:a)$. Rohde [6], Bhimasankaram [1], Carlson et al. [2, 3], and others obtained g-inverses of

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

from those of A. In this paper, we obtain formulae for computing symmetric (nnd) g-inverses of a symmetric (correspondingly nnd) matrix A from those of

$$\begin{pmatrix} A & x \\ x' & c \end{pmatrix},$$

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where x is a vector and c a scalar. We also obtain formulae to compute a g-inverse of A from that of

$$\begin{pmatrix} A & u \\ v' & c \end{pmatrix},$$

where A need not be symmetric. Finally, we give an application of the above results to the updating of BLUES in a general linear model.

In this paper we consider only real matrices. The extension to complex matrices is trivial. *In Sections* 2 *to* 6, *by* M *and* G *we mean the following*:

$$M = \begin{pmatrix} A & x \\ x' & c \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} B & y \\ y' & d \end{pmatrix} \tag{1.1}$$

where A and B are symmetric. We follow the same notation as in Rao and Mitra [5]. We use $\|u\|$ to denote the Euclidean norm of u, namely, $(u'u)^{1/2}$. $\mathbf{M}(A)$ denotes the column space of A.

2. WHEN DOES $x \in \mathbf{M}(A)$?

Let M and G be as specified in (1.1). In this section we obtain necessary and sufficient conditions under which $x \in \mathbf{M}(A)$ in terms of G and M. We prove

THEOREM 1. *Let* M *and* G *be as specified in* (1.1), *and let* $G = M^-$. *Then* $x \notin \mathbf{M}(A)$ *if and only if* $Ay = 0$, $y'x = 1$, *and* $d = 0$. *If* M *is nnd, then* $x \in \mathbf{M}(A)$.

Proof. $G = M^- \Rightarrow$

$$\left[\begin{aligned} &ABA + xy'A + Ayx' + dxx' = A, && (2.1)\\ &ABx + y'x \cdot x + c(Ay + dx) = x, && (2.2)\\ &x'Bx + cy'x + c(y'x + cd) = c. && (2.3)\end{aligned}\right.$$

"If" part: If $Ay = 0$, $y'x = 1$, and $d = 0$, then $ABA = A$ and $ABx = 0$ in view of (2.1) and (2.2). Let, if possible, $x \in \mathbf{M}(A)$. Now if $x \in \mathbf{M}(A)$, then $x = ABx = 0$. This is a contradiction to $y'x = 1$. Hence $x \notin \mathbf{M}(A)$.

"Only if" part:

$$x(Ay + dx)' = A(I - BA - yx')$$

from (2.1). Hence, $x \notin \mathbf{M}(A) \Rightarrow Ay + dx = 0 \Rightarrow dx = -Ay \Rightarrow d = 0$ and $Ay = 0$ [since $x \notin \mathbf{M}(A)$]. Now from (2.2) it follows that $ABx = (1 - y'x)x$ and hence $y'x = 1$.

It is well known that if M is nnd, then $x \in \mathbf{M}(A)$. This completes the proof of Theorem 1. ■

3. g-INVERSES OF A WHEN $x \notin \mathbf{M}(A)$

Consider M and G as in (1.1), and let $G = M^-$. In this section we obtain formulae for computing various types of g-inverses of A from the corresponding types of g-inverses of M.

THEOREM 2. *Let $G = M^-$ and let $x \notin \mathbf{M}(A)$. Then*

(i) $B = A^-$,
(ii) $B = A_r^-$ *if and only if* $Bx = 0$ *and* $G = M_r$,
(iii) $B = A_{lm}^-$ *if* $y = rx$ *for some* r *and* $G = M_{lm}^-$, *and*
(iv) $B = A^+$ *if* $c = 0$, $y = (x'x)^{-1}x$, *and* $G = M^+$.

Proof. (i) follows trivially from (2.1) since $Ay = 0$ and $d = 0$.
(ii): "If" part:

$$G = M_r^- \Rightarrow BAB + Bxy' + yx'B + cyy' = B. \tag{3.1}$$

$x \in \mathbf{M}(A) \Rightarrow c = -x'Bx$. Now $c = 0$, since $Bx = 0$. Hence from (3.1) it follows that $B = BAB$.

