Note
On the number of independent chorded cycles in a graph
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Abstract
Hajnal and Corrádi proved that any simple graph on at least 3k vertices with minimal degree at least 2k contains k independent cycles. We prove the analogous result for chorded cycles. Let G be a simple graph with \(|V(G)| \geq 4k\) and minimal degree \(\delta(G) \geq 3k\). Then G contains k independent chorded cycles. This result is sharp.
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1. Introduction
Hajnal and Corrádi proved the following result in 1963 [2].

Theorem 1. Let G be a graph with \(|V(G)| \geq 3k\) and \(\delta(G) \geq 2k\). Then G contains k independent cycles.

Their result was generalized and extended by Erdős [5], Dirac [4,5], Justesen [6], and Wang [8,9], who currently has the strongest generalization in [8]. Theorem 1 is in a sense a natural generalization of the well-known fact that any graph G with minimal degree \(\delta(G) \geq 2\) contains a cycle. Pósa posed the same question for chorded cycles in [7], and it is easy to show [3] that \(\delta(G) \geq 3\) gives the existence of a chorded cycle in G.

It is natural to ask whether a result analogous to Theorem 1 holds for chorded cycles. In this paper, we show that it does.

Theorem 2. Let G be a graph with \(|V(G)| \geq 4k\) and \(\delta(G) \geq 3k\). Then G contains k independent chorded cycles.

2. Notation and conventions
We will consider only simple graphs. A chorded cycle is a cycle with at least one chord. Let G be a graph, and H, H′ subgraphs of G. We denote by \(\delta(G)\) the minimal degree of the vertices of G. We will write \(G - H\) to denote the subgraph of G induced by \(V(G)\setminus V(H)\), and \(H + H'\) to denote the subgraph of G induced by \(V(H) \cup V(H')\). Write \(d(H; H')\) for the number of edges \(xy\) satisfying \(x \in V(H)\) and \(y \in V(H')\). Let a, b be vertices of G. For convenience we will write \(d(a, b; H, H')\) rather than the more cumbersome \(d((a, b); H + H)\). For positive integers m, n, let \([n, m]\)

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denote the complete bipartite graph with classes consisting of \( n \) and \( m \) vertices. By independent we mean vertex disjoint. We say \( G \) contains \( k \) independent chorded cycles if there is a subgraph \( H \) of \( G \) consisting of \( k \) independent chorded cycles.

3. Proof of main result

**Remark.** This result is sharp. Since a chorded cycle must have at least four vertices, the condition on \( |V(G)| \) is clearly necessary. For \( m \geq 6k - 2 \) the graph \([3k - 1, m - 3k + 1]\) has minimal degree \( 3k - 1 \) and contains no collection of \( k \) independent chorded cycles, as any chorded cycle must contain three vertices from the first class.

**Proof of Theorem 2.** We will proceed by induction on \( k \). Pósa posed the problem for \( k = 1 \) in [7], and it was proven in [3]. Suppose the theorem is true for all \( k \leq s - 1 \), and take \( G \) a graph with \( |V(G)| \geq 4s \) and \( \delta(G) \geq 3s \). Consider the set \( \mathcal{B} \) of all collections of \( s - 1 \) independent chorded cycles in \( G \). By the inductive hypothesis \( \mathcal{B} \) is nonempty, and certainly \( \mathcal{B} \) is finite. Thus, we may choose an element \( K \in \mathcal{B} \) satisfying the following two conditions:

1. For all \( K' \in \mathcal{B} \), \( |V(K)| \leq |V(K')| \). We refer to this condition as the minimality of \( K \).
2. For all \( K' \in \mathcal{B} \) that are minimal in the sense of the above condition, the length of a maximal path in \( G - K' \) is of equal or shorter length than a maximal path \( P \) in \( G - K \). We refer to this condition as the maximality of \( P \).

To prove the theorem, we show first that \( |V(P)| \geq 4 \), and then that this implies the existence of \( s \) independent chorded cycles in \( G \).

By the inductive hypothesis, we may remove any three vertices from \( V(G) \), and the graph induced by what remains contains \( s - 1 \) independent chorded cycles, so \( |V(G - K)| \geq 3 \). \( \square \)

**Lemma 1.** Let \( D \) be a chorded cycle and \( w \) a vertex not in \( V(D) \). Suppose \( |V(D)| \geq 5 \) and \( d(w; D) \geq 4 \). Then there is a chorded cycle \( D' \) on a subset of \( V(D) \cup w \) with \( |V(D')| < |V(D)| \).

