Blowing-up of Solution for a Nonlocal Reaction–Diffusion Problem in Combustion Theory

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In a nonlocal reaction–diffusion model in combustion theory the reaction function involves a physical parameter \( \sigma \) which is a measure of the strength of the reaction mechanism. The purpose of this paper is to show the existence of a critical value \( \sigma^* \) such that for \( \sigma < \sigma^* \) a unique global time-dependent solution exists and converges to a steady-state solution as \( t \to \infty \), and for \( \sigma > \sigma^* \) the solution blows-up in finite time. A characterization as well as upper and lower bounds of \( \sigma^* \) are given.

1. INTRODUCTION

In the combustion theory of thermal explosion an extended model for the temperature distribution in a bounded domain \( \Omega \) in \( \mathbb{R}^n \) is given by the integro-parabolic boundary value problem (cf. [3, 5])

\[
\begin{align*}
\frac{\partial u}{\partial t} - D \nabla^2 u &= \sigma \left[ \epsilon^{\gamma u} + b \int_{\Omega} \epsilon^{\gamma \alpha(x, \gamma)} \, dx' \right] & (t > 0, \, x \in \Omega) \\
Bu &= \alpha(x) \frac{\partial u}{\partial \nu} + \beta(x)u = h(x) & (t > 0, \, x \in \partial \Omega) \\
u(0, x) &= u_0(x) & (x \in \Omega),
\end{align*}
\]

where \( D, \sigma, \) and \( \gamma \) are positive constants, \( b \) is a nonnegative constant, and \( \partial / \partial \nu \) denotes the outward normal derivative on \( \partial \Omega \), the boundary of \( \Omega \). The boundary functions \( \alpha(x), \beta(x), \) and \( h(x) \) are nonnegative with either \( \alpha(x) \equiv 0, \beta(x) > 0 \) (Dirichlet condition) or \( \alpha(x) > 0, \beta(x) \geq 0 \) (Neumann or Robin condition). It is known that when \( b = 0 \) and \( h = u_0 = 0 \) there exists a critical value \( \sigma^* \) such that for \( \sigma < \sigma^* \) a unique global solution \( u(t, x) \) to (1.1) exists whilst for \( \sigma > \sigma^* \) the solution \( u(t, x) \) blows-up in finite time (cf. [2, 4, 6, 9]). In this paper we extend the global existence and blowing-up property of a solution to the problem (1.1) where \( b, h, \) and \( u_0 \) are...
nonnegative, not necessarily identically zero. In addition, we show that when \( \sigma < \sigma^* \) the corresponding steady-state problem

\[
-D\nabla^2 u = \sigma \left[ e^{\gamma u} + b \int_{\Omega} e^{\gamma u(x')} \, dx' \right] \quad \text{in } \Omega
\]

\[
Bu = h(x) \quad \text{on } \partial \Omega
\]

(1.2)

has a positive solution \( u_+(x) \); and when \( u_0 \leq u_+ \), including \( u_0 = 0 \), the time-dependent solution of (1.1) converges to \( u_+(x) \) as \( t \to \infty \). All these results are shown for the boundary condition which can be either Dirichlet type or Neumann–Robin type. As in the case \( h = 0 \) the critical value \( \sigma^* \) for the present problem is determined by the existence or nonexistence of a positive solution to (1.2). We characterize the value of \( \sigma^* \) and obtain upper and lower bounds for \( \sigma^* \) as well as for the blowing up time \( T^* \).

The global existence and the blowing-up behavior of the solution for the case \( b = 0 \) have been investigated by many researchers but are mostly for the Dirichlet boundary condition or for the case \( h(x) = 0 \) (cf. [2–7, 9–11]). The work in [2] summarizes much of the results for the Dirichlet boundary condition with \( u_0 = 0 \), whilst those in [10, 11] are devoted to various conditions on \( u_0 \) under the Robin boundary condition. In this paper we consider both type of boundary conditions, as well as an arbitrary nonnegative boundary function \( h(x) \), including \( h(x) \equiv 0 \). Since our results hold true for the case \( b = 0 \), the above consideration improves some of the conclusions in the earlier work in the above references.