"Only if" part:

$$B = A_r^- \Rightarrow yx'B + Bxy' + cyy' = 0,$$

$$\Rightarrow yx'Bx + Bx + cyy'x = 0,$$

$$\Rightarrow Bx = 0, \quad \text{since } x'Bx = -c.$$

Now, given that $G = M^-$, it is easy to establish that $B = A_r^-$ and $Bx = 0 \Rightarrow G = M_r^-$.

(iii): Since A and B are symmetric matrices, $B = A_l^-$ if and only if $B = A_m^-$. Now,

$$G = M_l^- \Rightarrow \begin{pmatrix} AB + xy' & Ay + dx \\ x'B + cy' & x'y + cd \end{pmatrix} \text{ is symmetric.}$$

So, if $y = rx$ then xy' is symmetric and hence AB is symmetric.

The proof of (iv) is trivial and is omitted. ■

Theorem 2 describes the conditions when B serves as A^-, A_r^-, A_{lm}^-, and A^+ and when G serves as M^-, M_r^-, M_{lm}^-, and M^+ respectively. Now we obtain formulae for various types of g-inverses of A when $x \notin \mathbf{M}(A)$.

Theorem 3. *Let $G = M^-$ and let $x \notin \mathbf{M}(A)$. Then*

(i) $$\left(I - \frac{yy'}{y'y}\right) B \left(I - \frac{yy'}{y'y}\right) = A_r^- \quad \text{if } G = M_r^-,$$

(ii) $$B\left(I - \frac{yy'}{y'y}\right) = A_l^- \quad \text{if } G = M_{lm}^-,$$

(iii) $$\left(I - \frac{yy'}{y'y}\right) B \left(I - \frac{yy'}{y'y}\right) = A^+ \quad \text{if } G = M^+.$$

Proof. The proof of (i) is easy.

(ii):

$$G = M_{lm}^- \Rightarrow \begin{pmatrix} AB + xy' & Ay + dx \\ x'B + cy' & x'y + cd \end{pmatrix} \text{ is symmetric.}$$

Now, $x \notin \mathbf{M}(A) \Rightarrow Ay = 0$, $d = 0 \Rightarrow Bx = -cy$. By Theorem 2, $B = A^-$. Now,

$$\begin{aligned} AAB\left(I - \frac{yy'}{y'y}\right) &= A(AB + xy')\left(I - \frac{yy'}{y'y}\right) \\ &= A(BA + yx')\left(I - \frac{yy'}{y'y}\right) \\ &= ABA = A. \end{aligned}$$

Hence $B(I - yy'/y'y)$ is A_l^-.

The proof of (iii) is easy and we omit it. ■

REMARK. If M is nnd, then clearly $x \in \mathbf{M}(A)$ and the results of this section are not needed in this case.

4. g-INVERSES OF A WHEN $y'x + cd \neq 1$

Notice that when $y'x + cd \neq 1$, $x \in \mathbf{M}(A)$.

Write $\theta = 1 - y'x - cd \neq 0$ and

$$T = B + \frac{1}{\theta}(Bx + cy)y' + \frac{1}{\theta}y(x'B + cy') + \frac{d}{\theta^2}(Bx + cy)(Bx + cy)'. \quad (4.1)$$

We prove:

THEOREM 4. *Let $G = M^-$, and let $y'x + cd \neq 1$. Let T be as in* (4.1). *Then*

(i) $T = A^-$,
(ii) $T = A_r^-$ *if* $G = M_r^-$,
(iii) $T = A_{lm}^-$ *if* $G = M_{lm}^-$, *and*
(iv) $T = A^+$ *if* $G = M^+$.

Proof. (i): From (2.2) it follows that

$$A(Bx + cy) = x(1 - y'x - cd).$$

So $x = (1/\theta)A(Bx + cy)$. Now the result follows easily from (2.1).

