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An Elucidation of "Infinite-Dimensional Algebras ... and the Very Strange Formula." $E_8^{(1)}$ and the Cube Root of the Modular Invariant j

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This is an addendum to my paper "Infinite-Dimensional Algebras, Dedekind's η-Function, Classical Möbius Function and the Very Strange Formula," published in *Advances in Math.* **30** (1978), 85–136, which is referred to here as [SF].

Recently McKay noticed that one of the coefficients in the q-series of the modular invariant

$$j(q) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots.$$

is the dimension of the sum of lowest nontrivial and one-dimensional representations of the Fisher–Griess Monster group, and Thomson found that the later coefficients are also dimensions of some representation of this group (see [25]). McKay also discovered that the same phenomenon takes place if one replaces the Monster by the Lie algebra E^8 and $j(q)$ by $(qj(q))^{1/3} = 1 + 248q + \cdots$ [29]. In this addendum I want to make some comments on [SF] and, in particular, to give an explanation of McKay's E_8-observation in the framework of infinite-dimensional Lie algebras. Though I do not know how to explain recent "Monstrous" discoveries [25], they have inspired some new η-function identities (Proposition 1) and multiplicity formulas (Proposition 2). The theory of infinite-dimensional Lie algebras suggests in turn some natural conjectures about the Monster.

1. Let L be a complex finite-dimensional simple Lie algebra of rank n and let $\mathscr{L}$ be the connected simply connected group with the Lie algebra L. Let $\langle\, ,\rangle$ denote the Killing form and let $A = (a_{ij})$ be the Cartan matrix of L. Let $L = N_- \oplus H \oplus N_+$ be a decomposition of L into a sum of subalgebras, where N_- and N_+ are maximal nilpotent subalgebras and H is the Cartan subalgebra. Let R be the set of all nonzero roots of L with respect to H. Let $\alpha_1, \ldots, \alpha_n$ be the system of simple roots, let Q be the lattice generated by them, and let $\theta = \sum a_i\alpha_i$ be the highest root; we write $\theta^\vee = \sum a_i^\vee \alpha_i^\vee$ for the dual root system. Let $h_1, \ldots, h_n$ be a basis of H such that $\alpha_i(h_j) = a_{ji}$ and let $t_1, \ldots, t_n$ be another basis, such that $\alpha_i(t_j) = \delta_{ij}$. Let W be the Weyl group of L.

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The infinite-dimensional Lie algebra $L^{(1)}$ of [SF] is the complex space $L^{(1)} = (\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} L) \oplus \mathbb{C}c$ with the following bracket:

$$[g_1 \oplus \mu_1 c, g_2 \oplus \mu_2 c] = [g_1, g_2] \oplus \frac{\langle\theta, \theta\rangle}{2} \operatorname{Res}\left\langle \frac{dg_1}{dt}, g_2 \right\rangle c,$$

where $g_1, g_2 \in \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} L$ and $\mu_1, \mu_2 \in \mathbb{C}$.

It is convenient to enlarge the algebra $L^{(1)}$ by adding a derivation $t(d/dt)$ which operates on $\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} L$ in an obvious way and sends c to 0. We denote the obtained Lie algebra by $\hat{L}$ and identify L with the subalgebra $1 \otimes_{\mathbb{C}} L$ of $\hat{L}$. We set $h_0 = c - \sum_{i=1}^{n} a_i^{\vee} h_i$ and $\hat{N} = t\mathbb{C}[t] \otimes_{\mathbb{C}} (N_- \oplus H) \oplus \mathbb{C}[t] \otimes_{\mathbb{C}} N_+$.

