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Computability on the probability measures on the Borel sets of the unit interval

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Abstract

While computability theory on many countable sets is well established and for computability on the real numbers several (mutually non-equivalent) definitions are applied, for most other uncountable sets, in particular for measures, no generally accepted computability concepts at all have been available until now. In this contribution we introduce computability on the set M of probability measures on the Borel subsets of the unit interval $[0;1]$. Its main purpose is to demonstrate that this concept of computability is not merely an ad hoc definition but has very natural properties. Although the definitions and many results can of course be transferred to more general spaces of measures, we restrict our attention to M in order to keep the technical details simple and concentrate on the central ideas. In particular, we show that simple obvious requirements exclude a number of similar definitions, that the definition leads to the expected computability results, that there are other natural definitions inducing the same computability theory and that the theory is embedded smoothly into classical measure theory. As background we consider TTE, Type 2 Theory of Effectivity [10, 11, 19], which provides a frame for very realistic computability definitions. In this approach, computability is defined on finite and infinite sequences of symbols explicitly by Turing machines and on other sets by means of notations and representations. Canonical representations are derived from computation spaces [18]. We introduce a standard representation $\delta_m :\subseteq \Sigma^\omega \to M$ via some natural computation space defined by a subbase σ (the atomic properties) of some topology τ on M and a standard notation of σ. While several modifications of δ_m suggesting themselves at first glance, violate simple and obvious requirements, δ_m has several very natural properties and hence should induce an important computability theory. Many interesting functions on measures turn out to be computable, in particular linear combination, integration of continuous functions and any transformation defined by a computable iterated function system with probabilities. Some other natural representations of M are introduced, among them a Cauchy representation associated with the Hutchinson metric, and proved to be equivalent to δ_m. As a corollary, the final topology τ of δ_m is the well-known weak topology on M. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Computable analysis; Probability measures; Computable operators

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1. Introduction

Measure and integration is a central branch of mathematics pervading almost all parts of abstract analysis. Several authors have already considered questions of effectivity, constructivity, computability or computational complexity in measure or integration theory. Kushner [12] studies computability and Ko [9] computational complexity of integration. Bishop and Bridges [3] present constructive measure theory extensively. Although they do not consider computability, certainly many of their concepts and results have computational counterparts. Edalat gives a domain theoretic approach to effective integration [4, 5]. He also does not consider computability, but it should be possible to extend his topological approach by computability concepts. Traub et al. [14] investigate the computational complexity of numerical algorithms for integration in the real number model of computation. However, this model is unrealistic in many situations and therefore not generally accepted. A systematic study of computability in integration and measure theory does not yet exist. In this paper, we introduce a very natural and realistic computability theory on probability measures. We achieve this by extending TTE, Type 2 Theory of Effectivity, to measure theory. Based on a definition by Grzegorczyk [6] of computable real functions and further work by Hauck [7] and others, TTE has been introduced by Kreitz and Weihrauch [10, 11] as a general framework for studying effectivity, i.e. continuity, computability and computational complexity, in Analysis. For details the reader is referred to the introduction [17] and a recent short survey [18] containing most of the notations we shall use in this paper. More details can be found in [11, 15, 19]. Since this paper is a first attempt, we consider only the space of probability measures on the Borel subsets of the real unit interval.

By $f :\subseteq A \to B$ we denote a partial function, i.e. a function from a subset of A to B. Throughout this paper let Σ be a sufficiently large finite alphabet. Let Σ^* be the set of finite and $\Sigma^\omega = \{p \mid p : \omega \to \Sigma\}$ the set of *"omega"* – words over Σ. On Σ^* we consider the discrete topology and on Σ^ω the Cantor topology defined by the basis $\{w\Sigma^\omega \mid w \in \Sigma^*\}$. For $Y_0, Y_1, \dots, Y_k \in \{\Sigma^*, \Sigma^\omega\}$, a function $f :\subseteq Y_1 \times \cdots \times Y_k \to Y_0$ is called computable, iff it is computed by a Turing machine with a one-way output tape. Every computable function is continuous. The basic idea of TTE is to use finite or infinite sequences as names of "abstract" objects. As naming systems we consider notations, i.e. surjections $\nu :\subseteq \Sigma^* \to S$, and representations, i.e. surjections $\delta :\subseteq \Sigma^\omega \to M$. Continuity and computability concepts are transferred from Σ^* and Σ^ω via notations and representations, respectively, to the named sets straightforwardly, see [11, 15, 17, 18]. Mainly notations or representations which are compatible with some relevant structure on the set under consideration are of practical interest. We do not discuss this for notations (see [13, 15] and Appendix C in [17]), but we will introduce "effective" notations explicitly whenever necessary. In particular, for the rational numbers let $\nu_Q :\subseteq \Sigma^* \to \mathbb{Q}$ be the standard representation via fractions of integers in binary notation. We shall abbreviate $\nu_Q(w)$ by $\bar{w}$. Standard notations of the natural numbers, pairs of rational numbers, etc. will be used without further definitions. For uncountable sets M we shall consider mainly representations derived from "computation spaces" (M,σ,ν), where σ is a countable subset of 2^M of "atomic properties" which identifies points, and ν is a notation of σ [18]. It is assumed that a computer (Turing machine) manipulates ν-names of atomic properties. As a name of an object $x\in M$ we consider any infinite list of all properties $A\in\sigma$ which hold for x. Concretely, the standard representation $\delta_\nu :\subseteq \Sigma^\omega \to M$ is defined by

$$\delta_\nu(p)=x \Leftrightarrow p=w_0\#w_1\#\dots \quad \text{and} \quad \{w_i \mid i\in\omega\}=\{w \mid x\in\nu(w)\}.$$

Every finite prefix of a δ_ν-name p of x contains finitely many atomic properties of x which "approximate" x. Mathematically, this kind of approximation is described by the topology τ_σ on M, which has σ as a subbase. Computability on σ and via δ_ν on M are fixed by the notation ν which expresses how atomic properties can be handled concretely. Thus, for any computation space (M,σ,ν), σ characterizes approximation and ν computability on M. The topology τ_σ and the standard representation δ_ν are closely related: $X\in\tau_\sigma \Leftrightarrow \delta_\nu^{-1}X$ is open in $dom(\delta_\nu)$ (for all $X\subseteq M$), i.e. τ_σ is the final topology of δ_ν. Let $\delta :\subseteq \Sigma^\omega \to M$ and $\delta' :\subseteq \Sigma^\omega \to M'$ be representations and let $f :\subseteq M \to M'$ be a function. An element $x\in M$ is called δ-computable, iff $\delta(p)=x$ for some computable sequence $p\in\Sigma^\omega$. By definition, $\delta\leqslant_t\delta'$ ($\delta\leqslant\delta'$), iff $\delta=\delta' g$ for some continuous (computable) function $g :\subseteq \Sigma^\omega \to \Sigma^\omega$, and f is (δ,δ')-continuous (-computable), iff $f\delta=\delta' g$ for some continuous (computable) function $g :\subseteq\Sigma^\omega\to\Sigma^\omega$. (Accordingly for functions with two or more arguments.) By the "main theorem for admissible representations" [11] a function is continuous relative to standard representations, iff it is continuous w.r.t. the associated final topologies in the usual sense. For more details see [11, 15, 17, 18]. For the real numbers, we need three representations $\rho_<,\rho_>,\rho :\subseteq\Sigma^\omega\to\mathbb{R}$, derived from computation spaces. They can be defined explicitly as follows [15, 18]:

$$\rho_<(p)=x :\Leftrightarrow p=w_0\#w_1\#\dots \text{ with } \{w_i \mid i\in\omega\}=\{w \mid \bar{w}<x\},$$

$$\rho_>(p)=x :\Leftrightarrow p=w_0\#w_1\#\dots \text{ with } \{w_i \mid i\in\omega\}=\{w \mid \bar{w}>x\},$$

$$\rho(p)=x :\Leftrightarrow p=v_0\#w_0\#v_1\#w_1\dots \text{ with } \{(v_i,w_i) \mid i\in\omega\}=\{(v,w) \mid \bar{v}<x<\bar{w}\}.$$

The final topologies are $\tau_<=\{(y;\infty) \mid y\in\mathbb{R}\}\cup\{\mathbb{R}\}, \tau_>=\{(-\infty;y) \mid y\in\mathbb{R}\}\cup\{\mathbb{R}\}$ and the set $\tau_\mathbb{R}$ of ordinary open subsets of $\mathbb{R}$, respectively. Notice that ρ induces the standard computability theory on the real line. The translatability or reducibility properties [15, 18] $\rho\leqslant\rho_<$, $\rho\leqslant\rho_>$, $\rho_<\not\leqslant_t\rho$, $\rho_>\not\leqslant_t\rho$, $\rho_<\not\leqslant_t\rho_>$, $\rho_>\not\leqslant_t\rho_<$ can be proved easily.