(ii):

$$AT = A\left(B + \frac{1}{\theta}(Bx + cy)y' + \frac{1}{\theta}y(x'B + cy') + \frac{d}{\theta^2}(Bx + cy)(Bx + cy)'\right)$$

$$= AB + xy' + \frac{1}{\theta}Ay(x'B + cy') + \frac{d}{\theta}x(Bx + cy)'$$

$$= AB + xy' + \frac{1}{\theta}(Ay + dx)(Bx + cy)'.$$

Now

$$(Bx+cy)'AT = (Bx+cy)'(AB+xy') + \frac{1}{\theta}(Bx+cy)'(Ay+dx)(Bx+cy)'$$

$$= (\theta x'B + \theta cy') + \frac{1}{\theta}(\theta x'y + \theta cd)(Bx+cy)'$$

from (2.1) and (2.3)

$$= \theta(Bx+cy)' + (1-\theta)(Bx+cy)' = (Bx+cy)'.$$

Now,

$$TAT = BAT + \frac{1}{\theta}(Bx+cy)'AT + \frac{1}{\theta}y(Bx+cy)'AT$$

$$+ \frac{d}{d^2}(Bx+cy)(Bx+cy)'AT.$$

We proved above that $(Bx+cy)'AT = (Bx+cy)'$. Hence

$$\frac{1}{\theta}y(Bx+cy)'AT = \frac{1}{\theta}y(Bx+cy)'$$

and

$$\frac{d}{\theta^2}(Bx+cy)(Bx+cy)'AT = \frac{d}{\theta^2}(Bx+cy)(Bx+cy)'.$$

Now,

$$BAT = BAB + Bxy' + y(Bx+cy)' = B,$$

since

$$y = \frac{1}{\theta}B(Ay+dx), \qquad \text{as} \quad G = M_r^-.$$

Also,

$$\frac{1}{\theta}(Bx + cy)y'AT = \frac{1}{\theta}(Bx + cy)[y' - d(Bx + cy)']$$

$$+ \frac{1}{\theta^2}(Bx + cy)\theta d(Bx + cy)'$$

$$= \frac{1}{\theta}(Bx + cy)y'.$$

Hence $TAT = T$.

(iii):

$$G = M_l^- \Rightarrow \begin{pmatrix} AB + xy' & Ay + dx \\ Bx + cy & x'y + cd \end{pmatrix} \text{ is symmetric.}$$

Further note that by construction T is symmetric. Hence T is A_m^- iff T is A_l^-. Now

$$AT = AB + xy' + \frac{1}{\theta}(Ay + dx)(Bx + cy)'.$$

Since $G = M_l^-$, $AB + xy'$ is symmetric and $(Ay + dx)' = (Bx + cy)$. Hence AT is symmetric.

(iv) follows trivially from (i)–(iii). ■

Remark. Let M and G be nnd. Then T in (4.1) can be rewritten as

$$T = \begin{pmatrix} I & \frac{1}{\theta}(Bx + cy) \end{pmatrix} \begin{pmatrix} B & y \\ y' & d \end{pmatrix} \begin{pmatrix} I \\ \frac{1}{\theta}(Bx + cy)' \end{pmatrix}.$$

So T is nnd. Hence the results of Theorem 4 give us various types of nnd g-inverses of A in this case.

5. g-INVERSES OF A WHEN $x'y + cd = 1$ AND $d \neq 0$

Let G be M^-, and let $x'y + cd = 1$ and $d \neq 0$. Then we have

$$ABx = -c \cdot Ay \tag{5.1}$$

and

$$x'Bx = -c \cdot y'x. \tag{5.2}$$

We consider two cases, namely, $Ay + dx = 0$ and $Ay + dx \neq 0$. We prove

THEOREM 5. *Let $G = M^-$, $x'y + cd = 1$, $d \neq 0$, and $Ay + dx = 0$. Then*

(i) $$B - \frac{1}{d}yy' = A^-,$$

(ii) $$B - \frac{1}{d}yy' = A_r \quad \textit{if} \quad G = M_r^-,$$

(iii) $$B - \frac{1}{d}yy' = A_{lm}^- \quad \textit{if} \quad G = M_{lm}^-,$$

(iv) $$B - \frac{1}{d}yy' = A^+ \quad \textit{if} \quad G = M^+.$$

Proof. (i): $Ay + dx = 0$ and $d \neq 0 \Rightarrow x = -(1/d)Ay$. Further, $ABA + Ayx' = A$ in view of (2.1), and $Ay + dx = 0$. It now clearly follows that $B - (1/d)yy' = A^-$.

