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The Grothendieck Ring of the Category of Endomorphisms

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1. INTRODUCTION

The aim of this note is to compute the "additive invariants" of an endomorphism $f: P \to P$ where P is a finitely generated projective A-module (A = commutative ring). Our main theorem states that *there are no other additive invariants of f than the class $[P]$ in the Grothendieck group $K_0(A)$ of A and the characteristic polynomial of f.* In order to make this statement precise we need the following definitions.

A will always denote a commutative ring with identity element 1. Following Bass' notation we have categories of A-modules

$$\mathsf{P}(A) \subseteq \mathsf{H}(A) \subseteq \mathsf{M}(A)$$

where M is in $\mathsf{P}(A)$ if M is finitely generated and projective, M is in $\mathsf{H}(A)$ if M has a finite resolution consisting of modules in $\mathsf{P}(A)$ and finally M is in $\mathsf{M}(A)$ if M is finitely generated.

Let end $\mathsf{M}(A)$ denote the category where the objects are endomorphisms $f: M \to M$ with M in $\mathsf{M}(A)$ and the morphisms are commutative diagrams

$$\begin{array}{ccc} M & \xrightarrow{u} & N \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ M & \xrightarrow{u} & N \end{array}$$

Analogously are end $\mathsf{P}(A)$ and end $\mathsf{H}(A)$ defined (M in $\mathsf{P}(A)$, $\mathsf{H}(A)$, respectively). The Grothendieck group $K_0(\text{end } \mathsf{M}(A))$ is then the free abelian group generated by all isomorphism classes of objects in end($\mathsf{M}(A)$) modulo the subgroup generated by all $[f] - [f'] - [f'']$ where

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{u} & M & \xrightarrow{v} & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & M' & \xrightarrow[u]{} & M & \xrightarrow[v]{} & M'' & \longrightarrow & 0 \end{array}$$

is exact and commutative. Similarly we define K_0(end $\mathsf{P}(A)$ and K_0(end $\mathsf{H}(A)$).

The problem we consider is to compute these groups. This is completely solved only for K_0(end $\mathsf{P}(A)$). The tensor product $f \otimes g$ induces a multiplication on K_0(end $\mathsf{P}(A)$) making it to a commutative ring.

We have two ring homomorphisms

$$K_0(\text{end } \mathsf{P}(A)) \underset{\psi}{\overset{\pi}{\rightleftarrows}} K_0(A)$$

where

$$\pi[M, f] = [M] \qquad (\text{forget } f)$$

and

$$\psi[M] = [M, 0].$$

(here $K_0(A) = K_0(\mathsf{P}(A))$ is the Grothendieck group of A). Clearly $\pi \circ \psi =$ identity so we get

$$K_0(\text{end } \mathsf{P}(A)) \simeq K_0(A) \times \tilde{K}_0(\text{end } \mathsf{P}(A))$$

as rings. This defines $\tilde{K}_0$(end $\mathsf{P}(A)$). Loosely speaking it is the Grothendieck group of endomorphisms where we consider the zero map as 0. Similarly we have

$$K_0(\text{end } \mathsf{M}(A)) \cong G_0(A) \times \tilde{K}_0(\text{end } \mathsf{M}(A))$$

as abelian groups where $G_0(A) = K_0(\mathsf{M}(A))$.

In order to state the main theorem we have to introduce the characteristic polynomial $\lambda_t(f)$ of an endomorphism $f: P \to P$ with P in $\mathsf{P}(A)$. It is defined as

$$\lambda_t(f) = \sum_{i \geqslant 0} \operatorname{Tr}\left(\bigwedge^i f\right) t^i$$

where Tr denotes the trace and t is an indeterminate (see [1] for details). If P is free then $\lambda_t(f) = \det(1 + tf)$. Using localization it is easily shown that:

(i) $\lambda_t(0) = 1$;