2. THE MAIN RESULTS

To ensure the existence of a classical solution to (1.1) and (1.2), we assume that the boundary \( \partial \Omega \) and the given functions, \( x, \beta, h, \) and \( u_0 \) are sufficiently smooth, and when \( x = 0 \) the compatibility condition \( \beta u_0 = h \) holds on \( \partial \Omega \). Consider the steady-state problem (1.2). Since the function

\[
(f(u))(x) = \sigma \left[ e^{\gamma u(x)} + b \int_{\Omega} e^{\gamma u(x')} \, dx' \right]
\]

(2.1)

is nondecreasing in \( u(x) \) and is Hölder continuous in \( \Omega \) whenever \( u \in C^2(\Omega) \) the sequence \( \{u^{(k)}\} \) given by the iteration process

\[
-D\nabla^2 u^{(k)} = \sigma \left[ e^{\gamma u^{(k-1)}} + b \int_{\Omega} e^{\gamma u^{(k-1)}(x')} \, dx' \right]
\]

\[
Bu^{(k-1)} = h(x)
\]

(2.2)
is well defined, and $u^{(k)} \in C^{2+\gamma}(\Omega)$ for every $k = 1, 2, \ldots$, when $u^{(0)} \in C^2(\Omega)$ (cf. [1]). Suppose Problem (1.2) has a positive upper solution $\check{u}_s$, that is, $\check{u}_s$ satisfies the relation

$$-D\nabla^2 \check{u}_s \geq \sigma \left[ e^{\check{u}_s} + b \int_{\Omega} e^{\check{u}_s(x')} dx' \right] \text{ in } \Omega$$

$$B\check{u}_s \geq h(x) \text{ on } \partial \Omega.$$  

By the well-known monotone argument for elliptic boundary value problems the sequence given by (2.2) with $u^{(0)} = \check{u}_s$ converges monotonically from above to a maximal solution $\check{u}_s(x)$ of (1.2) (cf. [1]). Since $\check{u}_s = 0$ is a lower solution of (1.2) the same monotone argument shows that the sequence corresponding to $u^{(0)} = 0$ converges monotonically from below to a minimal solution $\underline{u}_s(x)$. In view of $f(0) > 0$ the above monotone convergence yields $\check{u}_s(x) \geq \underline{u}_s(x) > 0$ in $\Omega$. Hence to guarantee the existence of a positive solution for the problem (1.2) it suffices to find a positive upper solution. Clearly, no positive upper solution can exist when $\beta(x) \equiv 0$. Suppose $\beta(x)$ is not identically zero. Then for any constant $\rho > 0$ the linear boundary-value problem

$$-D\nabla^2 w = \rho \text{ in } \Omega, \quad Bw = h(x) \text{ on } \partial \Omega,$$  

has a unique positive solution $w_\rho$. It is obvious that $\check{u}_s = w_\rho$ is an upper solution if

$$\rho \geq \sigma \left[ e^{w_\rho} + b \int_{\Omega} e^{w_\rho(x')} dx' \right].$$

Since $w_\rho$ is independent of $\sigma$ there exists $\sigma_0 > 0$ such that the above inequality holds for every $\sigma \leq \sigma_0$. This implies that for any $\sigma \leq \sigma_0$ Problem (1.2) has at least one positive solution which is denoted by $u_s(x; \sigma)$ to demonstrate its dependence on $\sigma$. It is clear that if $u_s(x; \sigma')$ is a positive solution of (1.2) corresponding to some $\sigma' > \sigma$ then it is an upper solution; and therefore the problem has a positive solution $u_s(x; \sigma)$ for every $\sigma < \sigma'$. This property implies that the constant $\sigma^*$ given by

$$\sigma^* = \sup\{ \sigma > 0; \text{ a positive solution to (1.2) exists} \}$$

is well defined and is positive. Moreover, since every positive solution of (1.2) is a positive upper solution of the same problem with $b = 0$, and since the problem with $b = 0$ has no positive solution when $\sigma$ is large, the constant $\sigma^*$ must be finite. This observation leads to the following conclusion.

**Theorem 1.** Let $\beta(x) \neq 0$, $h(x) \geq 0$, and let $\sigma^*$ be given by (2.5). Then $\sigma^*$ is positive, finite and for any $\sigma < \sigma^*$ the problem (1.2) has a maximal
solution \( \tilde{u}_*(x) \) and a minimal solution \( u_*(x) \) such that \( \tilde{u}_*(x) \gtrless u_*(x) > 0 \) in \( \Omega \). When \( \sigma \gtrsim \sigma^* \) the problem has no positive solution.