2. Let $\Lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ be an $(n + 1)$-tuple of nonnegative integers. Then there exists a unique irreducible $\hat{L}$-module $V(\Lambda)$ (cf. [SF]) for which there is a nonzero vector v_Λ (highest weight vector) such that

$$\hat{N}(v_\Lambda) = 0, \qquad t\frac{d}{dt}(v_\Lambda) = 0, \qquad h_i(v_\Lambda) = \lambda_i v_\Lambda, \qquad i = 0, 1, \ldots, n.$$

Setting $V_k := \{v \in V(\Lambda) \mid t(d/dt)(v) = -kv\}$ gives a $\mathbb{Z}_+$-gradation by finite-dimensional subspaces: $V(\Lambda) = \bigoplus_{k \in \mathbb{Z}_+} V_k$. Note that this gradation is L-invariant. One can show that the representation of L in the space $V(\Lambda)$ can be exponentiated to a representation of the semidirect product $\hat{\mathscr{L}}$ of the group $\mathscr{L}^{(1)}$, which is a central extension of $\mathscr{L}(\mathbb{C}((t)))$ by $\mathbb{C}^*$, and the group $\mathbb{C}^*$ corresponding to $-t(d/dt)$ (and even to the semidirect product of $\mathscr{L}^{(1)}$ and $\operatorname{Aut} \mathbb{C}((t))$).

Slightly changing the definition of ch $V(\Lambda)$ in [SF] (by a constant factor) we set

$$\operatorname{ch} V(\Lambda) := \sum_{k \geqslant 0} (\operatorname{ch} V_k)\, q^k,$$

where ch V_k is the "formal" character of the L-module V_k. Less formally ch $V(\Lambda)$ may be viewed as a function on the complex Lie group $\mathscr{L}(\mathbb{C})$ with the values in q-series:

$$(\operatorname{ch} V(\Lambda)(\sigma) := \sum_{k \geqslant 0} (\operatorname{tr} \sigma)|_{V_k}\, q^k, \qquad \sigma \in \mathscr{L}.$$

If $(e^{2\pi i\tau}, \sigma)$ is an element of the subgroup $\mathbb{C}^* \times \mathscr{L}(\mathbb{C})$ of the group $\hat{\mathscr{L}}$, then

$$\operatorname{ch} V(\Lambda)((e^{2\pi i\tau}, \sigma)) := \sum_{k \geqslant 0} (\operatorname{tr} \sigma)_{V_k}\, e^{2\pi i k\tau} = \operatorname{tr}(e^{2\pi i\tau}, \sigma)|_{V(\Lambda)}$$

is a convergent series on $\mathscr{H} = \{\tau \mid \operatorname{Im} \tau > 0\}$. (I am grateful to I. Frenkel for this remark.) This follows from the fact that we can rewrite formula (2.6) of [SF] as follows (we use Proposition 3.4(c)):

ch $V(\Lambda)((e^{2\pi i\tau}, e^{2\pi ih}))$

$$= \frac{\sum_{\gamma\in g_{\Lambda+\rho}Q^\vee} \mathrm{ch}(2\pi i(\bar{\Lambda}+\gamma)(h)) \exp \pi i g_{\Lambda+\rho}^{-1}(\|\gamma+\bar{\Lambda}+\bar{\rho}\|^2 - \|\bar{\Lambda}+\bar{\rho}\|^2)\tau}{\prod_{k\geqslant 1}((1-e^{2\pi ik\tau})^n \prod_{\alpha\in R}(1-e^{2\pi i(k\tau+\alpha(h))}))}. \tag{1}$$

Here $h \in H$, $\bar{\Lambda} \in H^*$ is such that $\bar{\Lambda}(h_i) = \lambda_i$, $i = 1,..., n$, $\bar{\rho}$ is the half-sum of positive roots of L, $g_{\Lambda+\rho} = \frac{1}{2}(\lambda_0+1)\|\theta\|^2 + (\bar{\Lambda}+\bar{\rho}, \theta)$, $Q^\vee$ is the dual root lattice $\mathrm{ch}(2\pi i\beta(h)) = \sum_{w\in W}(\det w)\, e^{2\pi iw(\beta+\rho)(h)}/\sum_{w\in W}(\det w)\, e^{2\pi iw(\rho)(h)}$.

Formula (1) has sense only for a regular $h \in H$. However, it follows from (4) below that $\mathrm{ch} V(\Lambda)$ is an analytic function on $\mathscr{H} \times \mathscr{L}(\mathbb{C})$.