(ii) if

$$\begin{tikzcd} 0 \arrow[r] & P' \arrow[r] \arrow[d, "f'"] & P \arrow[r] \arrow[d, "f"] & P'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & P' \arrow[r] & P \arrow[r] & P'' \arrow[r] & 0 \end{tikzcd}$$

is exact then $\lambda_t(f) = \lambda_t(f')\,\lambda_t(f'')$. It follows that λ_t is defined on $\tilde{K}_0(\text{end }\mathsf{P}(A))$ by

$$\lambda_t([f] - [g]) = \lambda_t(f)/\lambda_t(g).$$

Hence we have a group homomorphism

$$\lambda_t\colon \tilde{K}_0(\text{end }\mathsf{P}(A)) \to \tilde{A}_0 = \{(1 + a_1t + \cdots + a_nt^n)/1 + b_1t + \cdots b_mt^m;\ a_i, b_j \in A\},$$

where $\tilde{A}_0$ is a group under multiplication. It is easily seen that λ_t is surjective and we can define a $*$-multiplication on $\tilde{A}_0$ by the formula

$$\lambda_t(f) * \lambda_t(g) = \lambda_t(f \otimes g).$$

Then $\tilde{A}_0$ becomes a commutative ring. To indicate the flavor of the $*$-multiplication we write out the formula

$$\begin{aligned}(1 + a_1t + a_2t^2 + \cdots + a_nt^n) * (1 + b_1t + b_2t^2 + \cdots + b_mt^m) \\ = 1 + a_1b_1t + (a_1{}^2b_2 + a_2b_1{}^2 - 2a_2b_2)\,t^2 + \cdots + a_n{}^mb_m{}^nt^m\end{aligned}$$

(the coefficient of t^{10} contains more than 700 monomials!). The ring $\tilde{A}_0$ is isomorphic to a certain subring of the Witt ring $W(A)$ of A (see Almkvist [1] for details).

MAIN THEOREM. *Let A be any commutative ring. Then the map*

$$\lambda_t\colon \tilde{K}_0(\text{end }\mathsf{P}(A)) \to \tilde{A}_0$$

is a ring isomorphism.

This was proved for the cases $A =$ field by Kelley–Spanier [5, p. 327]; $A = PID$, $A = K[X_1, X_2]$ and $A =$ regular local ring of dimension less than 3 in Almkvist [1, 6.6].

The computation of $\tilde{K}_0(\text{end }\mathsf{M}(A))$ seems to be more difficult. We always have

$$K_0(\text{end }\mathsf{H}(A)) \simeq K_0(\text{end }\mathsf{P}(A))$$

which is proved by the use of resolutions (see [1, 5.2]). The ring A is called *regular* (in the sense of Bass) if $\mathsf{M}(A) = \mathsf{H}(A)$, i.e., every finitely generated A-module M has a finite resolution in $\mathsf{P}(A)$. This is the case if A is noetherian and $A_{\mathfrak{p}}$ is a regular local ring for every prime ideal $\mathfrak{p}$ in A. If A is regular we have of course

$$K_0(\text{end } \mathsf{M}(A)) \cong K_0(\text{end } \mathsf{P}(A)) \cong K_0(A) \times \tilde{A}_0 ,$$

but if A is not regular there is to my knowledge no definition of a characteristic polynomial of $f: M \to M$ for M in $\mathsf{M}(A)$. However, we have the following.

THEOREM. *If A is artinian then*

$$\tilde{K}_0(\text{end } \mathsf{M}(A)) \cong (\widetilde{A/J(A)})_0$$

where $J(A)$ is the Jacobson radical of A.

Hence if $A = Z/4Z$ we have

$$\tilde{K}_0(\text{end } \mathsf{P}(Z/4Z)) \cong (\widetilde{Z/4Z})_0$$

and

$$\tilde{K}_0(\text{end } \mathsf{M}(Z/4Z)) \cong (\widetilde{Z/2Z})_0 \cap (K_0(A) = G_0(A) = Z).$$

We use a method due to Claburn–Fossum [3] to compute $K_0(\text{end } \mathsf{M}(A))$ when A is the coordinate ring of the cusp $x^3 = y^2$. This can possibly be generalized to higher dimensional nonregular rings.

