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RAMSEY'S THEOREM AND SELF-COMPLEMENTARY GRAPHS

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Abstract. It is proved that, given any positive integer k, there exists a self-complementary graph with more than $4 \cdot 2^{\frac{1}{4}k}$ vertices which contains no complete subgraph with $k+1$ vertices. An application of this result to coding theory is mentioned.

A graph will be called *s-good* if it contains neither a complete subgraph with more than s vertices nor an independent set of more than s vertices. A special case of the celebrated Ramsey's theorem [7] asserts that given any positive integer s there is an $n = n(s)$ such that no graph with more than $n(s)$ vertices is s-good. Apart from the trivial $n(1) = 1$, only two exact values of $n(s)$ are known [4]; these are $n(2) = 5$ and $n(3) = 17$. Clearly, a graph G is s-good if and only if its complement $\bar{G}$ is s-good. It does not seem unlikely that for any s, there is an s-good self-complementary graph with $n(s)$ vertices. This is true at least for $s = 2$ and $s = 3$ (and in this case, the s-good graphs with $n(s)$ vertices are unique [6]). However, it seems quite difficult to prove this conjecture for all s. We shall denote by $n^*(s)$ the greatest integer n^* such that there is a self-complementary s-good graph with n^* vertices, trivially, $n^*(s) \leq n(s)$.

Theorem. $n^*(st) \geq (n^*(s) - 1)n(t)$.

Proof. Let $G_0 = (V_0, E_0)$ be an s-good self-complementary graph with

$n^*(s)$ vertices, let $f_0 : V_0 \to V_0$ be an isomorphism between G and $\overline{G}$. It is easy to see that the permutation f_0 has at most one fixed point and no odd cycles of length $\geqslant 3$. Therefore there is an s-good self-complementary graph $G_1 = (V_1, E_1)$ with $n^*(s)$ or $n^*(s)-1$ vertices and a permutation $f: V_1 \to V_1$ setting up an isomorphism between G_1 and $\overline{G}_1$ such that f has cycles of even length only (and no fixed points). Consequently, V_1 can be split into disjoint sets X and Y with $f(X) = Y$, $f(Y) = X$.

Let $G_2 = (V_2, E_2)$ be a t-good graph with $n(t)$ vertices. We shall consider the graph $G = (V_1 \times V_2, E)$ where $\{(u, v), (w, z)\}$ belongs to E if and only if either $\{u, w\} \in E_1$ or $u = w \in X$, $\{v, z\} \in E_2$ or finally $u = w \in Y$, $\{v, z\} \notin E_2$. G is self-complementary; indeed, the mapping $F: V_1 \times V_2 \to V_1 \times V_2$ defined by $F(u, v) = (f(u), v)$ is an isomorphism between G and $\overline{G}$.

If $Z \subset V_1 \times V_2$ spans a complete subgraph in G then at most s vertices in Z have distinct first coordinates (otherwise G_1 would not be s-good) and at most t vertices in Z have the same first coordinate (otherwise G_2 would not be t-good). Therefore $|Z| \leqslant st$ and G, being self-complementary, is st-good. Hence $n^*(st) \geqslant |V_1 \times V_2| \geqslant (n^*(s)-1)n(t)$ and the proof is finished.

Corollary. $n^*(2t) \geqslant 4n(t)$.