In analogy to the steady-state problem (1.2) the time-dependent problem (1.1) has a unique global solution if there exist a pair of ordered upper–lower solutions. Here upper and lower solutions \( \tilde{u}, \tilde{u} \) are required to satisfy the inequalities

\[
\tilde{u}_t - D\nabla^2 \tilde{u} \geq \sigma \left[ e^{\beta} + b \int_{\Omega} e^{\beta(x',x')} \, dx' \right] \quad (t > 0, \, x \in \Omega)
\]

\[
B\tilde{u} \geq h(x) \quad (t > 0, \, x \in \partial \Omega) \tag{2.6}
\]

and the reversed inequalities, respectively. Sinced for any \( u_0 \geq 0, \tilde{u} = 0 \) is a lower solution, the existence of a positive solution follows from the same monotone argument as for the standard parabolic boundary-value problem provided that there exists a positive upper solution (cf. [12]). It is obvious that if \( \tilde{u} \) is a positive upper solution of the steady-state problem (1.2) then it is an upper solution of (1.1) when \( u_0 \leq \tilde{u} \); and in this situation Problem (1.1) has a unique global solution \( u(t, x) \). To ensure the convergence of \( u(t, x) \) to a steady-state solution as \( t \to \infty \) we prepare the following positivity lemma concerning functions in \( C(\overline{D}_T) \cap C^{1,2}(D_T) \), where \( D_T = (0, T] \times \Omega \), \( \overline{D}_T = [0, T] \times \overline{\Omega} \), and \( C^{1,2}(D_T) \) denotes the set of functions which are once-differentiable in \( t \) and twice differentiable in \( x \).

**Lemma 1.** Let \( w \in C(\overline{D}_T) \cap C^{1,2}(D_T) \) and satisfy

\[
w_t - D\nabla^2 w \geq c_1(t, x) w + \int_{\Omega} c_2(t, x') w(t, x') \, dx' \text{ in } D_T,
\]

\[
Bw \geq 0 \text{ on } (0, T] \times \partial \Omega, \quad w(0, x) \geq 0 \text{ in } \Omega,
\]

where \( c_i \equiv c_i(t, x), \ i = 1, 2 \), are bounded functions in \( D_T \). Then \( w \geq 0 \) on \( \overline{D}_T \). Moreover, \( w > 0 \) in \( D_T \) if \( c_2 \geq 0 \) and \( w \) is not identically zero.

**Proof.** Let \( \tilde{c}_i \) be the last upper bound of \( c_i(t, x) \) in \( D_T \), \( i = 1, 2 \), and let \( v = e^{-\gamma t} w \) for some constant \( \gamma > \tilde{c}_1 + \tilde{c}_2 \vert \Omega \), where \( \vert \Omega \) is the “volume” of \( \Omega \). By the relation (2.7),

\[
L v \equiv v_t - D\nabla^2 v + (\gamma - c_1) v \geq \int_{\Omega} c_2(t, x') w(t, x') \, dx' \text{ in } D_T
\]

\[
Bv \geq 0 \text{ on } (0, T] \times \partial \Omega, \quad v(0, x) \geq 0 \text{ in } \Omega.
\]

Assume by contradiction that \( v \) has a negative minimum at some point \( (t_0, x_0) \in \overline{D}_T \). In view of \( v(0, x) \geq 0, \ t_0 > 0 \). Consider the case \( \beta(x) > 0 \) on
Then $x_0 \notin \partial \Omega$, for if $x_0$ were on $\partial \Omega$ then $v(t_0, x_0) \geq 0$ when $\alpha(x_0) = 0$ and

\[(\partial \upsilon/\partial v)(t_0, x_0) \geq - (\beta(x_0)/\alpha(x_0)) v(t_0, x_0) > 0 \quad \text{when } \alpha(x_0) > 0, \tag{2.9}\]

contradicting the negative minimum property of $v(t_0, x_0)$. Knowing $(t_0, x_0) \in D_T$ we have $v(t_0, x_0) \leq 0$, $v_{x_0}(t_0, x_0) \geq 0$ and hence

\[(Lv)(t_0, x_0) \leq (\gamma - c_1(t_0, x_0)) v(t_0, x_0).\]