In Section 2 we introduce a standard representation δ_m of the set $\mathbf{M}$ of probability measures on the Borel sets of the interval $[0;1]$ by a very natural computation space. We prove a stability theorem for this definition. We discuss some further modifications of the definition and show that they have undesirable properties. The results indicate that the computability theory on $\mathbf{M}$ induced by the representation δ_m is indeed very natural. In Section 3 we prove computability of several interesting functions on measures, in particular linear combination and integration of continuous functions. Also the measure transformation induced by a computable iterated function system with probabilities [1, 8] is computable. Finally, in Section 4 we introduce representations based on other natural computation spaces and a Cauchy representation for the Hutchinson metric [1, 8]. We prove that all these representations are equivalent and that their final topology is the well-known weak topology [2].

2. The standard representation of measures

In this section we introduce the standard representation δ_m of the probability measures and show that it induces a very natural computability theory. Let $Int := \{(a;b), [0;a), (b;1], [0;1] \mid a,b \in \mathbb{Q}, 0<a<b<1\}$ be the set of open subintervals of $[0;1]$ with rational boundaries, and let $I :\subseteq \Sigma^* \to Int$ be some standard notation of Int with $dom(I) \subseteq (\Sigma \setminus \{¢, \#\})^*$. We write I_w for $I(w)$. By $\mathbf{B}$ we denote the set of Borel subsets of $[0;1]$, i.e. the smallest σ-algebra containing Int. By $\mathbf{M}$ we denote the set of probability measures $\mu : \mathbf{B} \to \mathbb{R}$ on the space $([0;1], \mathbf{B})$. By a basic theorem of measure theory [2], every measure $\mu \in \mathbf{M}$ is defined uniquely by its values on the generating set Int. We introduce a standard representation of $\mathbf{M}$ via a computation space. The portions of information available from some standard name of a measure μ shall be all (r, J) with $r \in \mathbb{Q}$ and $J \in Int$ such that $r < \mu(J)$.

Definition 2.1. Define a computation space $(\mathbf{M}, \sigma, \nu)$ by $\sigma :=$ range(ν), where $\mu \in \nu(u¢v) :\Leftrightarrow \bar{u} < \mu(I_v)$ for all $u \in dom(\nu_Q)$, $v \in dom(I)$ and $\mu \in \mathbf{M}$. Let τ_m be the topology on $\mathbf{M}$ with subbase σ and let δ_m be the standard representation of $\mathbf{M}$ derived form ν.

It remains to show that σ identifies the points of $\mathbf{M}$. Consider measures $\mu, \mu' \in \mathbf{M}$ such that $r < \mu(J) \Leftrightarrow r < \mu'(J)$ for all $r \in \mathbb{Q}$ and $J \in Int$. Then obviously, $\mu(J) = \mu'(J)$ for all $J \in Int$, i.e. $\mu = \mu'$. The definition of the representation δ_m looks somewhat arbitrary. By the next stability lemma, we obtain an equivalent representation, if we replace ν_Q and I by adequate other notations. For any $X \subseteq \mathbb{R}$ let $cls(X)$ be the closure of X.

Lemma 2.2 (stability of δ_m). *Let $\nu_S :\subseteq \Sigma^* \to S$ be a notation of a set S which is dense in $\mathbb{R}$ such that $\{(u,v) \mid \nu_S(u) < \nu_Q(v)\}$ and $\{(u,v) \mid \nu_Q(u) < \nu_S(v)\}$ are r.e. Let D be a countable dense subset of* $[0,1]$ *and let I' be a notation of $Int' := \{(a;b), [0;a)(a;1], [0;1] \mid a,b \in D, 0<a<b<1\}$ such that $\{(u,v) \mid cls(I'_u) \subseteq I_v\}$ and $\{(u,v) \mid cls(I_u) \subseteq I'_v\}$ are r.e. Define τ'_m and δ'_m by substituting ν_S for ν_Q and I' for I in Definition* 2.1.

Then $\tau'_m = \tau_m$ and $\delta'_m \equiv \delta_m$.

Proof. We show that there is a machine which maps any $p \in \Sigma^\omega$ into some $q \in \Sigma^\omega$ such that the following holds: If p is a list of all (u,v) such that $\nu_Q(u) < \mu I_v$ for some

$\mu \in \mathbf{M}$, then q is a list of all (x, y) such that $v_S(x) < \mu I'_y$. We have $v_S(x) < \mu I'_y$, iff there are words u, v such that $v_S(x) < v_Q(u), v_Q(u) < \mu I_v$ and $cls(I_v) \subseteq I'_y$. Since the first and the third property are r.e. and the input p lists the second one by assumption, there is indeed a machine with the desired property. This shows $\delta_m \leqslant \delta'_m$. By symmetry we have also $\delta'_m \leqslant \delta'_m$, hence $\delta_m \equiv \delta'_m$. Since τ_m and τ'_m are the final topologies of δ_m and δ'_m [18], respectively, and since equivalent representations have the same final topology, we obtain $\tau_m = \tau'_m$. □

If we replace, for example, rational numbers by finite binary fractions or by finite decimal fractions in the definition of the set *Int* and in Definition 2.1, we obtain an equivalent representation with the same final topology.

If we replace the relation "$<$" in Definition 2.1 by "$\leqslant$", "$>$" or "$\geqslant$", we obtain representations which violate Lemma 2.2. Remember that by definition, the topology τ_m has the subbase $\sigma = \{U_{r,J} \mid r \in \mathbb{Q} \text{ and } J \in Int\}$ where $U_{r,J} = \{\mu \in \mathbf{M} \mid r < \mu(J)\}$. We prepare the proof of the theorem by two lemmas. First, we consider the cases "$r \leqslant \mu(J)$", "$r \geqslant \mu(J)$".

Lemma 2.3. *For $Q \subseteq \mathbb{R}$ let $\tau(Q)$ be the topology on $\mathbf{M}$ generated by the subbase $\sigma(Q) := \{U_{r,J} \mid r \in Q,\ J \in Int\}$, where $U_{r,J} = \{\mu \in \mathbf{M} \mid r \leqslant \mu(J)\}$.*
Then $\tau(P) \not\subseteq \tau(Q)$, if $t \in P \setminus Q$ for some $t \in (0;1)$ (for all $P, Q \subseteq \mathbb{R}$).

The statement holds accordingly, if "$\leqslant$" is replaced by "$\geqslant$".

Proof. The set $U := \{\mu \in \mathbf{M} \mid t \leqslant \mu[0;1)\}$ is in $\tau(P)$. For $x \in (0;1)$ define $\mu_x \in \mathbf{M}$ by $\mu_x\{0\} := x, \mu_x\{1\} := 1 - x$. Then $\mu_t \in U$. Assume $U \in \tau(Q)$. Then there are $V_1, \ldots, V_k \in \sigma(Q)$ with $\mu_t \in V_1 \cap \cdots \cap V_k \subseteq U$. For $i = 1, \ldots, k$ there are $r_i \in Q$ and $J_i \in Int$ with $V_i = \{\mu \mid r_i \leqslant \mu(J_i)\}$. If $1 \notin J_i$, from $\mu_t \in V_i$ we obtain $r_i \leqslant \mu_t(J_i)$, hence $r_i < t$ (since $t \notin Q$). Let $r := \max(\{r_i \mid 1 \notin J_i\} \cup \{0\})$. Then $r < t$. If $0, 1 \notin J_i$, then $r_i \leqslant \mu_t(J_i) = 0 = \mu_r(J_i)$, if $0 \in J_i$ and $1 \notin J_i$ then $r_i \leqslant r \leqslant \mu_r(J_i)$, if $0 \notin J_i$ and $1 \in J_i$ then $r_i \leqslant \mu_t(J_i) = 1 - t < 1 - r = \mu_r(J_i)$, and if $J_i = [0;1]$ then $r_i \leqslant \mu_t(J_i) = 1 = \mu_r(J_i)$. We obtain $\mu_r \in V_1 \cap \cdots \cap V_k$, but $t > r = \mu_r[0;1)$, hence $\mu_r \notin U$. This is a contradiction.