Finally it might be worth while to remark that, since the category of vector bundles $VB(X)$ over a compact Hausdorff space X is equivalent to $\mathsf{P}(C(X))$ where $C(X)$ is the ring of continuous complex-valued functions on X, we have also proved that

$$K_0(\text{end } VB(X)) \cong K_0(X) \times \widetilde{C(X)}_0 .$$

2. Proof of the Main Theorem

We are going to work in the category of $A[t]$-modules. If $f: P \to P$ is A-linear then P can be considered as an $A[t]$-module where the $A[t]$-action is defined by $t \cdot x = f(x)$ for all x in P. Conversely to any $A[t]$-module P

corresponds an endomorphism $P \xrightarrow{t\cdot} P$. The commutative exact diagram of A-modules

$$\begin{tikzcd} 0 \arrow[r] & P' \arrow[r, "u"] \arrow[d, "f'"] & P \arrow[r, "v"] \arrow[d, "f"] & P'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & P' \arrow[r, "u"'] & P \arrow[r, "v"'] & P'' \arrow[r] & 0 \end{tikzcd}$$

is the same thing as a short exact sequence.

$$0 \to P' \to P \to P'' \to 0$$

in mod $A[t]$. If P is projective as A-module then P is not necessarily projective as $A[t]$-module (rather $hd_{A[t]}P \leqslant 1$). But we may identify end $\mathsf{P}(A)$ with the category of $A[t]$-modules P such that P considered as A-module is in $\mathsf{P}(A)$. Call this category C.

First we reduce to the case when P is A-free. Given $f: P \to P$ with P in $\mathsf{P}(A)$ there exists Q in $\mathsf{P}(A)$ such that $P \oplus Q = F$ is free. We get an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & P \arrow[r] \arrow[d, "f"] & F \arrow[r] \arrow[d, "f_1"] & Q \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & P \arrow[r] & F \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

where $f_1 = f \oplus 0$. Hence in $K_0(\text{end } \mathsf{P}(A))$ we have

$$[f] = [f_1] - [0]$$

where $[0] = [0_Q]$ can be identified with $[Q]$ in $K_0(A)$ and $[f] \equiv [f_1] \bmod K_0(A)$.

Thus we may assume that $P = A^n$ and $f: P \to P$ is given by an $n \times n$-matrix $f = (a_{ik})$. We are going to use the "characteristic sequence of f" (see [2, p. 630]), i.e.,

$$0 \longrightarrow B^n \xrightarrow{g} B^n \xrightarrow{\pi} P \longrightarrow 0. \qquad (*)$$

Here $B = A[t]$ and the sequence is exact in mod B. The B-linear maps g and π are given by

$$g = (b_{ik}) \qquad \text{where} \quad b_{ik} = t\delta_{ik} - a_{ik}$$

$$\pi\left(\sum_{i=0}^{k} t^i u_i\right) = \sum_{i=0}^{k} f^i(u_i)$$

(a vector in B^n can be written $\sum_{i=0}^{k} t^i u_i$ with $u_i \in A^n = P$). Denote by S the multiplicative set consisting of all monic polynomials in $A[t] = B$.

Claim. *$\tilde{K}_0(\text{end } \mathsf{P}(A)) = K_0(\mathsf{C}) \bmod K_0(A)$ is generated by all $[B/(s)]$ where $s \in S$.*

Now $K_0(\text{end } \mathsf{P}(A)) = K_0(\text{end } \mathsf{H}(A))$ so sometimes it might be convenient to use modules in $\mathsf{H}(A)$ instead of $\mathsf{P}(A)$. Let C_1 be the category of B-modules M such that M considered as A-module is in $\mathsf{H}(A)$. We have $K_0(\mathsf{C}_1) = K_0(\text{end } \mathsf{H}(A)) = K_0(\text{end } \mathsf{P}(A))$. Starting with M in C_1 we resolve M in $\mathsf{P}(A)$ and modulo $K_0(A)$ we get $[M]$ is (in $K_0(\mathsf{C}_1)$) a sum of "modules" $\pm[P]$ having characteristic sequences of type $(*)$.

Consider the following statement L_n: If $0 \to B^n \to^g B^n \to M \to 0$ is exact in mod B where M is in $\mathsf{H}(A)$ and $g = (b_{ik})$ is an $n \times n$-matrix with b_{ik} in B such that $b_{ii} \in S$ and $\deg b_{ii} > \deg b_{jk}$ for all i, $j \neq k$, then $[M]$ can in $K_0(\mathsf{C}_1)$ be written as a finite sum $\sum_i \pm [B/(s_i)]$ with $s_i \in S$.

