Reversibility and potentiality of exclusion processes on countable discrete groups

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Abstract

Transition invariant exclusion processes on discrete groups, as examples of spatially extended systems, are discussed in this work. We show that there is reversibility if and only if the entropy production vanishes, or iff the speed function field has some bounded potential, which is expressed in terms of physical states rather than mathematical states in probability space. Moreover, we conclude that the only possible bounded potential is the constant one.

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1. Introduction

The question of whether a steady system is in equilibrium, or, correspondingly, whether a stationary stochastic process modelling the system is reversible, has attracted much interest from physicists as well as mathematicians. In fact, the theory of reversibility of Markov processes has been discussed for quite a long time, since Kolmogorov’s work. Many equivalent conditions for reversibility have been proved, such as the detailed balance, the vanishing of the entropy production and the existence of a potential. Here, we recommend reference to [2] for the systematic conclusions of this theory.

For example, suppose that \( X \) is an irreducible and positive-recurrent stationary Markov chain with continuous time, which has the finite state space \( S \), the transition density matrix \( Q = (q_{ij})_{i,j \in S} \) and the invariant probability distribution \( \Pi = (\pi_i)_{i \in S} \). Then the following statements are equivalent:

(i) The Markov chain \( X \) is reversible.
(ii) The Markov chain \( X \) is in detailed balance, that is, \( \pi_i q_{ij} = \pi_j q_{ji}, \forall i, j \in S \).
(iii) The entropy production rate \( e_p = \frac{1}{2} \sum_{i,j \in S} (\pi_i q_{ij} - \pi_j q_{ji}) \log \frac{\pi_i q_{ij}}{\pi_j q_{ji}} \) vanishes.
(iv) The transition density matrix \( Q \) of \( X \) satisfies the Kolmogorov cyclic condition:

\[
q_{i_1 i_2} q_{i_2 i_3} \cdots q_{i_{s-1} i_s} q_{i_s i_1} = q_{i_s i_1} q_{i_1 i_2} \cdots q_{i_{s-1} i_s} q_{i_s i_1},
\]

for any directed cycle \( c = (i_1, \ldots, i_s) \).

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Its field of speed functions has a bounded potential.

\[ \sum_{k=1}^{s-1} \log \frac{q(i_k, i_{k+1})}{q(i_{k+1}, i_k)} = \log \pi_{i_k} - \log \pi_{i_1}, \]

where the left side is regarded as the “force” in the sense of probability.

Therefore, we ask the same question for an interacting particle system. Interacting particle systems constitute a large and growing field of probability theory which is devoted to the rigorous analysis of certain types of models that arise in statistical physics, biology, economics, and other fields. Its concept of reversibility was put forward in [3,4], but only a few results about it were stated there. Chapter 11 of [1] deals with the reversibility of two important classes of particle systems, spin processes and exclusion processes where the relationships among reversibility, potentiality and Gibbs measures were discussed.

Recently, Zhang [5] defined the entropy production density of exclusion processes on countable discrete groups, and showed that it vanishes if and only if the process is reversible. This is the first case of expressing the equivalent conditions of reversibility in terms of the physical states \( \xi, \eta \) in probability space.

For example, consider the exclusion process \( \{\xi_t : t \geq 0\} \) on the discrete circle \( Z_N = \mathbb{Z}(\text{mod } N) \). Suppose the transition probability matrix on \( Z_N \) is transition invariant, i.e. \( p(x, y) = p(x + z, y + z) \) for any \( x, y, z \in Z_N \). Take the invariant measure as the product measure with density \( \alpha \) [3]. Then each state is a function on \( Z_N \) with image 0 or 1, the invariant measure is \( \pi_\xi = \alpha \sum_{x \in Z_N} \xi(x) (1 - \alpha)^{N - \sum_{x \in Z_N} \xi(x)} \), and the \( Q \)-matrix is

\[ q_{\xi,\eta} = \begin{cases} p(x, y), & \xi(x) = \eta(y) = 1, \xi(y) = \eta(x) = 0, \xi(z) = \eta(z), \forall z \neq x, y, \\ 0, & \text{otherwise}, \end{cases} \]

for any \( \eta \neq \xi \). The EPR (entropy production rate) of \( \{\xi_t\} \) is easily computed as

\[ epr = \sum_{\xi,\eta} \pi_\xi q_{\xi,\eta} \log \frac{\pi_\xi q_{\xi,\eta}}{\pi_\eta q_{\eta,\xi}} = N\alpha(1 - \alpha) \sum_x p(0, x) \log \frac{p(0, x)}{p(x, 0)}. \]