**Proof.** Suppose \( D \) and \( w \) be given satisfying the conditions of the lemma. Label five vertices \( \{y_1, y_2, y_3, y_4, y_5\} \subset V(D) \) such that \( wy_j \) is an edge for \( 1 \leq j \leq 4 \) and \( \{y_1, y_2, y_3, y_4, y_5\} \) is a list in cyclic order in \( D \). Then there is a chorded cycle \( D' \) induced by the vertices \( V(D+w-y_4-y_5) \), and \( |V(D')| \leq |V(D)| + 1 - 2 < |V(D)| \) is immediate. \( \square \)

The lemma implies that if \( w \in G - K \) is a vertex with \( d(w; D) \geq 4 \) for some chorded cycle \( D \in K \), then \( |V(D)| = 4 \) by the minimality of \( K \). In particular, we have the following corollary.

**Corollary 1.** For any vertex \( w \) in \( G - K \) and any chorded cycle \( D \in K \), \( d(w; D) \leq 4 \), with equality only if \( D \) is a chorded 4-cycle.

If \( G - K \) contains a chorded cycle, the theorem follows immediately. For the remainder of this proof, we assume that \( G - K \) does not contain a chorded cycle.

**Lemma 2.** \( V(P) = V(G - K) \).

**Proof.** Suppose to the contrary that \( P \) consists of a single point \( a \), and choose another vertex \( v \in G - K \). Then \( d(a, v; K) \geq 6s = 6(s - 1) + 6 \) by the maximality of \( P \), so there is a chorded cycle \( D \in K \) such that \( d(a, v; D) \geq 7 \).
Without loss of generality, \( d(a; D) = 4 \). It follows that \( D \) is a chorded 4-cycle, say \( V(D) = \{y_1, y_2, y_3, y_4\} \), and \( d(v; D) \geq 3 \). Without loss of generality, say \( y_1, y_2, \) and \( y_3 \) are neighbors of \( v \). Then there is a chorded cycle induced by \( \{v, y_1, y_2, y_3\} \) and \( ay_4 \) is an edge of \( G \). This gives a collection of \( s - 1 \) chorded cycles \( K' \in \mathcal{B} \) with \( |V(K)| = |V(K')| \).
But \( ay_4 \in G - K' \), so the maximal path in \( G - K' \) has nonzero length. This contradicts our assumption of the maximality of \( P \). Hence, \( P \) is not a single point.

Now suppose to the contrary that there was a point \( v \in G - K \) not in \( V(P) \). Then we can take a maximal path \( Q \) (possibly a point) in \( G - K - P \). Let \( a \) and \( b \) be the endpoints of \( P \), and let \( c \) be an endpoint of \( Q \). Note that \( d(a, b; K) \geq 6s - 4 \) and \( d(c; K) \geq 3s - 4 \), else we have a chorded cycle in \( G - K \). Hence, \( d(a, b, c; K) \geq 9s - 8 = 9(s - 1) + 1 \), so there exists a chorded cycle \( D \in K \) with \( d(a, b, c; D) \geq 10 \). By the pigeonhole principle and the above
corollary, \( d(x; D) = 4 \) for some \( x \in \{a, b, c\} \). Without loss of generality, it is sufficient to consider the cases where \( x = a \) and \( x = c \). If \( d(c; D) = 4 \), we know \( d(a; D) \geq 2 \), so there exists a vertex \( y \in D \) such that \( ay \) is an edge. This gives a chorded 4-cycle induced by \( V(D - y) \cup c \) and an edge \( ay \) left over. This contradicts the maximality of \( P \). If, on the other hand, \( d(a; D) = 4 \), then there is a vertex \( y \in D \) such that \( cy \) and \( by \) are edges of \( G \). This gives a chorded 4-cycle induced by \( V(D - y + a) \) and edges \( by \) and \( cy \) left over. This also contradicts the maximality of \( P \). Therefore, we have \( V(P) = V(G - K) \). □

It is immediate from the lemma that \( |V(P)| \geq 3 \). We prove now that equality does not hold. Suppose to the contrary that \( |V(P)| = 3 \). Let

\[
K_1 = \{ \text{chorded cycles } D \in K | d(y; P) = 3 \text{ for some } y \in D, \}
\]

and define iteratively

\[
K_i = \left\{ \text{chorded cycles } D \in K \left| \left( \bigcup_{j=1}^{i-1} K_j \right) \bigcap d(y; P) = 3 \text{ for some } y \in D, E \in K_{i-1} \right. \right\}.
\]

Obviously, \( K_i = \emptyset \) for some \( i \), since \( K \) contains only finitely many chorded cycles. Say \( K_i \) is the last nonempty set obtained from the process above. Define \( \bar{K} = P \cup \bigcup_i K_i \).