3. Among the nontrivial $\hat{L}$-modules $V(\Lambda)$ there is a canonically defined "simplest" module $V(\Lambda_0)$, corresponding to $\Lambda_0 = (1, 0,..., 0)$. This module is defined by the property that there exists a nonzero vector $v \in V(\Lambda_0)$ such that

$$\left(\mathbb{C}t\frac{d}{dt} \oplus (\mathbb{C}[t] \otimes_{\mathbb{C}} L)\right)(v) = 0 \qquad \text{and} \qquad c(v) = v.$$

Now let L be a simple Lie algebra of one of the types A_n, D_n, E_n. This is the case when the Cartan matrix A is symmetric; then $\|\theta\|^{-1} = h$ is the Coxeter number, $a_i^\vee = a_i$, and $\|\alpha\|^{-1} = h$ for $\alpha \in Q$ if and only if α is a root. In this case formula (3.37) of [SF] gives a very simple expression for ch $V(\Lambda_0)$. Set $q = e^{2\pi i\tau}$ and $\varphi(q) = \prod_{k\geqslant 1}(1-q^k)$ so that $\eta(\tau) = q^{1/24}\varphi(q)$ is the Dedekind η-function. Set $(\gamma_1, \gamma_2) = 2h\langle\gamma_1, \gamma_2\rangle$, $|\gamma|^2 = (\gamma, \gamma)$. Then for $l = \exp 2\pi i z \in \mathscr{L}(\mathbb{C})$ one has

$$\mathrm{ch}\, V(\Lambda_0)((q, l)) = \varphi(q)^{-n} \sum_{\gamma\in Q} q^{|\gamma|^2/2} e^{2\pi i\gamma(z)}. \tag{2}$$

Setting $z = 0$ in (2) we obtain the following formula (cf. the formula following (3.38) in [SF]):

$$\sum_{k\geqslant 0} (\dim V_k) q^k = \frac{\theta_Q(q)}{\varphi(q)^n}, \tag{3}$$

where $\theta_Q(q)$ is the θ-series of the lattice Q. In the case $L = E_8$ we obtain [30]

$$\sum_{k\geqslant 0} (\dim V_k) q^k = (q j(q))^{1/3}.$$

This follows from the usual formula for $j(q)$ (see, e.g., [26]). The latter formula explains McKay's E_8-observation.

Note that the numerator of (2) is a Θ-function. It is easy to see from (1) and Propositions 3.8(d) and 3.4(c) that in the general case one has

$$q^{-r}\, \mathrm{ch}\, V(\Lambda)((q, l)) = \sum_{\mu, w} a_\mu(\tau)\Theta_{\bar{\mu},w}(\tau, z). \tag{4}$$

Here μ ranges over maximal weights (see [SF] for definition), w ranges over $W/W_{\bar{\mu}}$, and $r = (\dim L)/24 - \| \bar{\Lambda} + \bar{\rho} \|^2/2g_{\Lambda+\rho}$. The functions $\Theta_{\bar{\mu},w}(\tau, z)$ are theta functions of the form

$$\Theta_{\bar{\mu},w}(\tau, z) = \sum_{\gamma - w(\bar{\mu}) \in g_\Lambda Q^\vee} \exp(\pi i g_\Lambda^{-1} \| \gamma \|^2 \tau + 2\pi i \gamma(z)).$$

The function $a_\mu(\tau)$ is of the form

$$a_\mu(\tau) = q^{-s} \sum_{k \geqslant 0} (\operatorname{mult} \mu - k\delta) q^k,$$

where s is a rational number (depending on μ). Moreover, $a_\mu(\tau)$ is a modular function of weight $-n/2$. Indeed, it follows from (1) and the denominator identity that $q^{-r} \operatorname{ch} V(\Lambda)((q, \sigma))$ is a modular function of weight 0 for any $\hat{L}$-module $V(\Lambda)$ and any element of finite order $\sigma \in \mathscr{L}$. From this and (4) it is easy to deduce now the latter statement. Moreover, $a_\mu(\tau)\eta(\tau)^{\dim L}$ is a cusp-form [31].

Modular functions also appear in the following situation. The $\mathbb{Z}$-gradation of type $s = (s_0, ..., s_n)$ of $\hat{L}$ (see [SF] for definition) induces a $\mathbb{Z}_+$-gradation $V(\Lambda) = \bigoplus_k V_k(s)$ (the $\mathbb{Z}_+$-gradation introduced in Section 1 corresponds to $s = (1,0,...,0)$). It follows from (4) that $q^{-r_1} \sum_{k \geqslant 0} \dim V_k(s) q^k$ is a modular function of weight 0 for $r_1 = r - 1/2g_\Lambda \bar{s}^t \bar{s}$, where $\bar{s} = (s_1, ..., s_n)$.