Then the site-average EPR, named the entropy production density, is defined as

\[ epd = \frac{epr}{N} = \alpha(1 - \alpha) \sum_x p(0, x) \log \frac{p(0, x)}{p(x, 0)} \]

\[ = \frac{1}{2} \alpha(1 - \alpha) \sum_x \frac{p(0, x) - p(x, 0)}{p(x, 0)} \log \frac{p(0, x)}{p(x, 0)}. \]

Each term in the summand is nonnegative, so the epd vanishes if and only if \( p(0, x) = p(x, 0) \), \( \forall x \), or iff \( \{\xi_t\} \) is reversible.

In this work, we continue Zhang’s work on considering the equivalence of reversibility and potentiality of exclusion processes on groups. The main result is proved in Section 2:

**Theorem 1.1.** \( \{\xi_t\} \) is a transition invariant exclusion process on groups. The following statements are equivalent:

1. \( \{\xi_t\} \) is reversible.
2. The generator \( A \) is symmetric, i.e. \( \int f A g d\mu = \int g A f d\mu \), where \( \mu \) is the invariant measure of \( X \).
3. It is in detailed balance: i.e. \( p(x, y) = p(y, x) \) for any \( x, y \).
4. The entropy production density vanishes.
5. Its field of speed functions has a bounded potential.
6. Its field of speed functions has a constant potential.

And finally, we apply Theorem 1.1 to amenable Cayley graphs in Section 3.

2. **Reversibility and potentiality of exclusion processes on groups**

First, we recall the definition of the exclusion process in the lattice gas interpretation.
Let $S$ be a countable set, $N \geq 1$ the maximum number of particles on each site, and $X = \{0, 1, \ldots, N\}^S$. For any $\xi \in X$, $\xi(x)$ denotes the number of particles on site $x$ for any $x \in S$. Let $\{p(x, y)\}$ be an irreducible transition probability matrix on $S$.

Each particle at $x$ waits for an exponential time with parameter one, then jumps to site $y$ with probability $p(x, y)$. The exclusion rule is that the jumping will be successful if $\xi(y) \leq N - 1$ at that time. Let $D$ be the set of all functions on $X$ taking values in $\{0, 1, 2, \ldots, N\}$ and each depending only on finitely many coordinates, and define

$$\xi^x(y)(z) = \begin{cases} 
\xi(x) - 1 & \text{if } z = x, \\
\xi(y) + 1 & \text{if } z = y, \\
\xi(z) & \text{otherwise},
\end{cases}$$

when $\xi(x) \geq 1$ and $\xi(y) \leq N - 1$.

An exclusion process $\{\xi(t) : t \geq 0\}$ on $S$ with transition rate $\{p(x, y)\}$ is a Feller process whose infinitesimal generator acts on $D$ as

$$Af(\xi) = \sum_{x, y \in S : \xi(x) \geq 1, \xi(y) \leq N - 1} p(x, y)(f(\xi^x(y)) - f(\xi)).$$

Suppose $S$ is a group, which we would like to denote by $G$. Suppose $\{p(x, y)\}$ is (left) translation invariant, i.e., $p(x, y) = p(zx, zy)$ for any $x, y, z \in G$. For any $\alpha > 0$, let $\alpha_k = \frac{e^{\alpha k}}{\sum_{k=0}^{N} e^{\alpha k}}$, $k = 0, 1, \ldots, N$, where $Z(\alpha) = \sum_{l=0}^{N} \alpha^l$ is a normalization constant. Let $\nu_{\alpha}$ be the product measure with marginal $\nu_{\alpha}(\{\xi(x) = k\}) = \alpha_k$, $k = 0, 1, \ldots, N$. A similar proof of Theorem VIII.2.1 in Ref. [3] shows that $\nu_{\alpha}$ is an invariant measure of $\{\xi(t)\}$.

Denote by $e$ the unit element of $G$, assume that $p(e, x) > 0 \iff p(x, e) > 0, \forall x \in G$.