The case "$\geqslant$" can be proved accordingly. □

The next lemma considers the case "$r > v(J)$".

Lemma 2.4. *For $D \subseteq (0;1)$ let $Int(D) := \{(a;b), [0;a), (a;1], [0;1] \mid a, b \in D, 0 < a < b < 1\}$. Let $\tau(D)$ be the topology on $\mathbf{M}$ generated by the subbase $\sigma(D) := \{U_{r,J} \mid r \in \mathbb{Q}, J \in Int(D)\}$ where $U_{r,J} = \{\mu \in \mathbf{M} \mid r > \mu(J)\}$. Then*

$$D \subseteq E \Leftrightarrow \tau(D) \subseteq \tau(E) \quad (\textit{for all } D, E \subseteq (0;1)).$$

Proof. "$\Rightarrow$": (obvious)
"$\Leftarrow$": Assume $D \not\subseteq E$. Then there is some $d \in D \setminus E$. The set $U := \{\mu \in \mathbf{M} \mid 1 > \mu[0;d)\}$ is an element of $\tau(D)$. For any x, $0 \leqslant x \leqslant 1$, define a measure $\mu_x \in M$ by $\mu_x(A) := (1 \text{ if } x \in A, 0 \text{ otherwise})$. In particular, we have $1 > 0 = \mu_d[0;d)$, hence $\mu_d \in U$.

We assume $U \in \tau(E)$. Then there are $V_1, \dots, V_k \in \tau(E)$ with $\mu_d \in V_1 \cap \cdots \cap V_k \subseteq U$. For $i=1,\dots,k$ there are $r_i > 0$ and $J_i \in Int(E)$ with $V_i = \{\mu \mid r_i > \mu(J_i)\}$. Define $e' := \max(\{\sup J_i \mid \sup J_i < d\} \cup \{\inf J_i \mid d \in J_i\})$. Then $e' \leqslant d$, hence $e' < d$ since $d \notin E$. Define $e := (e' + d)/2$. For $i = 1,\dots,k$ we have $d \in J_i \Leftrightarrow e \in J_i$, hence $\mu_d(J_i) = \mu_e(J_i)$ and $\mu_d \in V_i \Leftrightarrow \mu_e \in V_i$, consequently, $\mu_e \in V_1 \cap \cdots \cap V_k$. But $\mu_e[0;d) = 1$ since $e < d$, hence $\mu_e \notin U$. This is a contradiction. □

Theorem 2.5. *If in Definition* 2.1 *the relation "$\bar{u} < \mu(I_v)$" is replaced by "$\bar{u} > \mu(I_v)$", "$\bar{u} \geqslant \mu(I_v)$" or "$\bar{u} < \mu(I_v)$", the resulting representations δ_m violate the stability Lemma* 2.2.

Proof. By Lemma 2.3, in the cases "$\leqslant$" and "$\geqslant$" a change of the set Q of bounds changes the topology τ_m, by Lemma 2.4, in the case "$>$" a change of the set D of interval boundaries changes the topology τ_m. Since the equivalent representations have the same final topology, each of the modifications produces a non-equivalent representation. □

By Definition 2.1 and Lemmas 2.3 and 2.4, many different more or less natural representations and hence computability theories for the set $\mathbf{M}$ of probability measures on $([0;1], \mathbf{B})$ can be introduced. The "user" has to decide, which of them is adequate for his application. The stable representation δ_m from Definition 2.1 is certainly the most important one, since its computability theory will occur most frequently. We shall study it in the following exclusively.

As a simple consequence of Definition 2.1, all rational lower bounds of $\mu(J)$ can be obtained from any δ_m-name of μ and any I-name of J. This property characterizes the representation δ_m except for equivalence: The representation δ_m is $\leqslant$-complete in the set of all representations δ of $\mathbf{M}$, for which $(\mu, J) \mapsto \mu(J)$ is $(\delta, I, \rho_<)$-computable.

Theorem 2.6. *For any representation δ of* $\mathbf{M}$: $\delta \leqslant \delta_m \Leftrightarrow (\mu, J) \mapsto \mu(J)$ *is* $(\delta, I, \rho_<)$-*computable.*

Proof. Consider $\delta \leqslant \delta_m$. By definition there is some computable function $f :\subseteq \Sigma^\omega \to \Sigma^\omega$ with $\delta(p) = \delta_m f(p)$ for all $p \in dom(\delta)$. For $p \in dom(\delta)$, $f(p)$ is a list of all (u,v) with $\bar{u} < \delta_m f(p)(I_v)$. There is a Turing machine, which for any inputs $p \in dom(\delta)$ and $v \in dom(I)$ determines internally the sequence $f(p)$ and simultaneously writes all words u such that (u,v) is listed by $f(p)$. Therefore, $(\mu, J) \mapsto \mu(J)$ is $(\delta, I, \rho_<)$-computable.

Consider, on the other hand, that $(\mu, J) \mapsto \mu(J)$ is $(\delta, I, \rho_<)$-computable. Then there is a machine M_1 which for any $p \in dom(\delta)$ and $v \in Int$ produces a list of all $u \in \Sigma^*$ with $\bar{u} < \delta(p)(I_v)$. From this machine another machine M_2 can be constructed, which for any $p \in dom(\delta)$ constructs a list of all (u,v) with $\bar{u} < \delta(p)(I_w)$. This machine translates δ into δ_m. □

Notice, that in particular $(\mu, J) \mapsto \mu(J)$ is $(\delta_m, I, \rho_<)$-computable. Computing only lower rational bounds does not seem to be satisfactory. We would like to compute also arbitrarily close upper bounds of $\mu(I_v)$. We prove a negative and a positive answer. For any $x \in [0;1]$ define $\mu_x \in \mathbf{M}$ by $\mu_x(A) := (1 \text{if } x \in A, 0 \text{ otherwise})$. For any good and useful representation δ of $\mathbf{M}$ it should be possible to determine a δ-name of the measure μ_x effectively from a name of x. Let $M' := \{\mu_x \mid x \in [0;1]\}$.

Theorem 2.7. *For any representation δ of $\mathbf{M}$, for which $x \mapsto \mu_x$ is (ρ, δ)-continuous on $(0;1)$, $\mu \mapsto \mu[0;1/2)$ is not $(\delta, \rho_>)$-continuous on M'. δ_m is such a representation.*

Proof. Assume, $\mu \mapsto \mu[0;1/2)$ is $(\delta, \rho_>)$-continuous on M'. Then $f : x \mapsto \mu_x[0;1/2)$ is $(\rho, \rho_>)$-continuous, hence $(\tau_{\mathbb{R}}, \tau_>)$-continuous by the main theorem for admissible representations. Since $(-\infty, 1/2) \in \tau_>$, $f^{-1}(-\infty;1/2) \in \tau_{\mathbb{R}}$, but $f^{-1}(-\infty;1/2) = [1/2;1) \notin \tau_{\mathbb{R}}$. We prove now that $g : x \mapsto \mu_x$ is (ρ, δ_m)-continuous, i.e. $(\tau_{\mathbb{R}}, \tau_m)$-con -tinuous by the main theorem for admissible representations. Let $U_{r,J} := \{\mu \in \mathbf{M} \mid r < \mu(J)\}$, $r \in \mathbb{Q}$, $J \in Int$, be some arbitrary subbase element of τ_m. We have $g(x) \in U_{r,J}$, iff $r < \mu_x(J)$. For $r \geqslant 1$, $g^{-1}U_{r,J} = \emptyset$, for $r < 0$, $g^{-1}U_{r,J} = (0;1)$ and for $0 \leqslant r < 1$, $g^{-1} U_{r,J} = J$. Therefore, $g^{-1}U_{r,J}$ is $\tau_{\mathbb{R}}$-open in $dom(g) = (0;1)$. This shows that g is $(\tau_{\mathbb{R}}, \tau_m)$-continuous. □

Therefore, for reasonable representations δ of $\mathbf{M}$, in particular for our standard representation δ_m, arbitrarily close rational upper bounds of measures of open intervals cannot be computed. Although this contradicts intuition at first glance, it has to be accepted as a matter of fact. Notice, that for proving Lemmas 2.3, 2.4 and Theorem 2.7 we have used measures $\mu \in \mathbf{M}$ with $\mu\{x\} > 0$ for some $x \in \mathbb{R}$. Since the arguments have been purely topological without reference to computability, we have also shown that the final topology τ_m of the representation δ_m, which formalizes a concept of "approximation" on the set $\mathbf{M}$ of measures, is quite natural. If we exclude measures μ with $\mu\{x\} > 0$ for some $x \in [0;1]$, $(\mu, J) \mapsto \mu(J)$ becomes (δ_m, I, ρ)-computable. Let $\mathbf{M}^0 := \{\mu \in \mathbf{M} \mid \forall x \in [0;1].\mu\{x\} = 0\}$.