For an n-tuple of integers $s = (s_1, ..., s_n)$ let $\sigma = \exp(2\pi i/m) \sum_r s_r t_r$ be an element of period m of the group $\mathscr{L}$. Then setting $z = \sum s_r t_r$ in (2), we obtain the formula

$$e^{-\pi i n \tau/12} \sum_{k \geqslant 0} (\operatorname{tr} \sigma)|_{V_k} e^{2\pi i k \tau} = \sum_{\gamma \in Q} \epsilon^{\gamma(z)} e^{\pi i |\gamma|^2 \tau}, \tag{5}$$

where $\epsilon = \exp(2\pi i/m)$. The right-hand side of (5) is a modular function of weight 0. This is analogous to the experimental discoveries concerning the Monster group F_1 in [25].

In more detail, it is suggested in [25] that there is a sequence of F_1-modules V_k such that $q^{-1} \sum_{k \geqslant 0} (\operatorname{ch} V_k)(\sigma) q^k$ is the q-series of a modular function of weight 0 for any $\sigma \in F_1$. Moreover, these functions are Hauptmoduls.

In this connection, I would like to suggest the following conjecture. Suppose that n divides 24, say $24 = ns$. Denote by $f_\sigma(\tau)$ the right-hand side of (5) where q is replaced by q^s. Then $f_\sigma(\tau)$ is a Hauptmodul if and only if σ is a rational element (i.e., in the adjoint representation the characteristic polynomial of σ has rational coefficients, cf. [SF]).

Finally, in many cases, the right-hand side of (5) can be represented in the form $\prod_s \eta(q^s)^{n_s}$. One can show that in such a case one has (cf. [SF])

$$n_s = 0 \qquad \text{if} \quad s \mid m.$$

One can check that for the products in [25] this also holds.

These remarks suggest that the Monster group F_1 may be a subgroup of a "canonically" defined infinite group $\hat{F}_1$. The group $\hat{F}_1$ would have a "canonical" irreducible module V with an F_1-invariant $\mathbb{Z}_+$-gradation $V = \bigoplus V_k$ such that $q^{-1} \sum (\operatorname{ch} V_k) q^k$ is the Thompson series [25] for F_1(in particular, $q^{-1} \sum (\dim V_k) q^k = j(q) - 744$). Of course, one can ask an analogous question for any finite simple group.

4. The following relation of the Leech lattice Λ with the Monster is conjectured in [25]. Let σ be an automorphism of Λ, let Λ^σ denote the sublattice of Λ fixed by σ and $\theta_\sigma(\tau)$ be its θ-function. The characteristic polynomial of σ can be written in the following way: $\det(1 - \lambda\sigma) = \prod_s (1 - \lambda^s)^{k_s}$, where k_s are integers only a finite number of which are different from 0. Set $\eta_\sigma(\tau) = \prod_s \eta(s\tau)^{k_s}$. It is conjectured in [25] that there is always an element $\tilde{\sigma}$ in the Monster for which the Thompson series has a form $\theta_\sigma(\tau)/\eta_\sigma(\tau)$.

We will show that if one takes the root lattice Q of a simple Lie algebra L of type A_n, D_n, or E_n instead of the Leech lattice, then for any element σ of order m of the Weyl group there exists a rational element $\tilde{\sigma}$ of period $2m$ in the Lie group $\mathscr{L}$ such that an analogous statement takes place. This will give some new beautiful η-function identities.

So we see that the Leech lattice and the Conway group $\cdot 0$ play the same role for the Monster as the root lattice and the Weyl group do for a simple Lie group.