To assure that the EPD (entropy production density) is finite, we also assume that

$$\sum_{x \in G} p(e, x) \left| \log \frac{p(e, x)}{p(x, e)} \right| < \infty. \quad (1)$$

Let $\{\xi(t) : t \geq 0\}$ be an exclusion process on $G$ with translation invariant transition rate $\{p(x, y)\}$ and initial measure $\nu_{\alpha}$.

The definition of reversibility comes from [3].

**Definition 2.1.** The probability measure $\mu$ on $X$ is said to be reversible for the process with semigroup $S(t)$ if it satisfies

$$\int f S(t) g d\mu = \int g S(t) f d\mu$$

for all $f, g \in C(X)$.

The definition of EPD comes from [5], which has already been explained in the introduction.

**Definition 2.2.** The EPD of $\{\xi_t\}$ is defined as

$$\text{epd} := \frac{1}{2} (1 - \alpha_0)(1 - \alpha_N) \sup_{B \subset G, |B| \leq \infty} \left\{ \sum_{x \in G} \frac{|B \cap Bx^{-1}|}{|B|} (p(e, x) - p(x, e)) \log \frac{p(e, x)}{p(x, e)} \right\},$$

where $Bx^{-1} = \{yx^{-1} : y \in B\}$.

**Remark 2.3.** As in [5], if $G = \mathbb{Z}^d$, take $B_n = \{x \in \mathbb{Z}^d : |x| \leq n\}$; then one can directly define that $\text{epd} = \lim_{n \to \infty} \frac{e^{\text{epd}_n}}{|B_n|} = \frac{1}{2} (1 - \alpha_0)(1 - \alpha_N) \sum_{x \in \mathbb{Z}^d} (p(e, x) - p(x, e)) \log \frac{p(e, x)}{p(x, e)}$.

**Definition 2.4.** Given the speed functions $\{p(x, y)\}$, if there exists a function $\{V(x) : x \in G\}$ so that

$$\sum_{i=0}^{n-1} \log \frac{p(x_i, x_{i+1})}{p(x_{i+1}, x_i)} = V(x_n) - V(x_0), \quad \forall x_0, x_1, \ldots, x_n \in G \quad (2)$$

then we say that the field of speed functions $\{p(x, y)\}$ has a potential $\{V(\cdot)\}$.
Here, we refer the reader to [1][Chapter 7, 11] for the general field theory and potential theory.

**Proof of Theorem 1.1.** (1) $\Leftrightarrow$ (2): See proposition II.5.3 in [3].

(2) $\Leftrightarrow$ (3): From Theorem 1.1 in [5].

(3) $\Leftrightarrow$ (4): Since each term in the expression for the epd is nonnegative, and $[p(e, x) - p(x, e)] \log \frac{p(e, x)}{p(x, e)} = 0$ if and only if $p(e, x) = p(x, e)$.

(4) $\Rightarrow$ (5): We can take $V(x) = \text{constant}$ in Definition 2.4, which is bounded.

(5) $\Rightarrow$ (6): This is the heart of the proof. It means that if the bounded potential exists, then it must be the constant one.

For instance, we first consider the case of $Z^2$. As we all know, $Z^2$ is an Abelian (commutative) group with finite generator $x$ and $y$, and every site of $Z^2$ can be expressed as $x^k y^h$, for some integer $k$ and $h$.

When $k$ and $h$ are positive, take $x_0 = e$, $x_i = x^i$, $i = 1, 2, \ldots, k$, and $x_{j+k} = x^k y^j$, $j = 1, 2, \ldots, h$, $n = k + h$ in (2); one can get that

$$\sum_{i=0}^{n-1} \log \frac{p(x_i, x_{i+1})}{p(x_{i+1}, x_i)} = \sum_{i=0}^{k-1} \log \frac{p(x^i, x^{i+1})}{p(x^{i+1}, x^i)} + \sum_{j=0}^{h-1} \log \frac{p(x^k y^j, x^{k+1} y^j)}{p(x^{k+1} y^j, x^k y^j)} = V(x^k y^h) - V(e).$$