Theorem 2.8. *The function $(\mu, J) \mapsto \mu(J)$ is (δ_m, I, ρ)-computable for $J \in Int$ and $\mu \in \mathbf{M}^0$.*

Proof. By Theorem 2.6, $(\mu, J) \mapsto \mu(J)$ is $(\delta_m, I, \rho_<)$-computable. It suffices to show, that this function is $(\delta_m, I, \rho_>)$-computable for $\mu \in \mathbf{M}^0$. The algorithm branches into four cases depending on the type of the interval $J \in Int$. We discuss only the branch for $J = (a;b)$ with $0 < a < b < 1$. For $\mu \in \mathbf{M}^0$ we have $1 = \mu[0;a) + \mu(a;b) + \mu(b;1]$. If we can approximate $\mu[0;a)$ and $\mu(b;1]$ from below, we can approximate $\mu(J)$ from above. Consider $\delta_m(p) = \mu \in \mathbf{M}^0$ and $I_w = (a;b) \in Int$, $0 < a < b < 1$. Then for any $r \in \mathbb{Q}$ we have: $r > \mu(a;b)$, iff there are rational numbers s_l, s_r with $s_l < \mu[0;a)$ and $s_r < \mu(b;1]$ such that $r > 1 - s_l - s_r$. There is a machine, which for input p and w writes the rational number r to the output , iff it has found numbers s_l and s_r with the above properties by reading the input p. $\square$

3. Computable functions on measures

In this section we prove computability of some interesting functions on probability measures. By the next theorem, the linear combination of measures is computable in all variables.

Theorem 3.1. *The function* $(a, \mu, \mu') \mapsto a\mu + (1-a)\mu'$ *is* $(\rho, \delta_m, \delta_m, \delta_m)$*-computable for* $0 \leqslant a \leqslant 1$.

Proof. Consider $\delta_m(p) = \mu$, $\delta_m(p') = \mu'$ and $\rho(p_a) = a$. Then p is a list of all (r, J) with $r < \mu(J)$, and p' is a list of all (r, J) with $r < \mu'(J)$ $(r \in \mathbb{Q}, J \in Int)$. For $\bar{r} \in \mathbb{Q}$ and $J \in Int$ we have $\bar{r} < a\mu(J) + (1-a)\mu'(J)$, iff there are $r, r', s, s' \in \mathbb{Q}$ with $\bar{r} < sr + (1-s')r'$, $r < \mu(J)$ and $r' < \mu'(J)$, and $s < a < s'$. Therefore, there is a Turing machine, which for any input (p, p', p_a), $(p, p' \in dom(\delta_m), 0 \leqslant \rho(p_a) \leqslant 1)$ produces some list $q \in dom(\delta_m)$ of all $(\bar{r}, J) \in \mathbb{Q} \times Int$ for which there are $r, r', s, s' \in \mathbb{Q}$ such that the above properties hold or are listed by p, p' or p_a, respectively. $\square$

By Theorem 2.6, $(\mu, J) \mapsto \mu(J)$ is $(\delta_m, I, \rho_<)$-computable on $\mathbf{M} \times Int$. We extend this result to $\tau'_{\mathbb{R}} = \{U \cap [0;1] \mid U \in \tau_{\mathbb{R}}\}$, the set of all open subsets of $[0;1]$. First, we need a representation of this topology. For the set $\tau_{\mathbb{R}}$ of open subsets of $\mathbb{R}$, the following computation space $(\tau_{\mathbb{R}}, \sigma, \nu)$ and its derived representation δ_o and topology τ_o are natural (see [18]): For any $U \in \tau_{\mathbb{R}}$ and $u, v \in \Sigma^*$ let $U \in \nu(u¢v)$ iff $[\bar{u}; \bar{v}] \subseteq U$. Consequently, $\delta_o(p) = U$ iff p is a list of all closed intervals with rational boundaries contained in U. We define our standard representation of $\tau'_{\mathbb{R}}$ accordingly: $\delta'_o(p) = U :\Leftrightarrow p$ is a list of all $w \in \Sigma^*$ with $cls(I_w) \subseteq U$ $(p \in \Sigma^\omega, U \in \tau'_{\mathbb{R}})$. Let $\mu_L \in \mathbf{M}$ be the Lebesgue measure on $([0;1], \mathbf{B})$.

Theorem 3.2. (1) $(\mu, U) \mapsto \mu(U)$ *for* $\mu \in \mathbf{M}$ *and* $U \in \tau'_{\mathbb{R}}$ *is* $(\delta_m, \delta'_o, \rho_<)$*-computable.*
(2) $(\mu, U) \mapsto \mu(U)$ *for* $\mu = \mu_L$ *and* $U \in \tau'_{\mathbb{R}}$ *is not* $(\delta_m, \delta'_o, \rho_>)$*-continuous.*

Proof. Consider $\delta_m(p) = \mu$ and $\delta'_o(q) = U$. Then p lists all $(r, J) \in \mathbb{Q} \times Int$ with $r < \mu(J)$, and q lists all $K \in Int$ such that $cls(K) \subseteq U$. For $s \in \mathbb{Q}$ we have $s < \mu(U)$, iff there are intervals $J_1, \ldots, J_k \in Int$ with $cls(J_i) \cap cls(J_j) = \emptyset$ for $1 \leqslant i < j \leqslant k$ and rational numbers $r_1, \ldots, r_k$ such that: $s < r_1 + \cdots + r_k$, $r_i < \mu(J_i)$ and $cls(J_i) \subseteq U$ for $i = 1, \ldots, k$. Therefore, there is some machine, which from any $p \in dom(\delta_m)$ and $q \in dom(\delta'_o)$ computes some $q' \in \Sigma^\omega$ with $\rho_<(q') = \delta_m(p)(\delta'_o(q))$. This means that the function is $(\delta_m, \delta'_o, \rho_<)$-computable. Assume, $F : U \mapsto \mu_L(U)$ is $(\delta'_o, \rho_>)$-continuous, hence $(\tau', \tau_>)$-continuous, where τ' is the final topology of δ'_o defined by the subbase $\sigma = \{V_J \mid J \in Int\}$, where $U \in V_J$ iff $cls(J) \subseteq U$. Since $F[0; \frac{1}{2}) \in (-\infty; \frac{3}{4}) \in \tau_>$, by continuity of F there are intervals $J_1,\ldots,J_k \in Int$ such that $[0;1/2) \in V_{J_1} \cap \cdots \cap V_{J_k} \subseteq F^{-1}(-\infty;3/4)$. We obtain $[0;1] \in V_{J_i}$ for $i=1,\ldots,k$, but $F[0;1]=1 \notin (-\infty;3/4)$, a contradiction. $\square$

For uniform formulations in the next theorems we need a standard representation $\delta_{\rightarrow}$ of the set $C[0;1]$ of continuous functions $f:[0;1]\rightarrow \mathbb{R}$. We define $\delta_{\rightarrow}$ and the corresponding final topology $\tau_{\rightarrow}$ by the following computation space $(C[0;1],\sigma,v): f \in v\,(u¢v¢w) :\Leftrightarrow \bar{u} < f(cls\, I_v) < \bar{w}$ for all $f \in C[0;1]$ and $u,v,w \in \Sigma^*$. Properties of $\delta_{\rightarrow}$ are discussed in [15, 17, 18]. In particular, $\tau_{\rightarrow}$ is the compact-open topology on $C[0;1]$, which is also generated by the metric $d(f,g) := \max\{|f(x)-g(x)| \mid 0 \leqslant x \leqslant 1\}$ on $C[0;1]$. For any measure $\mu \in \mathbf{M}$ and any continuous function $f:[0;1]\rightarrow[0;1]$ define the measure $T_f(\mu)$ by $T_f(\mu)(A) := \mu f^{-1}(A)$ for every Borel set $A \subseteq [0;1]$ (see [2, p. 42]).