**Lemma 3.** Every chorded cycle \( D \in \bigcup K_i \) has exactly 4 vertices.

**Proof.** Label the vertices of \( P = x_0y_0z_0 \) and let \( D_1 \) be a chorded cycle in \( K_1 \). This means that there is a vertex \( v_1 \) in \( D_1 \) such that \( v_1x_0, v_1y_0, \) and \( v_1z_0 \) are edges. In particular, there is a chorded 4-cycle induced by the vertices \( \{v_1, x_0, y_0, z_0\} \). By minimality of \( K \), \( D_1 \) must have four vertices as well.

Let \( \{D_i\}_{1 \leq i \leq n} \) be a collection of chorded cycles with \( D_i \in K_i \) and \( v_i \in D_i \) satisfying \( d(v_i; D_{i-1}) = 4 \). Assume inductively that \( D_i \) is a chorded 4-cycle for \( 1 \leq i \leq n - 1 \), with vertices \( v_i, x_i, y_i, z_i \). By hypothesis, \( v_n \) is a neighbor to every vertex of \( D_{n-1} \), so \( v_n, x_{n-1}, y_{n-1}, z_{n-1} \) is a chorded 4-cycle. Repeating the argument, we have \( v_{n+1}, x_{n+1}, y_{n+1}, z_{n+1} \) as a chorded 4-cycle, for \( 1 \leq i \leq n - 1 \). This gives \( n \) chorded 4-cycles induced by \( V(P + D_1 + \cdots + D_{n-1} + v_n) \), so by the minimality of \( K \), \( D_n \) must have only four vertices as well. Applying the inductive step proves the lemma. □

Define \( G' = G - \bar{K} \). Now \( \bar{K} \) consists of \( t \leq s - 1 \) chorded cycles, all with four vertices, and \( P \), so \( |V(\bar{K})| = 4t + 3 \). It follows that \( |V(G')| \geq 4s - (4t + 3) \geq 4s - 4(s - 1) - 3 = 1 \), so in particular \( G' \) is nonempty. Consider any vertex \( w \in G' \). Our definition of \( \bar{K} \) and Lemma 2 give us that there is a chorded cycle \( E \in K \) such that \( w \in E \subseteq G' \). If \( d(w; P) = 3 \) then \( E \) would be a member of \( \bar{K} \), a contradiction. Therefore, since \( V(P) = V(G - K) \), we have \( d(w; G - K) \leq 2 \). Also, given \( D \in K \), it follows that \( d(w; D) \leq 3 \), else \( w \in \bar{K} \). Therefore, \( d(w; \bar{K}) \leq 3t + 2 \). This gives us that \( \delta(G') \geq 3s - (3t + 2) = 3(s - t - 1) + 1 \). Hence \( \delta(G' - w) \geq (s - t - 1) + 1 \), and \( |V(G' - w)| \geq 4\delta(G' - w) - 4s - 4t - 3 - 1 = 4s - 4t - 1 \). By the inductive hypothesis it follows that \( G' - w \) contains a collection of \( s - t - 1 \) independent chorded cycles. This means that \( (G' - w) + (\bar{K} - P) \subseteq G - P - w \) contains \( s - t - 1 + t = s - 1 \) chord cycles. But \( w \in K \), so this contradicts the minimality of \( K \). Therefore, \( |V(P)| > 3 \).

Let \( a \) and \( d \) be the endpoints of \( P \), and \( b \) and \( c \) the neighbors of \( a \) and \( d \), respectively, in \( P \). Since \( G - K \) does not contain a cycle with chord by assumption, \( d(a; P), d(d; P) \leq 2 \) and \( d(b; P), d(c; P) \leq 3 \). We have already observed that there are no vertices of \( G - K \) that are not in \( P \), so we must have \( d(a, b, c, d; K) \geq 12s - 10 = 12(s - 1) + 2. \) This means that for some chorded cycle \( D \in K \) we have \( d(a, b, c, d; D) = 13 \). Since \( d(x; D) \geq 4 \) for some \( x \in \{a, b, c, d\} \) we know that \( |V(D)| = 4 \).

Without loss of generality, we can assume \( d(a, b; D) \geq d(c, d; D) \). Counting gives \( d(a, b; D) \geq 7 \) and \( d(c, d; D) \geq 5 \). Consider \( D \) to be a 4-cycle, and name its vertices \( y_1, y_2, y_3, y_4 \) in cyclic order. Without loss of generality, for two neighboring vertices in \( D \), say \( y_3 \) and \( y_4 \), we have \( d(c, d; y_3, y_4) \geq 3 \). This means that a chorded cycle is induced on \( \{c, d, y_3, y_4\} \). But \( d(a, b; y_1, y_2) \geq 3 \) is immediate as well, so a chorded cycle is induced on these vertices as well. Hence, \( G \) contains a independent collection of \( s \) chorded cycles. Applying induction, this proves the theorem.
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References