In view of (2.8),

\[(\gamma - c_1(t_0, x_0)) v(t_0, x_0) \geq \int_{\Omega} c_2(t_0, x') v(t_0, x') \, dx' \geq \tilde{c}_2 |\Omega| v(t_0, x_0).\]

This leads to $\gamma - \tilde{c}_2 \leq \tilde{c}_2 |\Omega|$, which contradicts the relation $\gamma > \tilde{c}_1 + \tilde{c}_2 |\Omega|$. Hence $w(t, x) = e^{\gamma t} v(t, x) \geq 0$ in $D_T$ when $\beta(x) > 0$. In the general case $\beta(x) \geq 0$ the above conclusion remains true if either $x_0 \in \Omega$ or $x_0 \in \partial \Omega$ and $\beta(x_0) > 0$. In the event of $x_0 \in \partial \Omega$ and $\beta(x_0) = 0$, the relation (2.9) is reduced to $(\partial \upsilon/\partial v)(t_0, x_0) \geq 0$. In this situation we choose a small positive constant $\varepsilon$ satisfying

\[3\varepsilon < (\gamma - \tilde{c}_1 - \tilde{c}_2 |\Omega|)(-v(t_0, x_0)) \tag{2.10}\]

and find a point $x_\varepsilon \in \Omega$, sufficiently close to $x_0$, such that

\[v(t_0, x_\varepsilon) \leq v(t_0, x_0) + \varepsilon/(\gamma - \tilde{c}_1), \quad v_x(t_0, x_\varepsilon) \leq \varepsilon \]

\[Dv^2(t_0, x_\varepsilon) \geq -\varepsilon.\]

By the relation (2.8) with $x = x_\varepsilon$, we have

\[3\varepsilon + (\gamma - c_1(t_0, x_\varepsilon)) v(t_0, x_\varepsilon) \geq \int_{\Omega} c_2(t_0, x') v(t_0, x') \, dx' \geq \tilde{c}_2 |\Omega| v(t_0, x_0),\]

which contradicts the relation (2.10). This contradiction shows that $w = e^{\gamma t} v(t, x) \geq 0$ in $D_T$. Finally, when $c_2 \geq 0$ the integral term in (2.7) is nonnegative. The positive property of $w$ in $D_T$ follows from the maximum principle.

It is easily seen that when $\beta(x) = 0$ (Neumann boundary condition) the solution $u(t, x)$ of (1.1) blows-up in finite time for any $\sigma > 0$ (cf. [11]). Without loss of generality we may assume that $\beta(x)$ is not identically zero. This implies that the principle eigenvalue $\lambda_0$ of the eigenvalue problem

\[Dv^2 + \lambda v = 0 \text{ in } \Omega, \quad Bv = 0 \text{ on } \partial \Omega\]
is positive and its corresponding eigenfunction $\phi(x)$ is positive in $\Omega$. We normalize $\phi$ so that $|\Omega|^{-1} \int_{\Omega} \phi \, dx = 1$. Using the property of $\phi$ and Lemma 1 we show the following main result.

**Theorem 2.** Let $h(x) \geq 0$, $u_0 \geq 0$, and let $\sigma^*$ be given by (2.5). Then (i) for $\sigma < \sigma^*$ the steady-state problem (1.2) has a minimal solution $u_*(x) > 0$ in $\Omega$ and for any $u_0 \leq u_*$, a unique global solution $u(t, x)$ to (1.1) exists and converges to $u_*(x)$ as $t \to \infty$, and (ii) for $\sigma > \sigma^*$ no positive steady-state solution can exist and there exists a finite $T^*$ such that a unique solution $u(t, x)$ to (1.1) exists in $[0, T^*) \times \Omega$ and

$$\lim_{t \to T^*} \max_{x \in \Omega} u(t, x) = \infty. \quad (2.11)$$

**Proof.** (i) Consider Problem (1.1) with $u_0 = 0$. Since for $\sigma < \sigma^*$ the problem (1.2) has a positive minimal solution $u_*$, the pair $\bar{u} = u_*$ and $\bar{u} = 0$ are upper and lower solutions of (1.1). This implies that Problem (1.1) has a unique solution $u(t, x)$ and $0 \leq u(t, x) \leq u_*(x)$ (cf. [10, 12]). Let $\delta > 0$ be any constant and let $w(t, x) = u(t + \delta, x) - u(t, x)$. By (1.1) and the mean value theorem, $w$ satisfies the relation