Moreover, by the transition invariance,

$$p(x^i, x^{i+1}) = p((x^i)^{-1} x^i, (x^i)^{-1} x^{i+1}) = p(e, x),$$
$$p(x^{i+1}, x^i) = p((x^i)^{-1} x^{i+1}, (x^i)^{-1} x^i) = p(x, e),$$

and

$$p(x^k y^j, x^{k+1} y^j) = p((x^k y^j)^{-1} x^k y^j, (x^k y^j)^{-1} x^{k+1} y^j) = p(e, y),$$
$$p(x^{k+1} y^j, x^k y^j) = p((x^k y^j)^{-1} x^{k+1} y^j, (x^k y^j)^{-1} x^k y^j) = p(x, y),$$

so

$$V(x^k y^h) - V(e) = k \log \frac{p(e, x)}{p(x, e)} + h \log \frac{p(e, y)}{p(y, e)}.$$

When $h \equiv 0$, and if $p(e, x) > p(x, e)$, then $\lim_{k \to \infty} V(x^k) = +\infty$, which contradicts the boundedness of potential function $V()$. On the other hand, if $p(e, x) < p(x, e)$, then $\lim_{k \to \infty} V(x^k) = -\infty$, which is another contradiction.

Therefore, $p(e, x) = p(x, e)$. Following the same steps as above when $k \equiv 0$, one can get that $p(e, y) = p(y, e)$, too.

The case of $Z^d$ is just the same as that for $Z^2$, $d \geq 3$.

Now, imitate the proof above; we can prove that (5) $\Rightarrow$ (6) when $G$ is an arbitrary countable discrete group.

(6) $\Rightarrow$ (3): For each $n$, $\sum_{i=0}^{n-1} \log \frac{p(x_i, x_{i+1})}{p(x_{i+1}, x_i)}$ always vanishes. So for any $x \in G$, take $n = 1$; then $\log \frac{p(e, x)}{p(x, e)} = V(x) - V(e) = 0$, i.e. $p(e, x) = p(x, e)$. \qed

**Remark 2.5.** Following the same steps as in the proof, similar results also hold in the multi-colored exclusion processes whose reversibility and EPD have already been discussed in [5].

3. **Reversibility and potentiality of exclusion processes on amenable Cayley graphs**

Suppose that $G$ is a finitely generated group and $H$ is a finite set of generators. Without loss of generality, suppose that $e \notin H$, and $x \in H \iff x^{-1} \in H, \forall x \in G$.

Define the set of edges as $E = \{ (x, xh) : x \in G, h \in H \}$. Then $\{G\}$ and $E$ compose a graph. Such graphs induced from finitely generated groups are called Cayley graphs, denoted by $(G, E)$.

Furthermore, we assume that $(G, E)$ is amenable, which means

$$\inf \{|\partial B|/|B| : B \subset G, |B| < \infty\} = 0,$$
where $\partial B = \{ x \in B : N(x) \cap B^c \neq \emptyset \}$, and $N(x) = \{ y \in G, (x, y) \in E \}$ is the set of neighbors of $x$, for any $x \in G$. In [5], Zhang has proved that its entropy production density can be expressed as

$$\text{epd} = \frac{1}{2} (1 - \alpha_0) (1 - \alpha_N) \sum_{x \in G} \left( p(e, x) - p(x, e) \right) \log \frac{p(e, x)}{p(x, e)}.$$  \hspace{1cm} (3)

Applying Theorem 1.1, one can get another theorem that

**Theorem 3.1.** \{ξ_t\} is a transition invariant exclusion process on the amenable Cayley graph $(G, E)$. The following statements are equivalent:

1. \{ξ_t\} is reversible.
2. The generator $\mathcal{A}$ is symmetric, i.e. $\int f \mathcal{A} g \, d\mu = \int g \mathcal{A} f \, d\mu$, where $\mu$ is the invariant measure of $X$.
3. It is in detailed balance: i.e. $p(x, y) = p(y, x)$ for any $x, y \in G$.
4. The entropy production density (3) vanishes.
5. Its field of speed functions $(p(x, y))$ has a bounded potential.
6. Its field of speed functions has a constant potential.

**Remark 3.2.** Applying Theorem 1.1 to a Cayley graph relies on the associating group since a Cayley graph may be induced from different groups. The key point is under which group $(p(x, y))$ is translation invariant.

**References**