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Congruence of multilinear forms

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Abstract

Let

$$F : U \times \cdots \times U \to \mathbb{K}, \quad G : V \times \cdots \times V \to \mathbb{K}$$

be two n-linear forms with $n \geqslant 2$ on finite dimensional vector spaces U and V over a field $\mathbb{K}$. We say that F and G are symmetrically equivalent if there exist linear bijections $\varphi_1, \dots, \varphi_n : U \to V$ such that

$$F(u_1, \dots, u_n) = G(\varphi_{i_1} u_1, \dots, \varphi_{i_n} u_n)$$

for all $u_1, \dots, u_n \in U$ and each reordering $i_1, \dots, i_n$ of $1, \dots, n$. The forms are said to be congruent if $\varphi_1 = \cdots = \varphi_n$.
Let F and G be symmetrically equivalent. We prove that

(i) if $\mathbb{K} = \mathbb{C}$, then F and G are congruent;
(ii) if $\mathbb{K} = \mathbb{R}$, $F = F_1 \oplus \cdots \oplus F_s \oplus 0$, $G = G_1 \oplus \cdots \oplus G_r \oplus 0$, and all summands F_i and G_j are nonzero and direct-sum-indecomposable, then $s = r$ and, after a suitable reindexing, F_i is congruent to $\pm G_i$.

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1. Introduction

Two matrices A and B over a field $\mathbb{K}$ are called *congruent* if $A = S^{\mathrm{T}}BS$ for some nonsingular S. Two matrix pairs (A_1, B_1) and (A_2, B_2) are called *equivalent* if $A_1 = RA_2S$ and $B_1 = RB_2S$ for some nonsingular R and S. Clearly, if A and B are congruent, then (A, A^{T}) and (B, B^{T}) are equivalent. Quite unexpectedly, the inverse statement holds for complex matrices too: if (A, A^{T}) and (B, B^{T}) are equivalent, then A and B are congruent [2, Chapter VI, §3, Theorem 3]. This statement was extended in [3,4] to arbitrary systems of linear mappings and bilinear forms. In this article, we extend it to multilinear forms.

A *multilinear form* (or, more precisely, *n-linear form*, $n \geqslant 2$) on a finite dimensional vector space U over a field $\mathbb{K}$ is a mapping $F : U \times \cdots \times U \to \mathbb{K}$ such that

$$\begin{aligned}&F(u_1, \ldots, u_{i-1}, au_i' + bu_i'', u_{i+1}, \ldots, u_n)\\ &\quad = aF(u_1, \ldots, u_i', \ldots, u_n) + bF(u_1, \ldots, u_i'', \ldots, u_n)\end{aligned}$$

for all $i \in \{1, \ldots, n\}$, $a, b \in \mathbb{K}$, and $u_1, \ldots, u_i', u_i'', \ldots, u_n \in U$.

Definition 1. Let

$$F : U \times \cdots \times U \to \mathbb{K}, \quad G : V \times \cdots \times V \to \mathbb{K} \tag{1}$$

be two n-linear forms.

(a) F and G are called *equivalent* if there exist linear bijections $\varphi_1, \ldots, \varphi_n : U \to V$ such that

$$F(u_1, \ldots, u_n) = G(\varphi_1 u_1, \ldots, \varphi_n u_n)$$

for all $u_1, \ldots, u_n \in U$.

(b) F and G are called *symmetrically equivalent* if there exist linear bijections $\varphi_1, \ldots, \varphi_n : U \to V$ such that

$$F(u_1, \ldots, u_n) = G(\varphi_{i_1} u_1, \ldots, \varphi_{i_n} u_n) \tag{2}$$

for all $u_1, \ldots, u_n \in U$ and each reordering $i_1, \ldots, i_n$ of $1, \ldots, n$.

(c) F and G are called *congruent* if there exists a linear bijection $\varphi : U \to V$ such that

$$F(u_1, \ldots, u_n) = G(\varphi u_1, \ldots, \varphi u_n)$$

for all $u_1, \ldots, u_n \in U$.

The *direct sum* of forms (1) is the multilinear form

$$F \oplus G : (U \oplus V) \times \cdots \times (U \oplus V) \to \mathbb{K}$$

defined as follows:

$$(F \oplus G)(u_1 + v_1, \ldots, u_n + v_n) := F(u_1, \ldots, u_n) + G(v_1, \ldots, v_n)$$

for all $u_1, \ldots, u_n \in U$ and $v_1, \ldots, v_n \in V$.

We will use the internal definition: if $F : U \times \cdots \times U \to \mathbb{K}$ is a multilinear form, then $F = F_1 \oplus F_2$ means that there is a decomposition $U = U_1 \oplus U_2$ such that

(i) $F(x_1, \ldots, x_n) = 0$ as soon as $x_i \in U_1$ and $x_j \in U_2$ for some i and j.
(ii) $F_1 = F|U_1$ and $F_2 = F|U_2$ are the restrictions of F to U_1 and U_2.

A multilinear form $F : U \times \cdots \times U \to \mathbb{K}$ is *indecomposable* if for each decomposition $F = F_1 \oplus F_2$ and the corresponding decomposition $U = U_1 \oplus U_2$ we have $U_1 = 0$ or $U_2 = 0$.

Our main result is the following theorem.