We return to the simple Lie algebra L. Let α be a positive root of L. We choose the root vectors e_α and $e_{-\alpha}$ in such a way that $\alpha([e_\alpha, e_{-\alpha}]) = 2$ and set (cf. Lemma 1.2 of [SF])

$$r_\alpha = (\exp e_\alpha)(\exp(-e_{-\alpha}))(\exp e_\alpha) \in \mathscr{L}.$$

Then $\tilde{r}_\alpha$ lies in the normalizer of H in $\mathscr{L}$ and its image in the Weyl group W is the reflection r_α relative to the root α. The elements $\tilde{r}_\alpha$, $\alpha \in R$, generate in $\mathscr{L}$ a finite subgroup $\tilde{W}$. The group $\tilde{W}$ is an extension of W by the group of elements of order 2 of $\mathscr{H}$. Now with an element $\sigma \in W$ of order m we associate a well-defined element $\tilde{\sigma} \in \tilde{W}$ of period $2m$ as follows. Let $\sigma = r_{\gamma_1} \cdots r_{\gamma_s}$ be a shortest expression of σ in terms of reflections. We set $\tilde{\sigma} = \tilde{r}_{\gamma_1} \cdots \tilde{r}_{\gamma_s}$; it is easy to see that $\tilde{\sigma}$ does not depend on the choice of the expression of σ.

PROPOSITION 1. *Let L be a simple Lie algebra of type A_n, D_n, or E_n, H be the Cartan subalgebra, and W be the Weyl group. Let σ be an element of order m in W and Q^σ denote the sublattice of Q fixed by σ. The characteristic polynomial of σ on H^* can be written in the form* $\det(1 - \lambda\sigma) = \prod_s (1 - \lambda^s)^{k_s}$. *Let $\tilde{\sigma}$ be an element of $\mathscr{L}$ of period $2m$ associated with σ; $\tilde{\sigma}$ is conjugate in $\mathscr{L}$ to an element* $\exp(\pi i/m)\, z$, *where $z = \sum_s \mu_s t_s$, μ_s being integers.*

Then the following identity holds:

$$\frac{\sum_{\gamma\in Q}\exp((\pi i/m)\gamma(z))\,q^{|\gamma|^2/2}}{\varphi(q)^n}=\frac{\sum_{\gamma\in Q^\sigma}q^{|\gamma|^2/2}}{\prod_s\varphi(q^s)^{k_s}}. \tag{6}$$

Proof. By formula (5) the left-hand side of (6) is equal to

$$\sum_{k\geqslant 0}(\operatorname{tr}\tilde\sigma)|_{V_k}\,q^k. \tag{7}$$

We will compute (7) directly and show that it is equal to the right-hand side of (6).

We consider the following subalgebras in $\hat{L}$: $P_-=\bigoplus_{k<0}(t^k\otimes H)$, $P_+=\bigoplus_{k>0}(t^k\otimes H)$, $P=P_-\oplus\mathbb{C}c\oplus P_+$. The algebra P is isomorphic to the infinite-dimensional Heisenberg Lie algebra, P_- and P_+ being maximal commutative subalgebras in it. Let $U=\{v\in V(\Lambda_0)\mid P_+(v)=0\}$ be the subspace of "vacuum" vectors. Then clearly $U=\bigoplus_k(U\cap V_k)$.

Let $V_k=\bigoplus_\lambda V_k{}^\lambda$ be the weight decomposition with respect to H. Then it follows from [SF] that $U\cap V_k$ is a sum of those V_k for which $\lambda\in Q$ and $|\lambda|^2/2=k$, and that all these $V_k{}^\lambda$ are one dimensional: $V_k{}^\lambda=\mathbb{C}v_k{}^\lambda$. Let $U_k{}^\lambda$ be the P-module for which $v_k{}^\lambda$ is a cyclic vector; set $U_{k,s}^\lambda=U_k{}^\lambda\cap V_s$. Then $U_k{}^\lambda$ is a free $U(P_-)$-module and the P-module $V(\Lambda_0)$ is a direct sum of these modules.

Note now that $\tilde\sigma(V_k{}^\lambda)=V_k^{\sigma(\lambda)}$ and that $\tilde\sigma(v_k{}^\lambda)=v_k{}^\lambda$ if $\sigma(\lambda)=\lambda$. Therefore

(a) only those λ give a contribution to the $\operatorname{tr}\tilde\sigma\,|_{V_k}$ for which $\sigma(\lambda)=\lambda$ and

(b) provided that $\sigma(\lambda)=\lambda$, one has (by the Molien formula):

$$\sum_{s\geqslant 0}(\operatorname{tr}\tilde\sigma)|_{U_{k,s}^\lambda}\,q^s=q^k\Big/\Big(\prod_{s\geqslant 1}\det(1-q^s\sigma)|_H\Big).$$

This completes the proof of the proposition.