Theorem 3.3. *The function $(f,\mu) \mapsto T_f(\mu)$ for continuous $f:[0;1]\rightarrow[0;1]$ and $\mu \in \mathbf{M}$ is $(\delta_{\rightarrow},\delta_m,\delta_m)$-computable.*

Proof. An easy consideration shows that $(J,f) \mapsto f^{-1}(J)$ is $(I,\delta_{\rightarrow},\delta'_o)$-computable. By Theorem 3.2, $(\mu,U) \rightarrow \mu(U)$ is $(\delta_m,\delta'_o,\rho_<)$-computable. Therefore, $(\mu,f,J) \rightarrow T_f(\mu)(J)$ is $(\delta_m,\delta_{\rightarrow},I,\rho_<)$-computable, i.e. there is a computable function h such that $\delta_m(p)(\delta_{\rightarrow}(q))^{-1} I_v = \rho_< h(p,q,v)$. There is a machine which for any input (p,q,v) produces a list of all u such that $\bar{u} < \rho_< h(p,q,w)$. Therefore there is a machine, which for inputs p and q with $\delta_m(p)=\mu$ and $\delta_{\rightarrow}(q)=f$ produces a list of all (u,v) such that $\bar{u} < \mu f^{-1}(I_v)) = T_f(\mu)(I_v)$. This means that $(f,\mu) \mapsto T_f(\mu)$ is $(\delta_{\rightarrow},\delta_m,\delta_m)$-computable. $\square$

We apply this theorem to iterated function systems with probabilities [8, 1]. An iterated function system (IFS) on $[0;1]$ with probabilities is a tuple $\mathbf{S}=([0;1],f_1,\ldots,f_k,p_1,\ldots,p_k)$ where $f_1,\ldots,f_k:[0;1]\rightarrow[0;1]$ are continuous functions and $p_1,\ldots,p_k$ are positive real numbers with $p_1+\cdots+p_k=1$. With $\mathbf{S}$ one associates the function $T_{\mathbf{S}}:\mathbf{M}\rightarrow\mathbf{M}$ defined by $T_{\mathbf{S}}(\mu) := \sum_{i=1}^{k} p_i T_{f_i}(\mu)$

Corollary 3.4. *Let $\mathbf{S}=([0;1],f_1,\ldots,f_k,p_1,\ldots,p_k)$ be an IFS with probabilities such that $f_1,\ldots,f_k$ are $\delta_{\rightarrow}$-computable and $p_1,\ldots,p_k$ are ρ-computable. Then $T_{\mathbf{S}}:\mathbf{M}\rightarrow\mathbf{M}$ is (δ_m,δ_m)-computable.*

Proof. Since computable functions transform computable elements to computable elements, by Theorem 3.3 $T_{f_i}:\mathbf{M}\rightarrow\mathbf{M}$ is computable for $i=1,\ldots,k$. Since the composition of computable functions is computable, by Theorem 3.1 the operator $T_{\mathbf{S}}$ is computable. $\square$

Therefore, for any computable iterated function system $\mathbf{S}$ with probabilities, the associated measure transformation $T_{\mathbf{S}}:\mathbf{M}\rightarrow\mathbf{M}$ is a (δ_m,δ_m)-computable function. We shall show in Theorem 4.8 that its unique fixed point $\mu_{\mathbf{S}} \in \mathbf{M}$ is δ_m-computable, if the system $\mathbf{S}$ is hyperbolic [8].

Next we show that integration of continuous functions is computable in both arguments. The integral of a continuous function can be defined via summations over finite partitions. Consider $\mu \in \mathbf{M}$ and $f \in C[0;1]$. Let *Part* be the set of all finite partitions Z of $[0;1]$ into intervals with rational boundaries (remember: $\bigcup Z = [0;1]$ and $I \cap J = \emptyset$ for $I, J \in Z$). For $Z \in Part$ define $s_+(Z) := \sum_{J \in Z} \mu(J) \cdot \sup_{x \in J} f(x)$ and $s_-(Z) := \sum_{J \in Z} \mu(J) \cdot \inf_{x \in J} f(x)$. Since f is continuous, we have $\sup_{Z \in Part} s_-(Z) = \inf_{Z \in Part} s_+(Z) =: \int f \, \mathrm{d}\mu$. The following lemma is the key to the next proof.

Lemma 3.5. *For any $\beta, \gamma > 0$ there are a finite set $T \subseteq Int$ of (pairwise disjoint) open intervals and a finite set L of closed intervals such that $T \cup L \in Part$, $length(J) < \gamma$ for every $J \in T$ and $\mu(\bigcup L) < \beta$. (L can be chosen, such that each $J \in L$ has length 0.)*

Proof. Define $X_n := \{x \mid \mu\{x\} > 2^{-n}\}$ $(n \in \omega)$ and $X := \bigcup X_n$. Since $\mu(X)$ is finite and $\mu(X) = \sup \mu(X_n)$, there is some n such that $\mu(X \setminus X_n) < \beta$. Since X_n is finite, there are a finite set $T \subseteq Int$ of pairwise disjoint neighboured intervals with $\forall J \in T.\ length(J) < \gamma$ such that $L := \{[a;a] \mid a \in [0;1] \setminus \bigcup T\}$ is finite and $X_n \subseteq \bigcup T$. For each $y \in \bigcup L$ we have $y \in X \setminus X_n$ or $\mu\{y\} = 0$, hence $\mu(\bigcup L) < \beta$. □

Theorem 3.6. *The function $(f, \mu) \mapsto \int f \, \mathrm{d}\mu$ for $f \in C[0;1]$ and $\mu \in \mathbf{M}$ is $(\delta_{\to}, \delta_m, \rho)$-computable.*

Proof. For any $T \subseteq Int$ let $s_-(T) := \sum \{\mu(J) \cdot \inf f(J) \mid J \in T\}$. Consider $f \in C[0;1]$ and $\varepsilon > 0$. By uniform continuity of f there is some $\gamma > 0$ such that $|x - y| < \gamma \Rightarrow |fx - fy| < \varepsilon/4$. Let $M := \max\{|f(x)| \mid 0 \leqslant x \leqslant 1\}$, choose $\beta := \varepsilon/(4(1+M))$. By Lemma 3.5 there is some set $T \subseteq Int$ of pairwise disjoint intervals such that $1 - \beta < \mu \cup T \leqslant 1$ and $\forall J \in T. length(J) < \gamma$. Furthermore, there are $z_J \in \mathbb{Q}$ such that $z_J < \mu(J)$ for $J \in T$ and $1 - \beta < \sum \{z_J \mid J \in T\} < 1$. We describe a procedure for determining from (p, q, n) a number $r \in \mathbb{Q}$ with $|r - \in t f \, d\mu| < 2^{-n}$ where $\delta_{\to}(p) = f$ and $\delta_m(q) = \mu$.

- From p and n determine some $k \in \omega$ such that $|x - y| < 2^{-k} \Rightarrow |fx - fy| < 2^{-n-2}$ [17, 18].
- From p determine some integer upper bound m of M.
- Let $\beta := 2^{-n-2}/(1+m)$.
- By systematic search find a finite set $T \subseteq Int$ of pairwise disjoint intervals and rational numbers z_J $(J \in T)$ with $length(J) < 2^{-k}$ and $z_J < \mu(J)$ for $J \in T$ and $1 - \beta < \sum \{z_J \mid J \in T\}$.
- Determine some $r \in \mathbb{Q}$ such that $|\sum \{z_J \cdot \inf f(J) \mid J \in T\} - r| < 2^{-n-2}$.

The existence of T and the numbers z_J has already been shown. We prove $|r - \int f \, \mathrm{d}\mu| < 2^{-n}$. Let L be the set from Lemma 3.5 and let $T' := T \cup L$. We have:

$$\begin{aligned}\left| \int f \, d\mu - s_-(T') \right| &\leqslant |s_+(T') - s_-(T')| \\ &\leqslant |\textstyle\sum \{\mu(J)(\sup f(J) - \inf f(J))| \mid J \in T'\}\end{aligned}$$

$$< \sum\{\mu(J)\cdot 2^{-n-2} \mid J\in T'\}$$
$$< 2^{-n-2},$$

$$\begin{aligned}|s_-(T')-s_-(T)| &\leqslant \sum\{\mu(L)\cdot \inf f(J) | J\in L\}\\ &\leqslant \mu\bigcup L\cdot m\\ &< \beta\cdot m\\ &< 2^{-n-2},\end{aligned}$$

$$\begin{aligned}|s_-(T)-\sum\{z_J\cdot \inf f(J)\mid J\in T\}| &\leqslant \sum\{(\mu(J)-z_J)\inf f(J) | J\in T\}\\ &< \beta\cdot m\\ &< 2^{-n-2}.\end{aligned}$$

By the triangle inequality we obtain $|\int f\,\mathrm{d}\mu - r| < 2^{-n}$.