Theorem 2. (a) *If two multilinear forms over* $\mathbb{C}$ *are symmetrically equivalent, then they are congruent.*

(b) *If two multilinear forms* F *and* G *over* $\mathbb{R}$ *are symmetrically equivalent and*

$$F = F_1 \oplus \cdots \oplus F_s \oplus 0, \quad G = G_1 \oplus \cdots \oplus G_r \oplus 0$$

are their decompositions such that all summands F_i *and* G_j *are nonzero and indecomposable, then* $s = r$ *and, after a suitable reindexing, each* F_i *is congruent to* G_i *or* $-G_i$.

The statement (a) of this theorem is proved in the next section. We prove (b) in the end of Section 3 basing on Corollary 11, in which we argue that every n-linear form $F : U \times \cdots \times U \to \mathbb{K}$ with $n \geqslant 3$ over an arbitrary field $\mathbb{K}$ decomposes into a direct sum of indecomposable forms uniquely up to congruence of summands. Moreover, if $F = F_1 \oplus \cdots \oplus F_s \oplus 0$ is a decomposition in which $F_1, \ldots, F_s$ are nonzero and indecomposable, and $U = U_1 \oplus \cdots \oplus U_s \oplus U_0$ is the corresponding decomposition of U, then the sequence of subspaces $U_1 + U_0, \ldots, U_s + U_0, U_0$ is determined by F uniquely up to permutations of $U_1 + U_0, \ldots, U_s + U_0$.

2. Symmetric equivalence and congruence

In this section, we prove Theorem 2(a) and the following theorem, which is a weakened form of Theorem 2(b).

Theorem 3. *If two multilinear forms* F *and* G *over* $\mathbb{R}$ *are symmetrically equivalent, then there are decompositions*

$$F = F_1 \oplus F_2, \quad G = G_1 \oplus G_2$$

such that F_1 *is congruent to* G_1 *and* F_2 *is congruent to* $-G_2$.

Its proof is based on two lemmas.

Lemma 4. (a) *Let* T *be a nonsingular complex matrix having a single eigenvalue. Then*

$$\forall m \in \mathbb{N} \quad \exists f(x) \in \mathbb{C}[x] : \quad f(T)^m = T^{-1}.$$

(b) *Let* T *be a real matrix whose set of eigenvalues consists of one positive real number or a pair of distinct conjugate complex numbers. Then*

$$\forall m \in \mathbb{N} \quad \exists f(x) \in \mathbb{R}[x] : \quad f(T)^m = T^{-1}. \tag{3}$$

Proof. (a) Let T be a nonsingular complex matrix with a single eigenvalue λ. Since the matrix $T - \lambda I$ is nilpotent (this follows from its Jordan canonical form), the substitution of T for x into the Taylor expansion

$$\begin{aligned} x^{-\frac{1}{m}} =& \lambda^{-\frac{1}{m}} + \left(-\frac{1}{m}\right) \lambda^{-\frac{1}{m}-1}(x-\lambda) \\ &+ \frac{1}{2!}\left(-\frac{1}{m}\right)\left(-\frac{1}{m}-1\right) \lambda^{-\frac{1}{m}-2}(x-\lambda)^2 + \cdots \end{aligned} \tag{4}$$

gives some matrix

$$f(T), \quad f(x) \in \mathbb{C}[x], \tag{5}$$

satisfying $f(T)^m = T^{-1}$.

(b) Let T be a square real matrix. If it has a single eigenvalue that is a positive real number λ, then all coefficients in (4) are real, so the matrix (5) satisfies (3).

Let T have only two eigenvalues

$$\lambda = a + \mathrm{i}b, \quad \bar{\lambda} = a - \mathrm{i}b \qquad (a, b \in \mathbb{R},\ b > 0). \tag{6}$$

It suffices to prove (3) for any matrix that is similar to T over $\mathbb{R}$, so we may suppose that T is the real Jordan matrix

$$T = R^{-1} \begin{bmatrix} J & 0 \\ 0 & \bar{J} \end{bmatrix} R = \begin{bmatrix} aI + F & bI \\ -bI & aI + F \end{bmatrix}, \qquad R := \begin{bmatrix} I & -\mathrm{i}I \\ I & \mathrm{i}I \end{bmatrix}$$

in which $J = \lambda I + F$ is a direct sum of Jordan blocks with the same eigenvalue λ (and so F is a nilpotent upper triangular matrix).

It suffices to prove that

$$\forall m \in \mathbb{N} \quad \exists f(x) \in \mathbb{R}[x] : \ f(J)^m = J^{-1} \tag{7}$$

since such $f(x)$ satisfies (3):

$$\begin{aligned} f(T)^m &= f(R^{-1}(J \oplus \overline{J})R)^m = R^{-1} f(J \oplus \overline{J})^m R \\ &= R^{-1}(f(J)^m \oplus \overline{f(J)}^m)R = R^{-1}(J \oplus \overline{J})^{-1} R = T^{-1}. \end{aligned}$$