$$w_t - D\nabla^2 w = \sigma(e^{\gamma(t+\delta,x)} - e^{\gamma(t,x)}) + \sigma b \int_{\Omega} (e^{\gamma(t+\delta,x')} - e^{\gamma(t,x')}) \, dx'$$

$$= (\sigma \gamma e^{\gamma(t,x)}) w + b \sigma \int_{\Omega} e^{\gamma(t,x')} w(t, x') \, dx' \quad (2.12)$$

and the boundary-initial conditions

$$Bw = 0 \quad (t > 0, \ x \in \partial \Omega), \quad w(0, x) = u(\delta, x) \geq 0 \text{ in } \Omega,$$

where $\eta(t, x)$ is an intermediate value between $u(t, x)$ and $u(t + \delta, x)$. By an application of Lemma 1 with $c_1 = \sigma \gamma e^{\gamma(t,x)}$ and $c_2 = b \sigma \gamma e^{\gamma(t,x)}$ we have $w \geq 0$ on $\bar{\Omega}$. This shows that for each $x \in \partial \Omega$, $u(t, x)$ is nondecreasing in $t$, and therefore $u(t, x)$ converges to some function $u_*(x)$ as $t \to \infty$. Using the argument in [13] the limit $u_*(x)$ is a solution of (1.2) and $0 < u_*(x) \leq u_*(x)$. The minimal property of $u_*(x)$ implies that $u_*(x) = u_*(x)$ in $\bar{\Omega}$. Now for $0 \leq u_0 \leq u_*$ the pair $u_*(x)$ and $u(t, x)$ are upper and lower solutions of (1.1). This ensures that the corresponding solution $u(t, x)$ satisfies $u(t, x) \leq u(t, x) \leq u_*(x)$; and therefore $u(t, x) \to u_*(x)$ as $t \to \infty$, which proves the conclusion in (i).

(ii) It is clear that when $\sigma > \sigma^*$ no positive solution to (1.4) can exist. Moreover, the above proof shows that the solution $u(t, x)$ corresponding to $u_0 = 0$ is strictly increasing in $t$. Hence there exists $T^* \leq \infty$ such that $u(t, x) \to \infty$ at some point in $\Omega$ as $t \to T^*$, for otherwise, $u(t, x)$ would
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converge to a positive solution of (1.2), contrary to the hypothesis \( \sigma > \sigma^* \). Since for any \( u_0 \geq 0 \) the corresponding solution \( u(t, x) \) satisfies \( u(t, x) \geq u_0(t, x) \), it suffices to show that \( T^* \) is finite when \( u_0 = 0 \). Define

\[
q(t) = |\Omega| \int_{\Omega} u(t, x) \phi(x) \, dx,
\]

where \( u(0, x) = 0 \). In view of (1.1),

\[
|\Omega| \int_{\Omega} \phi'(t) \, dx = \int_{\Omega} u_t(t, x) \phi(x) \, dx
\]

\[
= D \int_{\Omega} \phi \nabla^2 u \, dx + \sigma \left( \int_{\Omega} \phi e^{u(t)} \, dx + b \int_{\Omega} \phi \, dx \int_{\Omega} e^{u(t, x')} \, dx' \right).
\]

Consider the case \( \alpha(x) > 0 \). By the Green’s theorem and the relation

\[
\phi \frac{\partial u}{\partial v} - u \frac{\partial \phi}{\partial v} = \alpha^{-1} [\phi(-\beta u + h) - u(-\beta \phi)] = \alpha^{-1} \phi h \geq 0
\]

we obtain

\[
D \int_{\Omega} \phi \nabla^2 u \, dx = D \int_{\Omega} u \nabla^2 \phi \, dx + D \int_{\partial \Omega} (\phi \frac{\partial u}{\partial v} - u \frac{\partial \phi}{\partial v}) \, dS
\]

\[
\geq -\lambda_0 \int_{\Omega} u \phi \, dx = -\lambda_0 |\Omega| \cdot q(t).
\]