By the above considerations the claim will be proved if we can prove L_n for all n. This we are going to by induction on n. If $n = 1$ then g is multiplication by $s \in S$ so $M \simeq B/(s)$ and we are done. Now assume L_{n-1} is true and let

$$0 \longrightarrow B^n \xrightarrow{g} B^n \longrightarrow M \longrightarrow 0$$

be exact with M in $\mathsf{H}(A)$ and $g = (b_{ik})$ satisfying the assumptions in L_n. Consider the following diagram in mod B:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & B^n \arrow[r, "g"] \arrow[d, equal] & B^n \arrow[r, "\pi"] \arrow[d, "h"] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B^n \arrow[r, "g'"] & B^n \arrow[r, "\pi'"] \arrow[d, "\phi"] & M' \arrow[r] \arrow[d] & {} \\
 & & C_1 \arrow[r, "\simeq"] \arrow[d] & C_2 \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

where

$$g = \begin{pmatrix} s & \mathbf{b}^{\mathrm{T}} \\ \mathbf{c} & g_0 \end{pmatrix} = (b_{ik}) \qquad \text{with} \quad b_{11} = s \in S;$$

$\mathbf{b}$, $\mathbf{c}$ are columnvectors of length $n-1$; $\mathbf{b}^{\mathrm{T}}$ means the transpose of $\mathbf{b}$, g_0 is an $(n-1)\times(n-1)$-matrix of the same type as g. Furthermore

$$g' := \begin{pmatrix} s & \mathbf{b}^{\mathrm{T}} \\ 0 & g_1 \end{pmatrix}$$

$$h := \begin{pmatrix} 1 & 0 \\ -\mathbf{c} & sI_{n-1} \end{pmatrix}$$

where I_{n-1} is the identity matrix of size $n-1$. We have

$$h \circ g = \begin{pmatrix} 1 & 0 \\ -\mathbf{c} & sI_{n-1} \end{pmatrix}\begin{pmatrix} s & \mathbf{b}^{\mathrm{T}} \\ \mathbf{c} & g_0 \end{pmatrix} = \begin{pmatrix} s & \mathbf{b}^{\mathrm{T}} \\ 0 & g_1 \end{pmatrix} = g'$$

with $g_1 = sg_0 - \mathbf{c}\mathbf{b}^{\mathrm{T}}$. Note that all entries in the main diagonal of g' and g_1 are in S and have degrees strictly larger than the degree of any entry outside the main diagonal (we want g_1 to satisfy the assumptions of L_{n-1}). Furthermore g', g_1, and h are injective since their determinants are in S and hence are nonzero divisors in B. We now compute $C_1 := \operatorname{coker} h$ by noting

$$h\mathbf{x} = \begin{pmatrix} 1 & 0 \\ -\mathbf{c} & sI_{n-1} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ sx_2 - b_{21}x_1 \\ sx_3 - b_{31}x_1 \\ \cdots\cdots \\ sx_n - b_{n1}x_1 \end{pmatrix}.$$

We claim that $C_1 = (B/(s))^{n-1}$ and $\phi\colon B^n \to (B/(s))^{n-1}$ given by

$$y := \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \longrightarrow \begin{bmatrix} \overline{y_2 + b_{21}y_1} \\ \overline{y_3 + b_{31}y_1} \\ \cdots\cdots \\ \overline{y_n + b_{n1}y_1} \end{bmatrix} \text{ works.}$$

Clearly ϕ is surjective and $\mathbf{y} \in \ker\phi$ if and only if

$$\begin{aligned} y_2 + b_{21}y_1 &= sx_2 \qquad \text{for some } x_2, \ldots, x_n \text{ in } B, \\ y_3 + b_{31}y_1 &= sx_3 \\ \cdots\cdots&\cdots \\ y_n + b_{n1}y_1 &= sx_n . \end{aligned}$$

This is equivalent to

$$\begin{aligned} y_1 &= x_1 \\ y_2 &= sx_2 - b_{21}x_1 \qquad \text{i.e., } \mathbf{y} \text{ is in } \operatorname{Im} h. \\ &\cdots\cdots\cdots \\ y_n &= sx_n - b_{n1}x_1 \end{aligned}$$

Hence the middle column is exact. Now $B/(s) \simeq A^d$ as A-modules where d is the degree of s (recall that s is monic). It follows that

$$C_2 \simeq C_1 = (B/(s))^{n-1}$$

is free as A-module and M' is in $\mathsf{H}(A)$. Thus

$$[M] = [M'] - (n-1)[B/(s)] \qquad \text{in } K_0(\mathsf{C}_1).$$

The next step is to reduce the size of M' to $n - 1$. This is done in the following commutative diagram.