The matrix F is nilpotent, so the substitution of $J = \lambda I + F$ into the Taylor expansion (4) gives some matrix $g(J)$ with $g(x) \in \mathbb{C}[x]$ satisfying $g(J)^m = J^{-1}$. Represent $g(x)$ in the form:

$$g(x) = g_0(x) + \mathrm{i}g_1(x), \quad g_0(x), g_1(x) \in \mathbb{R}[x].$$

It suffices to prove that J reduces to $\mathrm{i}I$ by a finite sequence of polynomial substitutions

$$J \longmapsto h(J), \quad h(x) \in \mathbb{R}[x].$$

Indeed, their composite is some polynomial $p(x) \in \mathbb{R}[x]$ such that $p(J) = \mathrm{i}I$, and then $f(x) := g_0(x) + p(x)g_1(x) \in \mathbb{R}[x]$ satisfies (7):

$$f(J)^m = (g_0(J) + p(J)g_1(J))^m = (g_0(J) + \mathrm{i}g_1(J))^m = g(J)^m = J^{-1}.$$

First, we replace J by $b^{-1}(J - aI)$ (see (6)) making $J = \mathrm{i}I + G$, where $G := b^{-1}F$. Next, we replace J by

$$\frac{3}{2}J + \frac{1}{2}J^3 = \frac{3}{2}(\mathrm{i}I + G) + \frac{1}{2}(-\mathrm{i}I - 3G + 3\mathrm{i}G^2 + G^3) = \mathrm{i}I + H,$$

where $H := (3\mathrm{i}G^2 + G^3)/2$. The degree of nilpotency of H is less than the degree of nilpotency of F; we repeat the last substitution until obtain $\mathrm{i}I$. $\square$

Definition 5. Let $G : V \times \cdots \times V \to \mathbb{K}$ be an n-linear form. We say that a linear mapping $\tau : V \to V$ is *G-selfadjoint* if

$$G(v_1, \ldots, v_{i-1}, \tau v_i, v_{i+1}, \ldots, v_n) = G(v_1, \ldots, v_{j-1}, \tau v_j, v_{j+1}, \ldots, v_n)$$

for all $v_1, \ldots, v_n \in V$ and all i and j.

If τ is G-selfadjoint, then for every $f(x) \in \mathbb{K}[x]$ the linear mapping $f(\tau)$ is G-selfadjoint too.

Lemma 6. *Let $G : V \times \cdots \times V \to \mathbb{K}$ be a multilinear form over a field $\mathbb{K}$ and let $\tau : V \to V$ be a G-selfadjoint linear mapping. If*

$$V = V_1 \oplus \cdots \oplus V_s \tag{8}$$

is a decomposition of V into a direct sum of τ-invariant subspaces such that the restrictions $\tau|V_i$ and $\tau|V_j$ of τ to V_i and V_j have no common eigenvalues for all $i \neq j$, then

$$G = G_1 \oplus \cdots \oplus G_s, \quad G_i := G|V_i. \tag{9}$$

Proof. It suffices to consider the case $s = 2$. To simplify the formulas, we assume that G is a bilinear form. Choose $v_1 \in V_1$ and $v_2 \in V_2$, we must prove that $G(v_1, v_2) = G(v_2, v_1) = 0$.

Let $f(x)$ be the minimal polynomial of $\tau|V_2$. Since $\tau|V_1$ and $\tau|V_2$ have no common eigenvalues, $f(\tau|V_1) : V_1 \to V_1$ is a bijection, so there exists $v_1' \in V_1$ such that $v_1 = f(\tau)v_1'$. Since τ is G-selfadjoint, $f(\tau)$ is G-selfadjoint too, and so

$$\begin{aligned} G(v_1, v_2) &= G(f(\tau)v_1', v_2) = G(v_1', f(\tau)v_2) \\ &= G(v_1', f(\tau|V_2)v_2) = G(v_1', 0v_2) = G(v_1', 0) = 0. \end{aligned}$$

Analogously, $G(v_2, v_1) = 0$. $\square$

Proof of Theorem 2(a). Let n-linear forms (1) over $\mathbb{K} = \mathbb{C}$ be symmetrically equivalent; this means that there exist linear bijections $\varphi_1, \ldots, \varphi_n : U \to V$ satisfying (2) for each reordering $i_1, \ldots, i_n$ of $1, \ldots, n$. Let us prove by induction that F and G are congruent. Assume that $\varphi := \varphi_1 = \cdots = \varphi_t$ for some $t < n$ and prove that there exist linear bijections

$$\psi_1 = \cdots = \psi_t = \psi_{t+1}, \psi_{t+2}, \ldots, \psi_n : U \to V$$

such that

$$F(u_1, \ldots, u_n) = G(\psi_{i_1} u_1, \ldots, \psi_{i_n} u_n) \tag{10}$$

for all $u_1, \ldots, u_n \in U$ and each reordering $i_1, \ldots, i_n$ of $1, \ldots, n$.

By (2) and since $\varphi_1, \ldots, \varphi_n$ are bijections, for every pair of indices i, j such that $i < j$ and for all $u_i, u_j \in U$ and $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n \in V$, we have

$$\begin{aligned} &G(v_1, \ldots, v_{i-1}, \varphi u_i, v_{i+1}, \ldots, v_{j-1}, \varphi_{t+1} u_j, v_{j+1}, \ldots, v_n) \\ &\quad = G(v_1, \ldots, v_{i-1}, \varphi_{t+1} u_i, v_{i+1}, \ldots, v_{j-1}, \varphi u_j, v_{j+1}, \ldots, v_n). \end{aligned} \tag{11}$$

Denote $v_i := \varphi_{t+1} u_i$ and $v_j := \varphi_{t+1} u_j$. Then (11) takes the form:

$$G(\ldots, \varphi\varphi_{t+1}^{-1} v_i, \ldots, v_j, \ldots) = G(\ldots, v_i, \ldots, \varphi\varphi_{t+1}^{-1} v_j, \ldots),$$

this means that the linear mapping $\tau := \varphi\varphi_{t+1}^{-1} : V \to V$ is G-selfadjoint.