Our original interest in this area was stimulated by the notion of the *capacity* of a graph as defined by Shannon [9]. One defines the *product* $G_1 \times G_2 \times \ldots \times G_k$ of graphs $G_i = (V_i, E_i)$, $i = 1, 2, \ldots, k$, as the graph $G = (V_1 \times V_2 \times \ldots \times V_k, E)$ where two distinct vertices $(u_1, u_2, \ldots, u_k)$, $(v_1, v_2, \ldots, v_k)$ of G are adjacent if and only if, for each $i = 1, 2, \ldots, k$, either $\{u_i, v_i\} \in E_i$ or else $u_i = v_i$. We denote the largest cardinality of an independent set in G by $\mu(G)$; evidently,

$$(1) \qquad \mu(G_1 \times G_2 \times \ldots \times G_k) \geqslant \mu(G_1)\,\mu(G_2) \ldots \mu(G_k)\,.$$

Considering noisy channels in information theory, Shannon [9] was led to the definition of the capacity $\theta(G)$ of a graph G,

$$\theta(G) = \sup_k (\mu(G^k))^{1/k}\,.$$

Obviously, $\theta(G) \geqslant \mu(G)$. However, one can have $\theta(G) > \mu(G)$; for instance, if G is the pentagon then $\mu(G) = 2$, $\mu(G^2) = 5$.

It can be shown that $\mu(G_1) = \mu(G_2) = k$ implies $\mu(G_1 \times G_2) \leqslant n(k)$ and this bound is best possible. Moreover, this inequality generalizes into the case of more graphs G_i with $\mu(G_i)$ not necessarily equal. Apparently Hedrlín [5] was the first to discover this relation between Ramsey numbers and the capacity problems. However, Hedrlín did not publish his result. Unaware of his contribution, Erdös, McEliece and Taylor [3] recently published an independent derivation of the equivalence.

If $G = (V, E)$ is a self-complementary graph with m vertices then $\mu(G^2) \geqslant m$. Indeed, if f is an isomorphism between G and $\bar{G}$ then the set $\{(u, f(u)) \mid u \in V\}$ is independent in $G^2 = G \times \bar{G}$. Hence $\mu(G^2) \geqslant m$. Consequently, one has

$$(2) \qquad \theta(G) \geqslant m^{\frac{1}{2}}$$

for any self-complementary graph G with m vertices. Rosenfeld [8] proved that given any k there is a graph G_k with $\theta(G_k) > k\,\mu(G_k)$. This proof is based on the inequality

$$(3) \qquad n^*(k) > ck^{\alpha}$$

where $\alpha = \log 5/\log 2$ and c is an absolute positive constant. Rosenfeld's proof of (3) is constructive and has been discovered independently by Abbott [1]. Our Corollary together with the probabilistic lower bound [2]

$$(4) \qquad n(k) > 2^{\frac{1}{2}(k+1)}\,, \qquad k \geqslant 2\,,$$

yields

$$n^*(k) > 4 \cdot 2^{\frac{1}{4}k}$$

which is better than (3). Rosenfeld's theorem also follows directly from (4) and [3, Theorem 3] which asserts the existence, for any k, of a graph G (with $2n(k)$ vertices) such that $\mu(G) = k$, $\mu(G^2) = n(k)$.

References

[1] H.L. Abbott, A note on Ramsey numbers, Discrete Math., to appear.
[2] P. Erdös, Some remarks on the theory of graphs, Bull. Amer. Math. Soc. 53 (1947) 292–294.
[3] P. Erdös, R.J. McEliece and H. Taylor, Ramsey bounds for graph products, Pacific J. Math. 37 (1971) 45–46.
[4] R.E. Greenwood and A.M. Gleason, Combinatorial relations and chromatic graphs, Canad. J. Math. 7 (1955) 1–7.
[5] Z. Hedrlín, Ramsey's theorem and information theory, Charles University, Prague, 1964.
[6] J.G. Kalbfleisch, A uniqueness theorem for edge-chromatic graphs, Pacific J. Math. 21 (1967) 503–509.
[7] F.P. Ramsey, On a problem of formal logic, Proc. London Math. Soc. 30 (1930) 264–286.
[8] M. Rosenfeld, Graphs with a large capacity, Proc. Amer. Math. Soc. 26 (1970) 57–59.
[9] C.E. Shannon, The zero error capacity of a noisy channel, IRE Trans. Inform. Theory IT-2 (1956) 8–9.