There is a computable procedure for determing r, i.e. there is some computable function $g :\subseteq \Sigma^\omega \times \Sigma^\omega \times \Sigma^* \to \Sigma^*$ such that for $f=\delta_\to(p)$, $\mu=\delta_m(q)$ and $n=\bar{u}$ we have $|\bar{v}-\int f\,\mathrm{d}\mu| < 2^{-n}$ where $v=g(p,q,u)$. Using a machine for g one can define easily a machine for a function $h :\subseteq \Sigma^\omega \times \Sigma^\omega \to \Sigma^\omega$ such that $\int \delta_\to(p)\,\mathrm{d}\delta_m(q) = \rho h(p,q)$ for all $p\in dom(\delta_\to)$ and $q\in dom(\delta_m)$. □

Notice that not only arbitrarily close lower bounds but also arbitrarily close upper bounds of the integral can be computed. As a corollary of Theorem 2.7, Theorem 3.6 cannot be extended from $C[0;1]$ to the measurable functions, not even to step functions.

Corollary 3.7. *Let* $f : [0;1] \to \mathbb{R}$ *be the characteristic function of* $[0;1/2)$. *Then* $\mu \mapsto \int f\,\mathrm{d}\mu$ *is not* $(\delta_m, \rho_>)$*-continuous on* $\mathbf{M}$.

Proof. $\int f\,\mathrm{d}\mu = \mu[0;1/2)$, apply Theorem 2.7. □

4. Further representations of measures

In Definition 2.1 we have used atomic properties $r<\mu(J)$ with $r\in\mathbb{Q}$ and $J\in Int$ for identifying measures. By Theorem 3.6, $(f,\mu)\mapsto \int f\,\mathrm{d}\mu$ is $(\delta_\to,\delta_m,\rho)$-computable for continuous functions. In the following, we identify measures μ by atomic properties $r<\int t\,\mathrm{d}\mu$ or $r<\int t\,\mathrm{d}\mu<s$, where $r,s\in\mathbb{Q}$ and t is from a set of simple continuous "test functions".

Definition 4.1. For $n\in\omega$ and $0\leqslant m\leqslant 2^n$ define the triangle function $t_{nm}\in C[0;1]$ by

$$t_{nm}(x) := \begin{cases} x-(m-1)2^{-n} & \text{if } (m-1)2^{-n}\leqslant x\leqslant m\cdot 2^{-n},\\ (m+1)2^{-n}-x & \text{if } m\cdot 2^{-n}< x\leqslant (m+1)\cdot 2^{-n},\\ 0 & \text{otherwise.}\end{cases}$$

Let δ'_m and δ''_m be the standard representation of $\mathbf{M}$ induced by the computation spaces $(\mathbf{M}, \sigma', \nu')$ and $(\mathbf{M}, \sigma'', \nu'')$, respectively, defined as follows: $\mu \in \nu'(0^n¢0^m¢u) :\Leftrightarrow \bar{u} < \int t_{nm}\,\mathrm{d}\mu$, $\mu \in \nu''(0^n¢0^m¢u¢v) :\Leftrightarrow \bar{u} < \int t_{nm}\,\mathrm{d}\mu < \bar{v}$ for all $\mu \in \mathbf{M}$, $n \in \omega$, $0 \leqslant m < 2^n$ and $u, v \in dom(\nu_Q)$.

We have not yet shown, that the systems σ' and σ'' from Definition 4.1 identify points, i.e. δ'_m and δ''_m may still be representations of partitions of $\mathbf{M}$ which are coarser than $\{\{\mu\} \mid \mu \in M\}$.

Theorem 4.2. *δ'_m and δ''_m are representations of $\mathbf{M}$ such that $\delta_m \equiv \delta'_m \equiv \delta''_m$.*

Proof. For any interval $J \in Int$ let $c_J : [0;1] \to \mathbb{R}$ be the characteristic function of J. Let T be the set of linear combinations with positive integer coefficients of functions t_{nm} $(n \in \omega, 0 \leqslant m \leqslant 2^n)$. An easy consideration shows, that there is a sequence $u_0, u_1, \ldots$ of functions from T such that $\forall x, k \cdot u_k(x) \leqslant u_{k+1}(x)$ and $\forall x \cdot c_J(x) = \sup_k u_k(x)$. By a basic property of integrals [2], $\int c_J\,\mathrm{d}\mu = \sup_k \int u_k\,\mathrm{d}\mu$, i.e. $r < \mu(J) \Leftrightarrow r < \int c_J\,\mathrm{d}\mu \Leftrightarrow \exists k \cdot r < \int u_k\,\mathrm{d}\mu$. Therefore, $r < \mu(J)$, iff there is some function $u \in T$ with $\forall x \cdot u(x) \leqslant c_J(x)$ and $r < \int u\,\mathrm{d}\mu$. In particular, for each $J \in Int$, $\mu(J)$ is defined uniquely by the set of all $\int t_{nm}\,\mathrm{d}\mu$, therefore σ' and σ'' from Definition 4.1 identify points and δ'_m and δ''_m are representations of $\mathbf{M}$. There is a machine which transforms any $p \in \Sigma^\omega$ to some $q \in \Sigma^\omega$ with the following property. If p is a list of all (n, m, u) such that $\bar{u} < \int t_{nm}\,\mathrm{d}\mu$ for some $\mu \in \mathbf{M}$, then q is a list of all (u, v) such that $\bar{v} < \mu(I_v)$. This proves $\delta'_m \leqslant \delta_m$. On the other hand, we know from Theorem 3.6, that $(f, \mu) \mapsto \int f\,\mathrm{d}\mu$ is $(\delta_\to, \delta_m, \rho)$-computable. Furthermore, $(n, m) \mapsto t_{nm}$ is $(\nu_Q, \nu_Q, \delta_\to)$-computable on $\omega \times \omega$. Therefore there is a machine which transforms any $p \in \Sigma^\omega$ to some $q \in \Sigma^\omega$ with the following property. If p is a list of all (u, v) with $\bar{u} < \mu(I_v)$ for some $\mu \in \mathbf{M}$, then q is a list of all (n, m, u, v) with $\bar{u} < \int t_{nm}\,\mathrm{d}\mu < \bar{v}$. This proves $\delta_m \leqslant \delta''_m$. An easy consideration shows $\delta''_m \leqslant \delta'_m$. □

By definition, the weak topology τ_w on the set $\mathbf{M}$ of probability measures on $([0;1], \mathbf{B})$ is the coarsest, i.e. smallest, topology τ, such that $\mu \mapsto \int f\,\mathrm{d}\mu$ is $(\tau, \tau_\mathbb{R})$-continuous for every $f \in C[0;1]$ [2]. As a corollary of Theorem 4.2 we obtain:

Corollary 4.3. *The weak topology τ_w is the final topology τ_m of the representation δ_m.*

Proof. By Theorem 3.6, $\mu \mapsto \int f\,\mathrm{d}\mu$ is (δ_m, ρ)-continuous, i.e. $(\tau_m, \tau_\mathbb{R})$-continuous for all $f \in C[0;1]$. We obtain $\tau_w \subseteq \tau_m$ by definition of the weak topology. Consider the functions $F_{nm} : \mu \mapsto \int t_{nm}\,\mathrm{d}\mu$. Let $V_{nmr} = \{\mu \mid r < \int t_{nm}\,\mathrm{d}\mu\}$ be a subbase element of τ'_m. Then $V_{nmr} = F_{nm}^{-1}(r; \infty)$, hence $V_{nmr} \in \tau_w$. This shows $\tau'_m \subseteq \tau_w$. From Theorem 4.2 we know $\tau_m = \tau'_m$, hence $\tau_m \subseteq \tau_w$. □

The weak topology τ_w on $([0;1], \mathbf{B})$ can be generated by a metric [2].

Definition 4.4 (*Hutchinson metric*). Let $Lip := \{f \in C[0;1] \mid f(0) = 0 \text{ and } \forall x, y. |f(x) - f(y)| \leqslant |x - y|\}$. Define $d^H : \mathbf{M} \times \mathbf{M} \to \mathbb{R}$ by $d^H(\mu, \mu') := \sup\{|\int f \, \mathrm{d}\mu - \int f \, \mathrm{d}\mu'| \mid f \in Lip\}$.