It is natural to suppose that identities similar to (6) hold also for the Leech lattice Λ: for each σ of order m in the Conway group $\cdot 0$ there exists a 24-tuple of integers μ such that (6) holds (here A will be the matrix of the lattice Λ and $n=24$). Probably there exists also a "canonical form" of an element of the Monster similar to one described by Proposition 3.5 in [SF]. Or in other words, there exists a natural bijection between conjugacy classes of cyclic subgroups of order m in the Monster and a set of epimorphisms of Λ on the group of the mth roots of unity, defined up to the action of the Conway group $\cdot 0$.

Finally, consider a special case of the proposition when σ is the Coxeter element. Then

(a) $\tilde\sigma$ is the "principal" element [3] and therefore $\tilde\sigma$ is conjugate to the element $\exp(2\pi i/h)\sum_s t_s$;

(b) $Q^\sigma=0$;

(c) k_s's can be computed from Proposition 3.7(f)(ii) and (3.31) of [SF].

As a result we obtain the following identity$_s$

$$\prod_{k|h} \varphi(q^k)^{n_k} = \sum_{\gamma \in Q} e^{(2\pi i/h)\mathcal{O}(\gamma)} q^{|\gamma|^2/2}. \tag{8}$$

Here $\mathcal{O}$ is the height: $\mathcal{O}(\sum k_i \alpha_i) = \sum k_i$ and n_k's can be computed in terms of the exponents $m_1, ..., m_n$ of the Lie algebra L as follows. Set $d_s = n$ if $s \not\equiv m_t \bmod h$ and $d_s = n +$ (multiplicity m_t) otherwise; then $n_k = \sum_{s|k} \mu(k/s) d_s$.

Note that for $L = A_1$ formula (8) is the Gauss identity: $\varphi(q)^2/\varphi(q)^2 = \sum_{k \in \mathbb{Z}} (-1)^k q^{k^2}$. Note also that (8) implies that

$$\sum_{\gamma \in R} e^{(2\pi i/h)\mathcal{O}(\gamma)} = -(n+1),$$

which is equivalent to the fact that the trace of the principal element in the adjoint representation is -1 (cf. [3]).

5. Let $F(\lambda)$ denote the finite-dimensional irreducible $\mathscr{L}(\mathbb{C})$-module with highest weight λ and let χ_λ be its character. Denote by $m_\lambda(\Lambda, s)$ the multiplicity of $F(\lambda)$ in the $\mathscr{L}(\mathbb{C})$-module $V_s \subset V(\Lambda)$ and set

$$\Phi_{\Lambda,\lambda}(q) = \sum_{s \geqslant 0} m_\lambda(\Lambda, s)\, q^s.$$

Let $\mathscr{K}$ be a maximal compact subgroup in $\mathscr{L}(\mathbb{C})$ and let $\mathscr{T}$ be a maximal torus in $\mathscr{K}$. Then one has

$$\begin{aligned} \Phi_{\Lambda,\lambda}(q) &= \int_{\mathscr{K}} \overline{\chi_\lambda(k)}\, \mathrm{ch}\, V(\Lambda)((q, k))\, dk \\ &= \frac{1}{\# W} \int_{\mathscr{T}} \prod_{\alpha \in R} (1 - e^{2\pi i \alpha(z)})\, \overline{x_\lambda(z)}\, \mathrm{ch}\, V(\Lambda)(q, z))\, dz. \end{aligned} \tag{9}$$

We apply this formula to the module $V(\Lambda_0)$. Denote by R_+ the subset of positive roots in the root system R of L and by $\bar\rho$ their half-sum. For $\lambda \in Q$ introduce the λ-height on Q by $\mathcal{O}_\lambda(\alpha_i) = (\lambda + \bar\rho, \alpha)$ (note that $\mathcal{O}_0 = \mathcal{O}$).