This leads to the relation

\[
q' + \lambda_0 q \geq \sigma \left( |\Omega|^{-1} \int_{\Omega} \phi e^{u(t)} \, dx + b \int_{\Omega} e^{u(t, x')} \, dx' \right)
\]

\[
\geq \sigma_1 |\Omega|^{-1} \int_{\Omega} u \phi e^{u(t, x)} \, dx,
\]

where

\[
\sigma_1 \equiv \sigma(1 + b |\Omega|/\phi) \quad \text{and} \quad \phi \equiv \max \{\phi(x); x \in \overline{\Omega}\}.
\]

In the case \( \alpha(x) = 0, \phi(x) = 0 \) on \( \partial \Omega \); and in this situation \( \partial \phi/\partial v \leq 0 \) and therefore

\[
D \int_{\Omega} \phi \nabla^2 u \, dx = D \int_{\Omega} u \nabla^2 \phi \, dx - D \int_{\partial \Omega} (u \partial \phi/\partial v) \, dS \geq -\lambda_0 |\Omega| \cdot q(t).
\]

This also leads to the relation (2.14). Since the function \( f(u) = e^{u(t)} \) is convex and \( |\Omega|^{-1} \int_{\Omega} \phi \, dx = 1 \), the Jensen’s inequality implies that (cf. [8])

\[
q' + \lambda_0 q \geq \sigma_1 \exp \left[ \gamma |\Omega|^{-1} \int_{\Omega} u \phi \, dx \right] = \sigma_1 \exp(\gamma q).
\]
Integration of the above inequality over \((t_1, T)\) for \(T > t_1 \geq 0\) gives
\[
T - t_1 \leq \int_{q(t_1)}^{q(T)} (\sigma_1 e^s - \lambda_0 s)^{-1} ds \leq \int_{q(t_1)}^{\gamma} (\sigma_1 e^s - \lambda_0 s)^{-1} ds. \tag{2.17}
\]
The integral at the right-side exists and is finite if \(H(s) = \sigma_1 e^s - \lambda_0 s > 0\).
Since \(H(s)\) is an increasing function of \(s\) when \(e^s > \lambda_0 / \sigma_1 \gamma\), we see that \(T\) is finite if either \(\lambda_0 \leq \sigma_1 \gamma\) and \(q(t_1) \geq 0\) or \(\lambda_0 > \sigma_1 \gamma\) and \(q(t_1) > \gamma^{-1} \ln(\lambda_0 / \sigma_1 \gamma)\). This shows that if there exists \(t_1 > 0\) such that \(q(t_1) > \gamma^{-1} \ln(\lambda_0 / \sigma_1 \gamma)\), the solution \(u(t, x)\) blows-up at some finite time. Now if \(T^*\) were not finite, then \(q(t)\) exists globally and is bounded by \(\gamma^{-1} \ln(\lambda_0 / \sigma_1 \gamma)\) for all \(t \geq 0\). Define
\[
\omega = \inf \left\{ |\Omega|^{-1} \int_\Omega u_1(t, x) \phi(x) \, dx; \ t > 0 \right\}.
\]
By the strict increasing property of \(u(t, x)\) in \(t\), \(u(t, x) > 0\); and by the mean-value theorem and \(u(0, x) = 0\),
\[
q(t) = |\Omega|^{-1} \int_\Omega u_1(\eta, x) \phi(x) \, dx \geq \omega t,
\]
where \(0 \leq \eta \leq t\). The bounded property of \(q(t)\) ensures that \(\omega = 0\), which yields the relation \(\lim u_1(t, x) = 0\) as \(t \to \infty\). Hence for any positive \(\epsilon < \sigma - \sigma^*\) there exists \(t_0 > 0\) such that \(u_1(t_0, x) \leq \epsilon\) in \(\Omega\). Set \(u_1(x) = u(t_0, x)\).
In view of (1.1) and (1.2),
\[
-D \nabla^2 \tilde{u}_\epsilon = \sigma \left[ e^{\tilde{u}_\epsilon} + b \int_\Omega e^{\tilde{u}_\epsilon(x')} \, dx' \right] - u_1(t_0, x) \\
\geq \sigma \left[ e^{\tilde{u}_\epsilon} + b \int_\Omega e^{\tilde{u}_\epsilon(x')} \, dx' \right] \quad \text{in} \ \Omega \\
B \tilde{u}_\epsilon = h(x) \quad \text{on} \ \partial \Omega
\]
where \(\sigma_\epsilon \equiv \sigma - \epsilon\). This shows that \(\tilde{u}_\epsilon\) is a positive upper solution of (1.2) corresponding to \(\sigma = \sigma_\epsilon\), and therefore there exists a positive steady-state solution. But this is impossible since \(\sigma_\epsilon > \sigma^*\). This contradiction shows that \(T^*\) must be finite. 