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & B \arrow[r, "s"] \arrow[d] & B \arrow[r] \arrow[d] & B/(s) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B^n \arrow[r, "g'"] \arrow[d] & B^n \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B^{n-1} \arrow[r, "g_1"] \arrow[d] & B^{n-1} \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

where the maps in the first two columns are the natural injections and surjections. Clearly M_1 is in $\mathsf{H}(A)$ since $B/(s)$ and M' are. Hence

$$[M'] = [M_1] + [B/(s)] \qquad \text{in } K_0(\mathsf{C}_1).$$

But the last row in the diagram satisfies the assumptions of L_{n-1} so by the induction hypothesis

$$[M_1] = \sum_i \pm [B/(s_i)] \qquad \text{in} \quad K_0(\mathsf{C}_1).$$

This proves the claim.

As mentioned in the introduction we have a surjective ring homomorphism

$$\lambda_t\colon \tilde{K}_0(\text{end } \mathsf{P}(A)) \to \tilde{A}_0 .$$

We have to construct an inverse

$$\sigma\colon \tilde{A}_0 \to \tilde{K}_0(\text{end } \mathsf{P}(A)).$$

Let $(r_1/r_2) = (1 + a_1 t + \cdots + a_n t^n/1 + b_1 t + \cdots + b_m t^m) \in \tilde{A}_0$. Then we must have $\sigma(r_1/r_2) = \sigma(r_1) - \sigma(r_2)$ so it is sufficient to define $\sigma(r)$ when $r = 1 + a_1 t + \cdots + a_n t^n$. Put $\tilde{r} = t^n - a_1 t^{n-1} + a_2 t^{n-2} + \cdots + (-1)^n a_n$. Then $\tilde{r} \in S$ and we define $\sigma(r) = [B/(\tilde{r})]$. Now,

$$t^d \widetilde{r_1 r_2} = \tilde{r}_1 \cdot \tilde{r}_2$$

which implies

$$[B/(\widetilde{r_1 r_2})] = [B/(\tilde{r}_1)] + [B/(\tilde{r}_2)] - [B/(t^d)] \qquad \text{in} \quad K_0(\mathsf{C}_1),$$

since

$$0 \to B/(s_1) \to B/(s_1 s_2) \to B/(s_2) \to 0$$

is exact because $s_1, s_2 \in S$ are nonzero divisors in B. But $[B/(t^d)] = d[B/(t)] = 0$ in $\tilde{K}_0(\text{end } \mathsf{P}(A))$ since $B/(t)$ corresponds to the zero-map $0\colon A \to A$. Thus we have proved

$$\sigma(r_1 r_2) = \sigma(r_1) + \sigma(r_2).$$

Similar computations show that σ is well defined. Now $\sigma(r) = [B/(\tilde{r})] = [A[t]/(t^n - a_1 t + \cdots \pm a_n)]$ corresponds to the $n \times n$-matrix

$$f = \begin{pmatrix} 0 & 0 & & 0 & 0 & \pm a_n \\ 1 & 0 & & 0 & 0 & \mp a_{n-1} \\ 0 & 1 & & 0 & 0 & \pm a_{n-2} \\ - & - & - & - & - & - \\ 0 & 0 & & 1 & 0 & -a_2 \\ 0 & 0 & & 0 & 1 & a_1 \end{pmatrix}.$$

Then

$$\lambda_t \cdot \sigma(r) = \lambda_t(f) = \det(1 + tf) = 1 + a_1 t + \cdots + a_n t^n = r$$

so

$$\lambda_t \cdot \sigma = \text{identity}.$$

We want $\sigma \cdot \lambda_t =$ identity. It is sufficient to check this on the generators of $\tilde{K}_0(\text{end } \mathsf{P}(A))$ which by the claim are of the type $[B/(s)]$ with $s \in S$ (modules of type $[B/(t^d)]$ don't occur since they are zero in $\tilde{K}_0(\text{end } \mathsf{P}(A))$). Doing exactly the same computations as above with s instead of $\tilde{r}$ we see that $\sigma \cdot \lambda_t =$ identity. This ends the proof.