PROPOSITION 2. *Let L be a simple Lie algebra of type A_n, D_n or E_n. For $\lambda \in Q$ we set*

$$P_\lambda(q) = q^{|\lambda|^2/2} \prod_{\alpha \in R_+} (1 - q^{\mathcal{O}_\lambda(\alpha)}). \tag{10}$$

Then for a dominant λ one has

$$\Phi_{\Lambda_0,\lambda} = P_\lambda(q)/\varphi(q)^n, \qquad \lambda \in Q;\ \Phi_{\Lambda_0,\lambda}(q) = 0,\ \lambda \notin Q. \tag{11}$$

Proof. From (2), (9), and the Weyl character and denominator formulas we obtain

$$\varphi(q)^n \Phi_{\Lambda_0,\lambda}(q) = \frac{1}{\# W} \sum_{w, w_1 \in W} (\det w w_1)\, q^{|w(\lambda+\rho)-w_1(\rho)|^2/2}$$

$$= \sum_{w \in W} (\det w)\, q^{|\lambda+\rho-w(\rho)|^2/2} = q^{|\lambda|^2/2} \prod_{\alpha \in R_+} (1 - q^{\mathcal{O}_\lambda(\alpha)}).$$

Remarks. (a) The lowest and the highest terms of $P_\lambda(q)$ are $q^{|\lambda|^2/2}$ and $(-1)^{\# R_+/2} q^{|\lambda+2\rho|^2/2}$, respectively. In particular, the minimal k for which $F(\lambda)$ appears in $V_k \subset V(\Lambda_0)$ is $k = \frac{1}{2} \mid \lambda \mid^2$ (and the multiplicity is 1).

(b) For $L = A_1$ one has $P_{k\alpha}(q) = q^{k^2}(1 - q^{2k+1})$. This appears in [27, Proposition 1].

(c) Because of the action of the Virasoro algebra in $V(\Lambda_0)$ (see the results of Section 3 in [27]) we obtain for $L \neq A_1$,

$$\Phi_{\Lambda_0,0}(q) = \frac{1-q}{\varphi(q)}(1 + a_1{}^0 q + \cdots); \quad \Phi_{\Lambda_0,\lambda} = \frac{q^{|\lambda|^2/2}}{\varphi(q)}(1 + a_1{}^\lambda q + \cdots), \quad \lambda \neq 0, \tag{12}$$

where $a_k{}^\lambda$ are nonnegative integers (which are the multiplicities of modules of the Virasoro algebra). It follows from (11) that

$$a_k{}^0 = 0 \qquad \text{for} \quad k \leqslant m_2 \text{ (the second exponent of } L). \tag{13}$$

(d) As a consequence of Proposition 2 we obtain the following relation between the multiplicities $\mathrm{mult}_\gamma(\lambda)$ of weight γ in $F(\lambda)$:

$$\sum_\lambda \mathrm{mult}_\gamma(\lambda)\, P_\lambda(q) = q^{|\gamma|^2/2}.$$

It is natural to suppose that similar statements take place for the Monster. More precisely, we suggest the following conjectures. Let Λ be the Leech lattice and let R be the set of vectors in Λ of square length 4. Then for a suitable choice of R_+ in R for any irreducible character χ of the Monster group there exists $\lambda \in \Lambda$ such that the series $\Phi_\chi(q) = \sum_k (\text{mult } \chi \text{ in } V_k) q^k$ is equal to $P_\lambda(q)/\varphi(q)^{24}$, where $P_\lambda(q)$ is given by an (infinite) product (note that $P_0(q)$ is the average Thompson series over F_1). Formula (12) also should hold. I checked (12) using Table 1a from [25] and it turned out that

$$\Phi_1(q) = \frac{1-q}{\varphi(q)}(1 + t^{12} + \cdots), \qquad \Phi_2(q) = \frac{q}{\varphi(q)}(1 + t^8 + \cdots), \text{ etc.}$$

6. Let Q be a positive definite, integral, even lattice of rank n in the n-dimensional complex space H. Let $\hat{W}$ be the group of all isometries of Q, let T be the (normal) subgroup of translations, and let W be the (finite) subgroup of linear isometries.