Let $\lambda_1, \ldots, \lambda_s$ be all distinct eigenvalues of τ and let (8) be the decomposition of V into the direct sum of τ-invariant subspaces $V_1, \ldots, V_s$ such that every $\tau_i := \tau|V_i$ has a single eigenvalue λ_i. Lemma 6 ensures (9). For every $f_i(x) \in \mathbb{C}[x]$, the linear mapping $f_i(\tau_i) : V_i \to V_i$ is G_i-selfadjoint. Using Lemma 4(a), we take $f_i(x)$ such that $f_i(\tau_i)^{t+1} = \tau_i^{-1}$. Then

$$\rho := f_1(\tau_1) \oplus \cdots \oplus f_s(\tau_s) : V \to V$$

is G-selfadjoint and $\rho^{t+1} = \tau^{-1}$.

Define

$$\psi_1 = \cdots = \psi_{t+1} := \rho\varphi, \quad \psi_{t+2} := \varphi_{t+2}, \ldots, \psi_n := \varphi_n. \tag{12}$$

Since ρ is G-selfadjoint and

$$\rho^{t+1}\varphi = \tau^{-1}\varphi = (\varphi\varphi_{t+1}^{-1})^{-1}\varphi = \varphi_{t+1},$$

we have

$$\begin{aligned} G(\psi_1 u_1, \ldots, \psi_n u_n) &= G(\rho\varphi u_1, \ldots, \rho\varphi u_t, \rho\varphi u_{t+1}, \varphi_{t+2}u_{t+2}, \ldots, \varphi_n u_n) \\ &= G(\varphi u_1, \ldots, \varphi u_t, \rho^{t+1}\varphi u_{t+1}, \varphi_{t+2}u_{t+2}, \ldots, \varphi_n u_n) \\ &= G(\varphi_1 u_1, \ldots, \varphi_n u_n) = F(u_1, \ldots, u_n). \end{aligned}$$

So (10) holds for $i_1 = 1, i_2 = 2, \ldots, i_n = n$. The equality (10) for an arbitrary reordering $i_1, \ldots, i_n$ of $1, \ldots, n$ is proved analogously. □

Proof of Theorem 3. Let n-linear forms (1) over $\mathbb{K} = \mathbb{R}$ be symmetrically equivalent; this means that there exist linear bijections $\varphi_1, \ldots, \varphi_n : U \to V$ satisfying (2) for each reordering $i_1, \ldots, i_n$ of $1, \ldots, n$. Assume that $\varphi := \varphi_1 = \cdots = \varphi_t$ for some $t < n$. Just as in the proof of Theorem 2(a), $\tau := \varphi\varphi_{t+1}^{-1}$ is G-selfadjoint. Let (8) be the decomposition of V into the direct sum of τ-invariant subspaces such that every $\tau_p := \tau|V_p$ has a single real eigenvalue λ_p or a pair of conjugate complex eigenvalues

$$\lambda_p = a_p + \mathrm{i}b_p, \quad \bar{\lambda}_p = a_p - \mathrm{i}b_p, \qquad b_p > 0,$$

and $\lambda_p \neq \lambda_q$ if $p \neq q$. Lemma 6 ensures the decomposition (9).

Define the G-selfadjoint linear bijection

$$\varepsilon = \varepsilon_1 1_{V_1} \oplus \cdots \oplus \varepsilon_s 1_{V_s} : V \to V$$

in which $\varepsilon_i = -1$ if λ_i is a negative real number, and $\varepsilon_i = 1$ otherwise. Replacing φ_{t+1} by $\varepsilon\varphi_{t+1}$, we obtain τ without negative real eigenvalues. But the right-hand member of the equality (2) may change its sign on some subspaces V_p. To preserve (2), we also replace φ_{t+2} with $\varepsilon\varphi_{t+2}$ if $t + 1 < n$ and replace $G = G_1 \oplus \cdots \oplus G_s$ (see (9)) with

$$\varepsilon_1 G_1 \oplus \cdots \oplus \varepsilon_s G_s \tag{13}$$

if $t + 1 = n$. By Lemma 4(b), for every i there exists $f_i(x) \in \mathbb{R}[x]$ such that $f_i(\tau_i)^{t+1} = \tau_i^{-1}$. Define

$$\rho = f_1(\tau_1) \oplus \cdots \oplus f_s(\tau_s) : V \to V,$$

then $\rho^{t+1} = \tau^{-1}$. Reasoning as in the proof of Theorem 2(a), we find that (10) with (13) instead of G holds for the linear mappings (12). □

We say that two systems of n-linear forms

$$F_1, \ldots, F_s : U \times \cdots \times U \to \mathbb{K}, \quad G_1, \ldots, G_s : V \times \cdots \times V \to \mathbb{K}$$

are *equivalent* if there exist linear bijections $\varphi_1, \ldots, \varphi_n : U \to V$ such that

$$F_i(u_1, \ldots, u_n) = G_i(\varphi_1 u_1, \ldots, \varphi_n u_n)$$

for each i and for all $u_1, \ldots, u_n \in U$. These systems are said to be *congruent* if $\varphi_1 = \cdots = \varphi_n$.