3. K_0(END $\mathsf{M}(A)$)

As mentioned in the introduction the computation of $K_0(\text{end } \mathsf{M}(A))$ is more difficult. First we note that every $f: M \to M$ with M in $\mathsf{M}(A)$ satisfies a monic equation $s(f) = 0$ (see [1, 4.1]). Moving over into mod $A[t]$ we get

$$K_0(\text{end } \mathsf{M}(A)) = G_0(A[t], S)$$

where $G_0(A[t], S)$ is the Grothendieck group of the category of all $A[t]$-modules such that $S^{-1}M = 0$ (which is equivalent to that M is killed by a monic polynomial).

Proposition 3.1. *Let J be a nilpotent ideal in A. Then*

$$K_0(\text{end } \mathsf{M}(A)) \cong K_0(\text{end } \mathsf{M}(A/J)).$$

Proof. Given M in mod $A[t]$ with $S^{-1}M = 0$ we have a filtration

$$M \supseteq JM \supseteq J^2M \supseteq \cdots \supset J^kM = 0.$$

Hence,

$$[M] = \sum_i [J^iM/J^{i+1}M] \qquad \text{in} \quad G_0(A[t], S)$$

since all J^iM and $J^iM/J^{i+1}M$ are killed by a monic polynomial. But all $J^iM/J^{i+1}M$ are killed by J and can be considered as $A/J[t]$-modules. It follows that

$$G_0(A[t], S) \cong G_0(A/J[t], \bar{S})$$

where $\bar{S}$ is the set of monic polynomials in $A/J[t]$, (compare [2, p. 402]).

Theorem 3.2. (i) *Let A be a local artinian ring, i.e., A has precisely one prime ideal $\mathfrak{p}$. Then*

$$\tilde{K}_0(\text{end } \mathsf{M}(A)) \cong \tilde{k}_0 \qquad \text{where} \quad k = A/\mathfrak{p}$$

(ii) *If A is artinian then*

$$\tilde{K}_0(\text{end}\ \mathsf{M}(A)) \simeq (\widetilde{A/J(A)})_0 \qquad \text{where} \quad J(A)$$

is the Jacobson radical of A.

Proof. Since (i) follows from (ii) with $\mathfrak{p} = J(A)$ it is sufficient to show (ii). The Jacobson radical $J(A)$ is nilpotent and by 3.1 we get

$$K_0(\text{end}\ \mathsf{M}(A)) \simeq K_0(\text{end}\ \mathsf{M}(A/J(A))).$$

But $A/J(A)$ is semisimple so every $A/J(A)$ module is projective, i.e., $\mathsf{M}(A/J(A)) = \mathsf{P}(A/J(A))$. It follows

$$\begin{aligned} K_0(\text{end}\ \mathsf{M}(A/J(A))) &= K_0(\text{end}\ \mathsf{P}(A/J(A))) \\ &\simeq K_0(A/J(A)) \times (\widetilde{A/J(A)})_0 \end{aligned}$$

by the main theorem. But $G_0(A) = G_0(A/J(A)) = K_0(A/J(A))$ so we are done.

The next case to try is A noetherian. One can possibly use a filtration à la Claburn–Fossum [3].

PROPOSITION 3.3. *Let A be noetherian. Then $K_0(\text{end}\ \mathsf{M}(A))$ is generated by all $[A[t]/\mathfrak{p}]$ where $\mathfrak{p}$ is a prime ideal in $A[t]$ such that $\mathfrak{p} \cap S \neq \varnothing$.*

Proof. $A[t]$ is noetherian and given M in $\mathsf{M}(A[t])$ such that $S^{-1}M = 0$ we have a filtration

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_k = 0$$

such that

$$M_i/M_{i+1} \simeq A[t]/\mathfrak{p}$$

where $\mathfrak{p}$ is a prime ideal in $A[t]$. Now M and hence M_i and M_i/M_{i+1} are killed by a monic polynomial. It follows that $\mathfrak{p}$ contains a monic polynomial.