We construct an infinite-dimensional representation π of the group $\hat{W}$ in a complex vector space V as follows. Set $P_+ = t\mathbb{C}[t] \otimes_{\mathbb{C}} H = \oplus_{k\geqslant 1} P_+{}^k$, where $P_+{}^k = t^k \otimes_{\mathbb{C}} H$. Let $S(P_+)$ be the symmetric algebra over the complex space P_+ and let $\mathbb{C}(Q)$ be the group algebra of the group Q. We set $V = S(P_+) \otimes_{\mathbb{C}} \mathbb{C}(Q)$. The action of W on H induces a representation π_1 of W in V. We define a representation π_2 of Q in V by $\pi_2(\gamma) = 1 \otimes$ (multiplication by e^γ). Now the representation π is defined by $\pi(wT_\gamma) = \pi_1(w)\,\pi_2(\gamma)$, where $w \in W$ and T_γ is a translation by γ.

We define a gradation $V = \oplus_{k\geqslant 0} V_k$ by $\deg a = km$ for $a \in S^m(P_+{}^k)$, $\deg e^\gamma = \frac{1}{2}(\gamma, \gamma)$. Clearly, this gradation is W-invariant. For $w \in W$ set $\operatorname{ch} V((q, w)) = \sum_{k\geqslant 0} (\operatorname{tr} w\,|_{V_k})q^k$. The same argument as that in the proof of Proposition 1 gives the following formula (we use the notations of Section 4):

$$q^{-n/24} \operatorname{ch} V((q, w)) = \theta_w(q)/\eta_w(q). \tag{14}$$

In particular, if Q is the Leech lattice, then $q^{-1} \operatorname{ch} V((q, 1)) = j(q) - 720$, and (14) turns into the formula conjectured in [25] (mentioned in Section 4). So we have constructed the graded space in which one should look for the Monster.

Note that our construction is similar to the first step of the construction applied to the realization of the $\hat{L}$-module $V(\Lambda_0)$ in [28]. An analog of the second step hopefully would lead to the construction of the Monster. In any event the problems of the explicit realization of the $\hat{\mathscr{L}}$-module $V(\Lambda_0)$ and the construction of the Monster are very similar.

7. In the remainder of this addendum I would like to make some corrections to [SF]. The third line of the statement of Proposition 2.2 should be: "of $\tilde{V}(\mu)$ such that $V(\mu)$ is a subquotient of V." There is a simpler proof of this proposition. Indeed, one clearly has $\operatorname{ch} V = \sum_\lambda \operatorname{ch} V(\lambda)$, where $c_\lambda \neq 0$ only if $V(\lambda)$ is a subquotient of V. Now, since $V(\lambda) = \tilde{V}(\lambda)/I$, one has $\operatorname{ch} V(\lambda) = \operatorname{ch} \tilde{V}(\lambda) - \operatorname{ch} I$. Applying the same argument to module I and iterating it completes the proof. Also, in the proof of Proposition 2.5 one should add that $W_{\Lambda+\rho} = \{e\}$ because of Proposition 1.8(f).

Further Corrections

Page 115, line 11↑, should be X^4 instead of X^0;

Page 127, line 6↓, should be g instead of δ;

Page 128, one should divide the second factor in (3.26) by $\# W_{\mu_i}$;

Page 129, line 1↓, one should add (0,..., 0, 1) for $B_n^{(1)}$ and $A_{2n}^{(2)}$. In these cases $\sum a(r)X^r$ is $\varphi(X^2)\varphi(X)^{-n-1}$ and $\varphi(X)^{-n}$ [31];

Page 129, line 2↓, one should add (1, 0,..., 0) for $A_{2n-1}^{(2)}$, $D_{n+1}^{(2)}$ and $E_6^{(2)}$. In these cases $\sum a(r)X^r$ is $\varphi(X)^{-n+1}\varphi(X^2)^{-1}$, $\varphi(X)^{-1}\varphi(X^2)^{-n+1}$ and $\varphi(X)^{-2}\varphi(X^2)^{-2}$ [31];

Page 130, line 10↓, one should cross out $(\delta, \rho) = h$;
Page 130, line 15↓, should be $I - \lambda\sigma$ instead of $\sigma - \lambda I$;
Page 131, lines 11↓, 14↓, should be γ instead of $(h + 1)\gamma$.

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