(ii):

$$\left(B - \frac{1}{d}yy'\right)A\left(B - \frac{1}{d}yy'\right) = \left(B - \frac{1}{d}yy'\right)(AB + xy')$$

$$= (B - yx'B - cyy') - \frac{1}{d}y(-dx'B + y'xy')$$

$$= B - \frac{1}{d}y[(cd + x'y)y'] = B - \frac{1}{d}yy'.$$

(iii): Notice that $B - (1/d)yy'$ is symmetric. Now $A[B - (1/d)yy'] = AB + xy'$ is symmetric, since $G = M_{lm}^-$. Hence $B - (1/d)yy' = A_{lm}^-$.

The proof of (iv) follows from (i)–(iii). ∎

Remark. Consider the same setup as in Theorem 5, and let G be nnd. Then, clearly the Schur complement of d in G, namely, $B-(1/d)yy'$, is also nnd. Hence Theorem 5 gives various types of nnd g-inverses of A in this case.

Theorem 6. *Let $G = M^-$, $x'y + cd = 1$, $d \neq 0$, and $\|Ay+dx\|^2 = t \neq 0$. Write $\xi = (1/t)(I - BA - yx')(Ay + dx)$ and $R = B + \xi y' + y\xi' + d\xi\xi'$. Then,*

(i) $R = A^-$,
(ii) $RAR = A_r^-$,
(iii) $R = A_{ml}^-$ *if* $G = M_{ml}^-$, *and*
(iv) $RAR = A^+$ *if* $G = M^+$.

Proof. (i) is computational, and (ii) follows trivially.

(iii): Notice that R is symmetric. So it suffices to show that AR is symmetric to prove that R is A_{ml}^-.

If $G = M_{ml}^-$, then

$$\begin{aligned}\xi &= \frac{1}{t}(I - BA - yx')(Ay + dx)\\ &= \frac{1}{t}(I - AB - xy')(Ay + dx)\\ &= \frac{1}{t}[(Ayx' + dxx')y + dc(Ay + dx)]\\ &= \frac{1}{t}[(x'y + cd)(Ay + dx)]\\ &= \frac{1}{t}(Ay + dx).\end{aligned}$$

Further, by construction $A\xi = x$. Now

$$\begin{aligned}AR &= AB + xy' + Ay\xi' + dx\xi'\\ &= AB + xy' + \frac{1}{t}(Ay + dx)(Ay + dx).\end{aligned}$$

Now $AB + xy'$ is symmetric and hence AR is symmetric.

(iv): Notice that $RAR = A_r^-$ and $ARAR = AR$ is symmetric. The result now follows trivially. ■

Remark. Consider the same setup as in Theorem 6, and let G be nnd. Then

$$R=(I \quad \xi)\begin{pmatrix} B & y \\ y' & d \end{pmatrix}\begin{pmatrix} I \\ \xi' \end{pmatrix}$$

is also nnd. Hence Theorem 6 gives various types of nnd g-inverses in this case.

6. g-INVERSES OF A WHEN $x'y=1$, $d=0$, AND $Ay \neq 0$

Clearly, with this case all possible cases are exhausted. g-inverses of A in this case are obtained in a similar manner to those in Theorem 6. We only state the relevent theorem below. The proof is similar to that of Theorem 6.

Theorem 7. *Let $G=M^{-}$, $x'y=1$, $d=0$, and $Ay \neq 0$. Write $\|Ay\|^2 = t \neq 0$. Write $\xi=(1/t)(I-BA-yx')Ay$ and $R=B+\xi y'+y\xi'$. Then*

(i) $R=A^{-}$,
(ii) $RAR=A_r^{-}$,
(iii) $R=A_{lm}^{-}$ *if* $G=M_{lm}^{-}$, *and*
(iv) $RAR=A^{+}$ *if* $G=M^{+}$.

Remark. If G is nnd, then $d=0 \Rightarrow y=0$. Hence the situation in Theorem 7 never arises when G is nnd.