Remarks. (a) Since the function \(H(s) = \sigma_1 \gamma e^s - \lambda_0 s\) has a minimum at \(s_0 = \ln(\lambda_0 / \sigma_1 \gamma)\) and \(H(s_0) = \lambda_0 (1 - s_0) > 0\) when \(s_0 < 1\), the relation (2.17) implies that \(T < \infty\) when \(\sigma_1 \gamma > \lambda_0 / \epsilon\). In this situation, \(q(t)\) grows unbounded at some finite \(T^*\) for any \(q(0) \geq 0\), and therefore \(u(t, x)\) satisfies (2.11) for any \(u_0 \geq 0\). Hence no positive solution to (1.2) can exist if
\[ \sigma_1 > \lambda_0 / \gamma \varepsilon. \] Since there is always a positive steady-state solution when \( \sigma < \sigma_0 \) we conclude that
\[ \sigma_0 < \sigma^* < \lambda_0 / \gamma \varepsilon (1 + b \vert \Omega \vert / \phi). \] (2.18)

Note that \( \sigma_0 \) depends on \( \sigma \gamma \) as well as on \( h(x) \).

(b) It is obvious that an upper bound of the blowing-up time \( T^* \) can be obtained from (2.17). Since the solution of the equation \( Q' = \sigma \gamma e^{\gamma Q} \) is an upper solution of (1.1) for every \( T < T_1 \equiv (\sigma \gamma)^{-1} \gamma^{-1} \) when \( Q(0) \geq u_0 \) and \( \beta Q(0) \geq h \), we see that if \( h / \beta \) is finite then for any \( Q(0) \geq \max \{ \bar{u}_0, \bar{h} \} \), where \( \bar{u}_0, \bar{h} \) are the respective least upper bounds of \( u_0 \) and \( h / \beta \), the solution \( u(t, x) \) of (1.1) exists in \([0, T] \times \Omega \) and \( u(t, x) \leq Q(t) \). This implies that the blowing-up time \( T^* \) is bounded below by \( (\sigma \gamma e^{\gamma Q(0)})^{-1} \). Note that for the homogeneous Dirichlet boundary condition \( u = 0 \), \( Q(0) = u_0 \).

(c) For \( \sigma < \sigma^* \) a unique global solution \( u(t, x) \) to (1.1) exists and converges to the minimal solution \( u_+ \) of (1.2) when \( u_0 \leq u_+ \). In the case \( u_0 \leq \bar{u}_+ \), where \( \bar{u}_+ \) is the maximal solution of (1.2), the pair \( u = \bar{u}_+ \) and \( \bar{u} = 0 \) are upper and lower solutions, and therefore a unique global solution \( u(t, x) \) to (1.1) also exists and \( 0 \leq u(t, x) \leq \bar{u}_+(x) \). It is not clear in this situation whether the solution \( u(t, x) \) converges to a steady-state solution as \( t \to \infty \). However, if \( u(t_1, x) \geq 0 \) in \( \Omega \) at some \( t_1 > 0 \) then \( u(t_1, x) \) is a lower solution of (1.2), that is, \( v(x) \equiv u(t_1, x) \) satisfies
\[ -D \nabla^2 v \leq \sigma \left[ e^{\gamma v} + \int_{\Omega} e^{\gamma v(x')} dx' \right] \text{ in } \Omega, \quad Bv = h(x) \text{ on } \partial \Omega. \]

In this situation, \( u(t, x) \) is nondecreasing in \( t \) for \( t > t_1 \), and therefore it converges to a steady-state solution as \( t \to \infty \). The proof of this conclusion follows from the same reasoning as in the proof of Theorem 2.

(d) All the conclusions in Theorem 1, Theorem 2, and the above remarks hold true for the case \( b = 0 \). Therefore, these results are directly applicable to the classical thermal explosion model considered in [2–7, 10, 11].

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