For every n-linear form F, we construct the system of n-linear forms

$$\mathscr{S}(F) = \{F^\sigma | \sigma \in S_n\}, \quad F^\sigma(u_1, \dots, u_n) := F(u_{\sigma(1)}, \dots, u_{\sigma(n)}), \tag{14}$$

where S_n denotes the set of all substitutions on $1, \dots, n$.

The next corollary is another form of Theorem 2(a).

Corollary 7. *Two multilinear forms F and G over $\mathbb{C}$ are congruent if and only if the systems of multilinear forms $\mathscr{S}(F)$ and $\mathscr{S}(G)$ are equivalent.*

To each substitution $\sigma \in S_n$, we assign some $\varepsilon(\sigma) \in \{1, -1\}$. Generalizing the notions of symmetric and skew-symmetric bilinear forms, we say that an n-linear form F is *ε-symmetric* if $F^\sigma = \varepsilon(\sigma)F$ for all $\sigma \in S_n$. If G is another ε-symmetric n-linear form, then $\mathscr{S}(F)$ and $\mathscr{S}(G)$ are equivalent if and only if F and G are equivalent. So the next corollary follows from Corollary 7.

Corollary 8. *Two ε-symmetric multilinear forms over $\mathbb{C}$ are equivalent if and only if they are congruent.*

3. Direct decompositions

Every bilinear form over $\mathbb{C}$ or $\mathbb{R}$ decomposes into a direct sum of indecomposable forms uniquely up to congruence of summands; see the classification of bilinear forms in [1,5,4]. In [4, Theorem 2 and §2] this statement was extended to all systems of linear mappings and bilinear forms over $\mathbb{C}$ or $\mathbb{R}$. The next theorem shows that a stronger statement holds for n-linear forms with $n \geqslant 3$ over all fields.

Theorem 9. *Let $F : U \times \cdots \times U \to \mathbb{K}$ be an n-linear form with $n \geqslant 3$ over a field $\mathbb{K}$.*

(a) *Let $F = F' \oplus 0$ and let F' have no zero direct summands. If $U = U' \oplus U_0$ is the corresponding decomposition of U, then U_0 is uniquely determined by F and F' is determined up to congruence.*
(b) *Let F have no zero direct summands and let $F = F_1 \oplus \cdots \oplus F_s$ be its decomposition into a direct sum of indecomposable forms. If $U = U_1 \oplus \cdots \oplus U_s$ is the corresponding decomposition of U, then the sequence $U_1, \dots, U_s$ is determined by F uniquely up to permutations.*

Proof. (a) The subspace U_0 is uniquely determined by F since U_0 is the set of all $u \in U$ satisfying

$$F(u, x_1, \dots, x_{n-1}) = F(x_1, u, x_2 \dots, x_{n-1}) = \cdots = F(x_1, \dots, x_{n-1}, u) = 0$$

for all $x_1, \dots, x_{n-1} \in U$.

Let $F = F' \oplus 0 = G' \oplus 0$ be two decompositions in which F' and G' have no zero direct summands, and let $U = U' \oplus U_0 = V' \oplus U_0$ be the corresponding decompositions of U. Choose bases $u_1, \dots, u_m$ of U' and $v_1, \dots, v_m$ of V' such that $u_1 - v_1, \dots, u_m - v_m$ belong to U_0. Then

$$F(u_{i_1}, \dots, u_{i_n}) = F(v_{i_1}, \dots, v_{i_n})$$

for all $i_1, \dots, i_n \in \{1, \dots, m\}$, and so the linear bijection

$$\varphi : U' \longrightarrow V', \quad u_1 \mapsto v_1, \dots, u_m \mapsto v_m,$$

gives the congruence of F' and G'.

(b) Let $F : U \times \cdots \times U \to \mathbb{K}$ be an n-linear form with $n \geqslant 3$ that has no zero direct summands, let

$$F = F_1 \oplus \cdots \oplus F_s = G_1 \oplus \cdots \oplus G_r \tag{15}$$

be two decompositions of F into direct sums of indecomposable forms, and let

$$U = U_1 \oplus \cdots \oplus U_s = V_1 \oplus \cdots \oplus V_r \tag{16}$$

be the corresponding decompositions of U.