The metric d^H is called the Hutchinson metric [1, 8].

Lemma 4.5. *d^H is a metric on* $\mathbf{M}$.

Proof. For any $f \in Lip$, $\forall x. |f(x)| \leqslant 1$, hence $\int f \, \mathrm{d}\mu \leqslant \int 1 \, \mathrm{d}\mu = \mu[0;1] = 1$. Therefore, $d^H(\mu, \mu') \in \mathbb{R}$ is well defined. We have $d^H(\mu, \mu) = 0$ by definition. Consider $d^H(\mu, \mu') = 0$. Then by $t_{nm} \in Lip$, $\int f_{nm} \, \mathrm{d}\mu = \int f_{nm} \, \mathrm{d}\mu'$ for all n and $0 \leqslant m \leqslant 2^n$, hence $\mu = \mu'$ since the set σ' from Definition 4.1 identifies points. For any $f \in Lip$ we have $|\int f \, \mathrm{d}\mu - \int f \, \mathrm{d}\mu'| \leqslant |\int f \, \mathrm{d}\mu - \int f \, \mathrm{d}\mu''| + |\int f \, \mathrm{d}\mu'' - \int f \, \mathrm{d}\mu|$. Taking sups first on the right-hand side and then on the left-hand side, we obtain the triangle inequality for d^H. $\square$

Theorem 4.6. *$d^H : \mathbf{M} \times \mathbf{M} \to \mathbb{R}$ is $(\delta_m, \delta_m, \rho)$-computable.*

Proof. We introduce a simple dense subset of *Lip*. Let S_n be the set of all polygon functions $f \in C[0;1]$ with break points $m2^{-n}$, $0 \leqslant m \leqslant 2^n$, such that $f(0) = 0$ and $f(m2^{-n}) - f((m-1)2^{-n}) \in \{-2^{-n}, 0, 2^{-n}\}$. For all $g \in Lip$ and all $n \in \omega$ there is some $f \in S_n$ with $d(f, g) \leqslant 2^{-n}$. Let ν be some standard notation of $S := \bigcup_n S_n$. Then $\nu \leqslant \delta_{\to}$. It suffices, to describe a method for determining $d^H(\mu, \mu')$ with error $< 2^{-n}$ from $p \in \delta_m^{-1}(\mu)$, $p' \in \delta_m^{-1}(\mu')$ and $n \in \omega$. From p, p' and n determine for all $f \in S_{n+2}$ some $r_f \in \mathbb{Q}$ such that $|r_f - |\int f \, \mathrm{d}\mu - \int f \, \mathrm{d}\mu'|| < 2^{-n-2}$. Let $r := \max\{r_f \mid f \in S_{n+2}\}$. We show $|r - d^H(\mu, \mu')| < 2^{-n}$. Let $f \in S_{n+2}$ such that $r = r_f$. First of all, $r - 2^{-n-2} < |\int f \, \mathrm{d}\mu - \int f \, \mathrm{d}\mu'| \leqslant d^H(\mu, \mu')$. There is some $g \in Lip$ with $d^H(\mu, \mu') - 2^{-n-2} < |\int g \, \mathrm{d}\mu - \int g \, \mathrm{d}\mu'|$. There is some $h \in S_{n+2}$ with $d(g, h) \leqslant 2^{-n-2}$. We obtain

$$
\begin{aligned}
d_H(\mu, \mu') &< \left| \int g \, \mathrm{d}\mu - \int g \, \mathrm{d}\mu' \right| + 2^{-n-2} \\
&\leqslant \left| \int g \, \mathrm{d}\mu - \int h \, \mathrm{d}\mu \right| + \left| \int h \, \mathrm{d}\mu - \int h \, \mathrm{d}\mu' \right| \\
&\quad + \left| \int h \, \mathrm{d}\mu' - \int g \, \mathrm{d}\mu' \right| + 2^{-n-2} \\
&\leqslant 2^{-n-2} + \left| \int h \, \mathrm{d}\mu - \int h \, \mathrm{d}\mu' \right| + 2^{-n-2} + 2^{-n-2} \\
&\leqslant 3 \cdot 2^{-n-2} + r_h + 2^{-n-2} \\
&< 2^{-n} + r.
\end{aligned}
$$

Therefore, $|r - d^H(\mu, \mu')| < 2^{-n}$. Since $\nu \leqslant \delta_{\to}$ and $(f, \mu) \mapsto \int f \, \mathrm{d}\mu$ is $(\delta_{\to}, \delta_m, \rho)$-computable, there is a machine which determines some r with the above properties form p, p' and n. $\square$

By Lemma 2.1 from [16], the metric space $(\mathbf{M}, d^H)$ has a countable dense subset. By Corollary 45.4 from [2], the discrete measures are dense. We shall use the discrete measures determined by rational numbers as a dense subset. Let $\mathbf{M}_\alpha$ be the set of all probability measures $\mu \in \mathbf{M}$ such that there are a finite set K and rational numbers $r_k, s_k \in [0;1]$ for all $k \in K$ such that $\sum\{s_k \mid k \in K\} = 1$ and $\mu = \sum s_k \mu_{r_k}$, where $\mu_x(A) = (1$ if $x \in A, 0$ otherwise). Let v_d be a standard notation of $\mathbf{M}_\alpha$. A computable metric space is a quadruple (M, d, A, v) such that (M, d) is a metric space, A is a dense countable subset and v is a notation $v: \subseteq \Sigma^* \to A$ of A such that the set $\{(u, v, w, x) \mid \bar{u} < d(v(v), v(w)) < \bar{x}\}$ is r.e. [16]. This definition is somewhat stronger than that in [15]. For a computable metric space (M, d, A, v), the Cauchy representation δ_C [18] is defined as follows (we assume w.l.o.g. $dom(v) \subseteq (\Sigma \setminus \{\#\})^*) : \delta_C(p) = x :\Leftrightarrow$ $p = u_0 \# u_1 \# \ldots$ such that $\forall i > k d(v(u_i),\ v(u_k)) < 2^{-k}$ and $x = \lim_{i \to \infty} v(u_i)$.

Theorem 4.7. (1) $v_d \leqslant \delta_m$ (2) $(\mathbf{M}, d^H, \mathbf{M}_d, v_d)$ *is a computable metric space.* (3) *The Cauchy representation* δ_m^C *for this space is equivalent to* δ_m.

Proof. 1. This can be proved easily.

2. From (1) and Theorem 4.6 we conclude, that $\bar{u} < d^H(v_d(v), v_d(w)) < \bar{x}$ is r.e. It remains to show, that $\mathbf{M}_\alpha$ is dense in $\mathbf{M}$. Consider $\mu \in \mathbf{M}$ and $n \in \omega$. By Lemma 3.5, there are a finite set T and a finite set of closed intervals L such that $T' := T \cup L$ is a partition of $[0;1]$, $length(J) < 2^{-n-3}$ for all $J \in T$ and $\mu \cup L < 2^{-n-3}$. Since $1 - 2^{-n-3} < \mu \cup T \leqslant 1$, there are rational numbers $t_J < \mu(J)$ with $1 - 2^{-n-3} < \sum\{t_J \mid J \in T\}$ $< \sum\{\mu(J) \mid J \in T\} \leqslant 1$. Define $S := \sum\{t_J \mid J \in T\}$. Define $\mu' \in \mathbf{M}_d$ by $K := T$, $r_J :=$ the center of J, $s_J := t_J/S$ for all $J \in T$. For any $f \in Lip$ we obtain:

$$
\begin{aligned}
\left|\int f\,\mathrm{d}\mu - s_-(T')\right| &\leqslant |s_+(T') - s_-(T')| \\
&\leqslant \textstyle\sum\{|\sup f(J) - \inf f(J)|\mu(J) \mid J \in T'\} \\
&\leqslant 2^{-n-3}, \\
|s_-(T') - s_-(T)| &= \textstyle\sum\{|\inf f(J)|\mu(J) \mid J \in L\} \\
&\leqslant \mu \cup L \\
&\leqslant 2^{-n-3}, \\
\left|s_-(T) - \int f\,\mathrm{d}\mu'\right| &= |\textstyle\sum\{\inf f(J)\mu(J) - f(r_J)s_J \mid J \in T\}| \\
&\leqslant \textstyle\sum\{|\inf f(J)\mu(J) - f(r_J)\mu(J)| + |f(r_J)\mu(J) \\
&\qquad - f(r_J)s_J| \mid J \in T\} \\
&\leqslant 2^{-n-3} + \textstyle\sum\{|\mu(J) - s_J| \mid J \in T\} \\
&\leqslant 2^{-n-3} + \textstyle\sum\{|\mu(J) - t_J| \mid J \in T\} + \sum\{|t_J - t_J/S| \mid J \in T\} \\
&\leqslant 2 \cdot 2^{-n-3} + \textstyle\sum\{|t_J \mid J \in T\} \cdot (1/S - 1) \\
&< 4 \cdot 2^{-n-3} \quad (\text{since } 7/8 < S \leqslant 1).
\end{aligned}
$$

Combining the three inequalities, we obtain $|\int f\,\mathrm{d}\mu - \int f\,\mathrm{d}\mu'| < 2^{-n}$. Therefore, $\mathbf{M}_\alpha$ is dense in $\mathbf{M}$.