Now we try to copy some of the computations made in [3]. Let ht $\mathfrak{p}$ denote the height of a prime ideal $\mathfrak{p}$ in $A[t]$ and let i be a nonnegative integer. We define

$$\mathscr{N}_i = \{M \in \mathsf{M}(A\,[t]);\ S^{-1}M = 0 \text{ and } \mathfrak{p} \in \operatorname{supp} M \Rightarrow \operatorname{ht} \mathfrak{p} \geq i\}.$$

Thus we have a filtration

$$\mathscr{N}_1 \supset \mathscr{N}_2 \supset \mathscr{N}_3 \supset \cdots$$

of subcategories of $\mathcal{N}_1 \simeq \operatorname{end} \mathsf{M}(A)$ (the case $\operatorname{ht}\mathfrak{p} = 0$ and $\mathfrak{p} \cap S \neq \varnothing$ cannot occur). Clearly $\mathcal{N}_{i+1}$ is a Serre subcategory of the abelian category $\mathcal{N}_i$. Then we can form the quotient category $\mathcal{N}_i/\mathcal{N}_{i+1}$ (compare [3, p. 230]).

PROPOSITION 3.4. *Let A be noetherian. Then*

$$K_0(\mathcal{N}_i/\mathcal{N}_{i+1}) \simeq D_i(A[t], S)$$

where $D_i(A[t], S)$ is the free abelian group generated by all prime ideals $\mathfrak{p}$ in $A[t]$ such that $\mathfrak{p} \cap S \neq \varnothing$ and $\operatorname{ht}\mathfrak{p} = i$. The isomorphism is given by

$$[M] \mapsto \sum 1_{\mathfrak{p}}(M_{\mathfrak{p}})\langle \mathfrak{p} \rangle$$

where the sum is extended over all $\mathfrak{p} \in \operatorname{Spec} A[t]$ such that $\mathfrak{p} \cap S \neq \varnothing$ and $\operatorname{ht}\mathfrak{p} = i$ (and $1_{\mathfrak{p}}(M_{\mathfrak{p}})$ means the length of $M_{\mathfrak{p}}$ as $A[t]_{\mathfrak{p}}$-module).

Proof. The proofs of 2.1–2.4 in Claburn–Fossum [3, p. 230–231] go through word for word. The only thing we have to change is that prime ideal should now mean prime ideal $\mathfrak{p}$ such that $\mathfrak{p} \cap S \neq \varnothing$. Similarly one gets

$$K_1(\mathcal{N}_i/\mathcal{N}_{i+1}) \simeq \bigoplus k(\mathfrak{p})^*$$

where $k(\mathfrak{p})^*$ is the multiplicative group of nonzero elements of $k(\mathfrak{p})$ = the quotient field of $A[t]/\mathfrak{p}$ and the direct sum is extended over all $\mathfrak{p}$ in spec $A[t]$ with $\operatorname{ht}\mathfrak{p} = i$ and $\mathfrak{p} \cap S \neq \varnothing$.

For all i, j with $i < j$ we have an exact sequence of categories

$$0 \to \mathcal{N}_{i+1}/\mathcal{N}_j \to \mathcal{N}_i/\mathcal{N}_j \to \mathcal{N}_i/\mathcal{N}_{i+1} \to 0.$$

This gives rise to an exact sequence of abelian groups

$$K_1(\mathcal{N}_i/\mathcal{N}_{i+1}) \xrightarrow{\partial} K_0(\mathcal{N}_{i+1}/\mathcal{N}_j) \xrightarrow{\mu} K_0(\mathcal{N}_i/\mathcal{N}_j) \xrightarrow{\nu} K_0(\mathcal{N}_i/\mathcal{N}_{i+1}) \longrightarrow 0$$

since each object of $\mathcal{N}_i/\mathcal{N}_{i+1}$ is of finite length (see Heller [4, Theorem 10.2, p. 406]). Now $K_0(\mathcal{N}_i/\mathcal{N}_{i+1}) \simeq D_i(A[t], S)$ is free and hence,

$$K_0(\mathcal{N}_i/\mathcal{N}_j) \simeq D_i(A[t], S) \oplus \operatorname{Im}\mu$$

since $\ker\nu = \operatorname{Im}\mu$. But $\operatorname{Im}\mu \simeq K_0(\mathcal{N}_{i+1}/\mathcal{N}_j)/\operatorname{Im}\partial$ since $\ker\mu = \operatorname{Im}\partial$. Thus it is important to know the image of

$$\partial \colon \bigoplus k(\mathfrak{p})^* \to K_0(\mathcal{N}_{i+1}/\mathcal{N}_j).$$