Put

$$d_1 = \dim U_1, \ldots, d_s = \dim U_s \tag{17}$$

and choose two bases

$$u_1, \ldots, u_m \in U_1 \cup \cdots \cup U_s, \quad v_1, \ldots, v_m \in V_1 \cup \cdots \cup V_r \tag{18}$$

of the space U with the following ordering of the first basis:

$$u_1, \ldots, u_{d_1} \text{ is a basis of } U_1, \quad u_{d_1+1}, \ldots, u_{d_1+d_2} \text{ is a basis of } U_2, \ldots \tag{19}$$

Let C be the transition matrix from $u_1, \ldots, u_m$ to $v_1, \ldots, v_m$. Partition it into s horizontal and s vertical strips of sizes $d_1, d_2, \ldots, d_s$. Since C is nonsingular, by interchanging its columns (i.e., reindexing $v_1, \ldots, v_m$) we make nonsingular all diagonal blocks. Changing the bases (19), we make elementary transformations within the horizontal strips of C and reduce it to the form:

$$C = \begin{bmatrix} I_{d_1} & C_{12} & \ldots & C_{1s} \\ C_{21} & I_{d_2} & \ldots & C_{2s} \\ \ldots & \ldots & \ldots & \ldots \\ C_{s1} & C_{s2} & \ldots & I_{d_s} \end{bmatrix}. \tag{20}$$

It suffices to prove that $u_1 = v_1, \ldots, u_m = v_m$, that is,

$$C_{pq} = 0 \quad \text{if } p \neq q. \tag{21}$$

Indeed, by (18) $v_1 \in V_p$ for some p. Since F_1 is indecomposable, if $d_1 > 1$ then $u_1, u_2 \in U_1$ and

$$F(\ldots, u_1, \ldots, u_2, \ldots) \neq 0 \quad \text{or} \quad F(\ldots, u_2, \ldots, u_1, \ldots) \neq 0 \tag{22}$$

for some elements of U denoted by points. If (21) holds, then $u_1 = v_1$ and $u_2 = v_2$. Since $v_1 \in V_p$, (22) ensures that $v_2 \notin V_q$ for all $q \neq p$, and so $v_2 \in V_p$. This means that $U_1 \subset V_p$. Therefore, after a suitable reindexing of $V_1, \ldots, V_s$ we obtain $U_1 \subset V_1, \ldots, U_r \subset V_r$. By (16), $r = s$ and $U_1 = V_1, \ldots, U_r = V_r$; so the statement (b) follows from (22).

Let us prove (21). For each substitution $\sigma \in S_n$, the n-linear form F^σ defined in (14) can be given by the n-dimensional matrix

$$\mathbb{A}^\sigma = [a^\sigma_{ij,\ldots,k}]^m_{i,j,\ldots,k=1}, \quad a^\sigma_{ij,\ldots,k} := F^\sigma(u_i, u_j, \ldots, u_k)$$

in the basis $u_1, \ldots, u_m$, or by the n-dimensional matrix

$$\mathbb{B}^\sigma = [b^\sigma_{ij,\ldots,k}]^m_{i,j,\ldots,k=1}, \quad b^\sigma_{ij,\ldots,k} := F^\sigma(v_i, v_j, \ldots, v_k)$$

in the basis $v_1, \ldots, v_m$. Then for all $x_1, \ldots, x_n \in U$ and their coordinate vectors $[x_i] = (x_{1i}, \ldots, x_{mi})^{\mathrm{T}}$ in the basis $u_1, \ldots, u_m$, we have

$$F^\sigma(x_1, \ldots, x_n) = \sum_{i,j,\ldots,k=1}^{m} a^\sigma_{ij\ldots k} x_{i1} x_{j2} \cdots x_{kn}. \tag{23}$$

If $C = [c_{ij}]$ is the transition matrix (20), then

$$b^{\sigma}_{i'j',\dots,k'} = \sum_{i,j,\dots,k=1}^{m} a^{\sigma}_{ij,\dots,k} c_{ii'} c_{jj'} \cdots c_{kk'}. \tag{24}$$

By (15), $a^{\sigma}_{ij,\dots,k} = F^{\sigma}(u_i, u_j, \dots, u_k) \neq 0$ only if all $u_i, u_j, \dots, u_k$ belong to the same space U_l. Hence $\mathbb{A}^{\sigma}$ and, analogously, $\mathbb{B}^{\sigma}$ decompose into the direct sums of n-dimensional matrices:

$$\mathbb{A}^{\sigma} = \mathbb{A}^{\sigma}_1 \oplus \cdots \oplus \mathbb{A}^{\sigma}_s, \quad \mathbb{B}^{\sigma} = \mathbb{B}^{\sigma}_1 \oplus \cdots \oplus \mathbb{B}^{\sigma}_r, \tag{25}$$

in which every $\mathbb{A}^{\sigma}_i$ has size $d_i \times \cdots \times d_i$ and every $\mathbb{B}^{\sigma}_j$ has size $\dim V_j \times \cdots \times \dim V_j$.

We prove (21) using induction in n.

Base of induction: $n = 3$. The three-dimensional matrices $\mathbb{A}^{\sigma}$ and $\mathbb{B}^{\sigma}$ can be given by the sequences of m-by-m matrices

$$\begin{aligned} A^{\sigma}_1 &= [a^{\sigma}_{ij1}]^m_{i,j=1}, \dots, A^{\sigma}_m = [a^{\sigma}_{ijm}]^m_{i,j=1}, \\ B^{\sigma}_1 &= [b^{\sigma}_{ij1}]^m_{i,j=1}, \dots, B^{\sigma}_m = [b^{\sigma}_{ijm}]^m_{i,j=1}; \end{aligned}$$

we call these matrices the *layers* of $\mathbb{A}^{\sigma}$ and $\mathbb{B}^{\sigma}$. The equality (23) takes the form:

$$F^{\sigma}(x_1, x_2, x_3) = [x_1]^{\mathrm{T}} (A^{\sigma}_1 x_{13} + \cdots + A^{\sigma}_m x_{m3})[x_2] \tag{26}$$

for all $x_1, x_2, x_3 \in U$ and their coordinate vectors $[x_i] = (x_{1i}, \dots, x_{mi})^{\mathrm{T}}$ in the basis $u_1, \dots, u_m$. Put

$$\begin{aligned} H^{\sigma}_1 &:= A^{\sigma}_1 c_{11} + \cdots + A^{\sigma}_m c_{m1}, \\ &\cdots \\ H^{\sigma}_m &:= A^{\sigma}_1 c_{1m} + \cdots + A^{\sigma}_m c_{mm} \end{aligned} \tag{27}$$

By (24)

$$b^{\sigma}_{i'j'k'} = \sum_{i,j=1}^{m} (a^{\sigma}_{ij1} c_{1k'} + \cdots + a^{\sigma}_{ijm} c_{mk'}) c_{ii'} c_{jj'}$$

and so

$$B^{\sigma}_1 = C^{\mathrm{T}} H^{\sigma}_1 C, \dots, \quad B^{\sigma}_m = C^{\mathrm{T}} H^{\sigma}_m C. \tag{28}$$

Partition $\{1, \dots, m\}$ into the subsets

$$\mathscr{I}_1 = \{1, \dots, d_1\}, \quad \mathscr{I}_2 = \{d_1 + 1, \dots, d_1 + d_2\}, \dots \tag{29}$$

(see (17)). By (25), if $k \in \mathscr{I}_q$ for some q, then the kth layer of $\mathbb{A}^{\sigma}$ has the form

$$A^{\sigma}_k = 0_{d_1} \oplus \cdots \oplus 0_{d_{q-1}} \oplus \widetilde{A}^{\sigma}_k \oplus 0_{d_{q+1}} \oplus \cdots \oplus 0_{d_s} \tag{30}$$

in which $\widetilde{A}^{\sigma}_k$ is d_q-by-d_q. So by (27) and since all diagonal blocks of the matrix (20) are the identity matrices, we have

$$H^{\sigma}_k = \sum_{i \in \mathscr{I}_1} \widetilde{A}^{\sigma}_i c_{ik} \oplus \cdots \oplus \sum_{i \in \mathscr{I}_{q-1}} \widetilde{A}^{\sigma}_i c_{ik} \oplus \widetilde{A}^{\sigma}_k \oplus \sum_{i \in \mathscr{I}_{q+1}} \widetilde{A}^{\sigma}_i c_{ik} \oplus \cdots \oplus \sum_{i \in \mathscr{I}_s} \widetilde{A}^{\sigma}_i c_{ik}. \tag{31}$$

We may suppose that

$$\sum_{\sigma \in S_3} \sum_{i=1}^{m} \operatorname{rank} A^{\sigma}_i \geqslant \sum_{\sigma \in S_3} \sum_{i=1}^{m} \operatorname{rank} B^{\sigma}_i; \tag{32}$$

otherwise we interchange the direct sums in (15). By (30) and (28),

$$\sum_{\sigma\in S_3}\sum_{i=1}^{m}\operatorname{rank}\widetilde{A}_i^{\sigma} \geqslant \sum_{\sigma\in S_3}\sum_{i=1}^{m}\operatorname{rank}H_i^{\sigma}. \tag{33}$$

Let us fix distinct p and q and prove that $C_{pq}=0$ in (20). Due to (31), (33), and (30),

$$\forall k\in\mathscr{I}_q:\quad \sum_{i\in\mathscr{I}_p}A_i^{\sigma}c_{ik}=0. \tag{34}$$

Replacing in this sum each A_i^{σ} by the basis vector u_i, we define

$$u:=\sum_{i\in\mathscr{I}_p}u_ic_{ik}\in U_p. \tag{35}$$

Since

$$[u]=(0,\dots,0,c_{d+1,k},\dots,c_{d+d_p,k},0,\dots,0)^{\mathrm{T}},\quad d:=d_1+\dots+d_{p-1}$$

by (26) and (34) we have $F^{\sigma}(x,y,u)=0$ for all $x,y\in U_p$. This equality holds for all substitutions $\sigma\in S_3$, hence

$$F(u,x,y)=F(x,u,y)=F(x,y,u)=0, \tag{36}$$

and so $F|u\mathbb{K}$ is a zero direct summand of $F_p=F|U_p$. Since F_p is indecomposable, $u=0$; that is, $c_{d+1,k}=\dots=c_{d+d_p,k}=0$. These equalities hold for all $k\in\mathscr{I}_q$, hence $C_{pq}=0$. This proves (21) for $n=3$.

Induction step. Let $n\geqslant 4$ and assume that (21) holds for all $(n-1)$-linear forms.