3. For each function t_{nm} from Definition 4.1 and each $\mu \in \mathbf{M}_\alpha$ with $\mu = \sum\{r_k \cdot \mu_{s_k} \mid k \in K\}$, $\int t_{nm}\,\mathrm{d}\mu = \sum r_k \cdot t_{nm}(s_k) \in \mathbb{Q}$. Obviously, $(n,m,\mu) \mapsto \int t_{nm}\,\mathrm{d}\mu$ is (v_Q, v_Q, v_d, v_Q)-computable for $n \in \omega$, $0 \leqslant m \leqslant 2^n$ and $\mu \in \mathbf{M}_\alpha$. Let $\delta_m^C(p) = \mu$ with $p = u_0\sharp u_1\sharp u_2\sharp\ldots$. We have $|\int t_{nm}\,\mathrm{d}\mu - \int t_{nm} dv_d(u_k)| \leqslant d^H(\mu, v(u_k)) \leqslant 2^{-k}$ for all k, therefore $\bar{v} < \int t_{nm}\,\mathrm{d}\mu \Leftrightarrow \exists k.\bar{v} < \int t_{nm}\,\mathrm{d}v_d(u_k) - 2^{-k}$. By the above observation, $\bar{v} < \int t_{nm}\,\mathrm{d}v_d(w) < 2^{-k}$ is decidable in v, w, n, m, k. Therefore, from p a list of all (n, m, v) can be computed such that $\bar{v} < \int t_{nm}\,\mathrm{d}\mu$. This shows $\delta_m^C \leqslant \delta_m'$.

We prove $\delta_m \leqslant \delta_m^C$. It sufficies to show, that there is a machine, which for any $p \in dom(\delta_m)$ and $n \in \omega$ determines some $v \in \Sigma^*$ such that $d^H(\delta_m(p), v_d(v)) < 2^{-n}$. The method is already outlined in (2) above. By definition, p is a list of all (u, J) with $\bar{u} < \mu(J)$ ($\mu := \delta_m(p)$). Compute v as follows:

- By exhaustive search determine a finite set $T \subset Int$ of pairwise disjoint intervals and rational numbers t_J ($J \in T$) such that $length\,\mu(J) < 2^{-n-3}$ and $t_J < \mu(J)$ for all $J \in T$ and $1 - 2^{-n-3} < \sum\{t_J \mid J \in T\}$.
- Determine v such that $v_d(v)$ is the measure determined by the numbers $r_J, s_J (J \in T)$ with $s_J = t_J / \sum\{t_J \mid J \in J\}$.

The existence of T and the numbers $t_J (J \in T)$ has been shown in (2). Also the property $d^H(\mu, v_d(v)) < 2^{-n}$ has been proved in (2). Therefore, $\delta_m \leqslant \delta_m^C$. □

Since $\delta_m \equiv \delta_m' \equiv \delta_m'' \equiv \delta_m^C$, these four representations of the probability measures $\mathbf{M}$ on the space $([0;1], \mathbf{B})$ induce the same computability theory and in particular have the same final topology, which is the topology τ generated by the Hutchinson-metric. As a consequence, for a hyperbolic [8] computable IFS with probabilities as in Corollary 3.4 the unique invariant measure is computable w.r.t. any of these representations. For a domain-theoretic approach see [5].

Theorem 4.8. *Let $\mathbf{S} = ([0;1], f_1, \ldots, f_k, p_1, \ldots, p_k)$ be a hyperbolic IFS with probabilities such that $f_1, \ldots, f_k$ are $\delta_\rightarrow$-computable and $p_1, \ldots, p_k$ are ρ-computable. Then the unique fixed point $\mu_\mathbf{S}$ of the operator $T_\mathbf{S} : \mathbf{M} \to \mathbf{M}$ defined by $T_\mathbf{S}(\mu)(A) := \sum_{i=1}^k p_i\, \mu(f_i^{-1}(A))$ is δ_m-computable.*

Proof. By Corollary 3.4 and Theorem 4.7 the operator $T_\mathbf{S}$ is continuous on the metric space $(\mathbf{M}, d^H)$. Since the system $\mathbf{S}$ is hyperbolic, $T_\mathbf{S}$ is contracting [8]. Its unique fixed point is the limit of the sequence $(\mu_n)_{n\in\omega}$ with $\mu_n = T_\mathbf{S}^n(\mu_0)$ where μ_0 is the (δ_m^C-computable) equidistribution measure. Since $(\mathbf{M}, d^H)$ is a computable metric space, the sequence $(\mu_n)_{n\in\omega}$ is (v_Q, δ_m^C)-computable by Corollary 3.4 and Theorem 4.7. Since $T_\mathbf{S}$ is contracting, $d^H(\mu_n, \mu_{n+1}) < r^{-n}$ for some positive $r < 1$. Hence the limit $\mu_\mathbf{S}$ of the sequence $(\mu_n)_{n\in\omega}$ is δ_m^C-computable and δ_m-computable (by Theorem 4.7). □

In measure theory not only probability measures but arbitrary measures $\mu : \mathbf{B} \to \mathbb{R} \cup \{\infty\}$ are studied. Let $\mathbf{M}^\mathrm{b}$ be the set of all measures $\mu : \mathbf{B} \to \mathbb{R}$, i.e. all bounded measures on $([0;1],\mathbf{B})$. Let $\delta^<$ be the representation of M^b obtained from Definition 2.1, where $\mathbf{M}$ is replaced by $\mathbf{M}^b$. While $\delta_m(p)[0;1]=1$, $\delta^<(p)[0;1]$ may be any non-negative real number. An easy proof shows that $\mu\mapsto\mu[0;1]$ is only $(\delta^<,\rho_<)$-computable and not $(\delta^<,\rho)$-continuous. This means, that portions of information about upper bounds of $\delta^<(p)[0;1]$ are not available from prefixes of p. As a consequence, Theorem 3.6 on integration fails for $\delta^<$. Only the following weak version can be proved: $(f,\mu)\mapsto\int f\,\mathrm{d}\mu$ for non-negative $f\in C[0;1]$ and $\mu\in\mathbf{M}^b$ is $(\delta_\to,\delta^<,\rho_<)$-computable. We can, however, include portions of information about upper bounds of $\mu[0;1]$ in the names. Let δ^b be the representation of $\mathbf{M}^b$ defined by the following notation v of atomic pieces of information: $\mu\in v(u\mathbin{\not c}v\mathbin{\not c}w)\Leftrightarrow\bar{u}<\mu(I_v)$ and $\mu[0;1]<\bar{w}$. Then the theorems we have proved for δ_m hold accordingly for δ^b, in particular Theorem 3.6 on integration. The connection to δ_m is given by the following lemma.

Lemma 4.9. *The function $\mu\mapsto\mu[0;1]$ on $\mathbf{M}^b$ is (δ^b,ρ)-computable, and the function $\mu\mapsto\mu/\mu[0;1]$ is (δ^b,δ_m)-computable for $\mu\in\mathbf{M}^b$, $\mu[0;1]\neq0$.*

5. Conclusion

In this paper, we have introduced and discussed a very natural and canonical computability theory on the set $\mathbf{M}$ of probability measures on the Borel subsets of the unit interval $[0;1]$. In particular, we have shown that simple obvious requirements exclude a number of similar definitions, that the definition leads to the expected computability results, that there are other natural definitions inducing the same computability theory and that the theory is embedded smoothly into classical measure theory. Although we have only stated the existence of computable functions throughout the paper, all the proofs provide algorithms, which can be realized by programs from some common programming language like PASCAL or C. Of course the basic definitions and many results can be transferred from the space $\mathbf{M}$ to more general spaces of measures.

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