Let $(\bar{x}_{\mathfrak{p}}) \in \bigoplus k(\mathfrak{p})^*$ be given with $\bar{x}_{\mathfrak{p}} = \bar{u}_{\mathfrak{p}}/\bar{v}_{\mathfrak{p}}$ where $u_{\mathfrak{p}}, v_{\mathfrak{p}} \in A[t]$ both not in $\mathfrak{p}$, (here $\mathfrak{p}$ runs through all primes in $A[t]$ such that ht $\mathfrak{p} = i$ and $\mathfrak{p} \cap S \neq \varnothing$. Put

$$\partial((\bar{x}_{\mathfrak{p}})) = \sum_{\mathfrak{p}} ([A[t]/(\mathfrak{p} + u_{\mathfrak{p}}A[t])] - [A[t]/(\mathfrak{p} + v_{\mathfrak{p}}A[t])]).$$

Now assume that A has finite Krull dimension d. Then $\mathcal{N}_j = 0$ if $j = d + 2$ and $K_0(\mathcal{N}_i/\mathcal{N}_j) = K_0(\mathcal{N}_i)$. Therefore, it is possible to compute $K_0(\mathcal{N}_{d+1})$, $K_0(\mathcal{N}_d)$, $K_0(\mathcal{N}_{d-1})$,..., and finally $K_0(\mathcal{N}_1) = K_0(\text{end } \mathsf{M}(A))$ if one knows the structure of the prime ideals in $A[t]$. This may, however, be rather complicated so we will do the computation only for an example.

THEOREM 3.5. *Let k be an algebraically closed field and $A = k[s^2, s^3]$, i.e., A is the coordinate ring of the cusp $x^3 = y^2$. Then*

$$K_0(\text{end } \mathsf{M}(A)) \cong D_1(A[t], S) = Z \times \widetilde{k[s]}_0 .$$

Proof. It is sufficient to show that

$$\partial\colon K_1(\mathcal{N}_1/\mathcal{N}_2) \to K_0(\mathcal{N}_2)$$

is surjective. Let $\mathfrak{q}$ be a prime ideal in $A[t]$ of ht 2. Then $\mathfrak{q}$ is maximal and corresponds to a point $(\alpha^2, \alpha^3, \beta)$ on the surface $x^3 = y^2$. Thus we can write

$$\mathfrak{q} = (s^2 - \alpha^2, s^3 - \alpha^3, t - \beta).$$

Take $\mathfrak{p} = ((t - \beta + \alpha)^2 - s^2,\ (t - \beta + \alpha)^3 - s^3)$. Then $\mathfrak{p}$ is prime of ht one in $A[t]$ since

$$A[t]/\mathfrak{p} = k[s^2, s^3, t]/\mathfrak{p} \cong k[(t - \beta + \alpha)^2, (t - \beta + \alpha)^3, t] = k[t].$$

Furthermore $\mathfrak{p} \cap S \neq \varnothing$ and $\mathfrak{q} = (\mathfrak{p}, t - \beta)$ is in Im ∂. We have proved

$$K_0(\text{end } \mathsf{M}(A)) \cong D_1(A[t], S).$$

Now $k[s, t]$ is the integral closure of $k[s^2, s^3, t]$. The map $D_1(k[s, t], S') \to D_1(A[t], S)$ induced by

$$\mathfrak{q} \mapsto \mathfrak{q} \cap A[t]$$

is surjective by the Cohen–Seidenberg going-up theorem. It is also injective since the primes of ht one in $k[s, t]$ are principal and

$$(f) \cap k[s^2, s^3, t] = (g) \cap k[s^2, s^3, t]$$

implies $(f) = (g)$ (here f and g are monic in t). This finishes the proof.

Remark. The method in the proof works for any curve which can be parametrized $x = g_1(s)$, $y = g_2(s)$ where g_1, g_2 are polynomials (e.g., the node $y^2 = x^2 - x^3$ with $x = 1 - s^2$, $y = s - s^3$).

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