The n-dimensional matrices $\mathbb{A}^{\sigma}$ and $\mathbb{B}^{\sigma}$ can be given by the sequences of $(n-1)$-dimensional matrices

$$A_1^{\sigma}=[a^{\sigma}_{i,\dots,j1}]_{i,\dots,j=1}^{m},\dots,A_m^{\sigma}=[a^{\sigma}_{i,\dots,jm}]_{i,\dots,j=1}^{m},$$
$$B_1^{\sigma}=[b^{\sigma}_{i,\dots,j1}]_{i,\dots,j=1}^{m},\dots,B_m^{\sigma}=[b^{\sigma}_{i,\dots,jm}]_{i,\dots,j=1}^{m}.$$

By (24)

$$\begin{aligned}
b^{\sigma}_{i',\dots,j'1}&=\sum_{i,\dots,j}(a^{\sigma}_{i,\dots,j1}c_{11}+\dots+a^{\sigma}_{i,\dots,jm}c_{m1})c'_{ii}\cdots c'_{jj},\\
&\dots\\
b^{\sigma}_{i',\dots,j'm}&=\sum_{i,\dots,j}(a^{\sigma}_{i,\dots,j1}c_{1m}+\dots+a^{\sigma}_{i,\dots,jm}c_{mm})c'_{ii}\cdots c'_{jj}.
\end{aligned} \tag{37}$$

Due to (25) and analogous to (30), each A_k^{σ} with $k\in\mathscr{I}_q$ (see (29)) is a direct sum of $d_1\times\dots\times d_1,\dots,d_s\times\dots\times d_s$ matrices, and only the qth summand $\widetilde{A}_k^{\sigma}$ may be nonzero. This implies (31) for each k and for H_k^{σ} defined in (27).

For each $(n-1)$-linear form G, denote by $s(G)$ the number of *nonzero* summands in a decomposition of G into a direct sum of indecomposable forms; this number is uniquely determined by G due to induction hypothesis. Put $s(M):=s(G)$ if G is given by an $(n-1)$-dimensional matrix M. By (37), the set of $(n-1)$-linear forms given by $(n-1)$-dimensional matrices (27) is congruent to the set of $(n-1)$-linear forms given by $B_1^{\sigma},\dots,B_m^{\sigma}$. Hence

$$s(H_1^{\sigma})=s(B_1^{\sigma}),\dots,s(H_m^{\sigma})=s(B_m^{\sigma}). \tag{38}$$

We suppose that

$$\sum_{\sigma\in S_n}\sum_{k=1}^{m} s(A_k^\sigma) \geqslant \sum_{\sigma\in S_n}\sum_{k=1}^{m} s(B_k^\sigma)$$

otherwise we interchange the direct sums in (15). Then by (38)

$$\sum_{\sigma\in S_n}\sum_{k=1}^{m} s(\widetilde{A}_k^\sigma) \geqslant \sum_{\sigma\in S_n}\sum_{k=1}^{m} s(H_k^\sigma). \tag{39}$$

Let us fix distinct p and q and prove that $C_{pq}=0$ in (20). By (31),

$$s(H_k^\sigma) = s(\widetilde{A}_k^\sigma) + \sum_{p\neq q} s\left(\sum_{i\in\mathscr{I}_p} \widetilde{A}_i^\sigma c_{ik}\right)$$

for each $k\in\mathscr{I}_q$. Combining it with (39), we have

$$\sum_{i\in\mathscr{I}_p} A_i^\sigma c_{ik} = \sum_{i\in\mathscr{I}_p} \widetilde{A}_i^\sigma c_{ik} = 0$$

for each $k\in\mathscr{I}_q$. Define u by (35). As in (36), we obtain

$$F(u,x,\dots,y) = F(x,u,\dots,y) = \cdots = F(x,\dots,y,u) = 0$$

for all $x,\dots,y\in U_p$ and so $F|u\mathbb{K}$ is a zero direct summand of $F_p = F|U_p$. Since F_p is indecomposable, $u=0$; so $C_{pq}=0$. This proves (21) for $n>3$. $\square$

Remark 10. Theorem 9(b) does not hold for bilinear forms: for example, the matrix of scalar product is the identity in each orthonormal basis of a Euclidean space. This distinction between bilinear and n-linear forms with $n\geqslant 3$ may be explained by the fact that decomposable bilinear forms are more frequent. Let us consider forms in a two-dimensional vector space. To decompose a bilinear form, we must make zero two entries in its 2×2 matrix. To decompose a trilinear form, we must make zero six entries in its $2\times 2\times 2$ matrix. In both the cases, these zeros are made by transition matrices, which have four entries.

Corollary 11. *Let $F: U\times\cdots\times U\to\mathbb{K}$ be an n-linear form with $n\geqslant 3$ over a field $\mathbb{K}$. If*

$$F = F_1\oplus\cdots\oplus F_s\oplus 0 \tag{40}$$

and the summands $F_1,\dots,F_s$ are nonzero and indecomposable, then these summands are determined by F uniquely up to congruence. Moreover, if $U=U_1\oplus\cdots\oplus U_s\oplus U_0$ is the corresponding decomposition of U, then the sequence of subspaces

$$U_1+U_0,\dots,U_s+U_0,U_0 \tag{41}$$

is determined by F uniquely up to permutations of $U_1+U_0,\dots,U_s+U_0$.

Proof of Theorem 2(b). For $n=2$ this theorem was proved in [4, Section 2.1] (and was extended to arbitrary systems of forms and linear mappings in [4, Theorem 2]). For $n\geqslant 3$ this theorem follows from Theorem 3 and Corollary 11. $\square$

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