7. g-INVERSE OF A FROM THAT OF $\begin{pmatrix} A & u \\ v' & c \end{pmatrix}$, WHERE A MAY NOT BE SYMMETRIC

In this section we consider the nonsymmetric case and give formulae to compute a g-inverse of A from that of $\begin{pmatrix} A & u \\ v' & c \end{pmatrix}$. We state

Theorem 8. *Let*

$$\begin{pmatrix} B & \alpha \\ \beta' & d \end{pmatrix}=\begin{pmatrix} A & u \\ v' & c \end{pmatrix}^{-}.$$

Then:

(i) *We have*

$$B + \frac{1}{t_1}(I - BA - \alpha v')(A'\beta + dv)\beta' + \frac{1}{t_2}\alpha(\alpha' A' + du')(I - AB - u\beta') = A^-$$

if $A\alpha + du \neq 0$ *and* $A'\beta + dv \neq 0$, *where*

$$t_1 = \|A'\beta + dv\|^2 \quad \text{and} \quad t_2 = \|A\alpha + du\|^2.$$

(ii) *Let* $A'\beta + dv = 0$. *Then* $B = A^-$ *if* $A\alpha = 0$.
(iii) *Let* $A'\beta + dv = 0$ *and* $A\alpha \neq 0$. *Then*

$$B + \frac{1}{t_3}\alpha\alpha' A'(I - AB) = A^-, \qquad \text{where} \quad t_3 = \|A\alpha\|^2.$$

(iv) *Let* $A'\beta + dv \neq 0$ *and* $A\alpha + du = 0$. *Then* $B = A^-$ *if* $A'\beta = 0$.
(v) *Let* $A'\beta + dv \neq 0$, $A\alpha + du = 0$, $A'\beta \neq 0$. *Then*

$$B + \frac{1}{t_4}(I - BA)A'\beta\beta' = A^-, \qquad \text{where} \quad t_4 = \|A'\beta\|^2.$$

The proof is computational and is omitted.

8. TABLE OF RESULTS

Case	Theorem(s) to be consulted
$x \notin \mathrm{M}(A)$	2, 3
$y'x + cd \neq 1$	4
$y'x + cd = 1,\ d \neq 0$	5, 6
$x'y = 1,\ d = 0,\ Ay \neq 0$	7
A nonsymmetric	8

9. AN APPLICATION

Here we briefly mention an application of the results of this paper. The detailed analysis is considered elsewhere. Consider the model

$$Y = x\beta + \epsilon, \qquad E(\epsilon) = 0, \qquad D(\epsilon) = \sigma^2 V,$$

where V is a known pd matrix. X may be deficient in rank. The following computations are made to get a least squares estimator of β:

$$X'V^{-1}Y, \qquad X'V^{-1}X, \qquad (X'V^{-1}X)^{-}, \quad \text{and} \quad \hat{\beta} = (X'V^{-1}X)^{-} X'V^{-1}Y.$$

In the process of model building, it is now found that the last component of β is not a suitable parameter and has to be deleted from the analysis. Partition $X = (X_1 : x)$ where x is the last column of X. We have computed

$$(X'V^{-1}X)^{-} = \left[\begin{pmatrix} X_1' \\ x' \end{pmatrix} V^{-1}(X_1 : x) \right]^{-1}$$

$$= \begin{pmatrix} X_1'V^{-1}X_1 & X_1'V^{-1}X \\ x'V^{-1}X_1 & x'V^{-1}X \end{pmatrix}.$$

When we delete the last component of β, we need

$$(X_1'V^{-1}X_1)^{-} \quad \text{and} \quad X_1'V^{-1}Y.$$

Using formulae of this paper, one can easily compute $(X_1'V^{-1}X_1)^{-}$ from $(XV^{-1}X)^{-}$ and $X_1'V^{-1}Y$ is obtained from $X'V^{-1}Y$ by deleting the last component.

10. CONCLUDING REMARKS

Let

$$M = \begin{pmatrix} A & x \\ x' & c \end{pmatrix}$$

be a symmetric matrix, and suppose G is a g-inverse of A but G is not symmetric. Then, clearly, $\frac{1}{2}(G + G')$ is a symmetric g-inverse of A. Again, if M is nnd, and G is any g-inverse of M, then GMG' is a nnd g-inverse of M. Hence if *some g-inverse* of a symmetric (or nnd) matrix M is available, one can use the results in this paper to obtain a symmetric (or nnd) g-inverse of A.

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