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Conjugacy invariants of $Sl(2, \mathbb{H})$

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Abstract

In this paper, we use the quaternionic formalism of Möbius transformations on $\widehat{\mathbb{R}}^4$ to derive conjugacy invariants on $Sl(2, \mathbb{H})$ (and hence on $PSl(2, \mathbb{H})$). We then use these invariants to distinguish between the various conjugacy classes of $PSl(2, \mathbb{H})$.

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1. Introduction

Let $\widehat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\} \cong S^n$ be the one-point compactification of n-dimensional Euclidean space. A *Möbius transformation* on $\widehat{\mathbb{R}}^n$ is the composition of an even number of inversions through spheres or hyperplanes. Let $\mathscr{M}_n$ be the space of all such transformations. It is well known that, via the Poincaré disk model B^{n+1}, each element of $\mathscr{M}_n$, which acts on $S^n = \partial B^{n+1} \subset \mathbb{R}^{n+1}$, can be extended to a unique orientation-preserving isometry of $(n+1)$-dimensional hyperbolic space (see [16, Chapter 4]).

If we restrict ourselves to $n = 2$, each Möbius transformation acts on $\widehat{\mathbb{C}} = \widehat{\mathbb{R}}^2$. It has been known for some time (see [4]) that each element of $\mathscr{M}_2$ can be represented by a linear fractional from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$ given by $z \mapsto \frac{az+b}{cz+d}$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2, \mathbb{C})$. This gives a homomorphic map from $\mathscr{M}_2$ to $Sl(2, \mathbb{C})$, which is unique up $\pm I$ in $Sl(2, \mathbb{C})$.

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Thus, the study of the elements of $\mathscr{M}_2$ becomes essentially the study of $Sl(2, \mathbb{C})$ (see [3,13]).

A classical result of this complex formalism is the classification of the Möbius transformations on the plane by the trace function, essentially the only conjugate invariant of $Sl(2, \mathbb{C})$. For a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2, \mathbb{C})$, we define $\mathrm{tr}(A) = a + d$. Then, for $g \in \mathscr{M}_2$ represented by A,

(1) g is parabolic, i.e. g has exactly one fixed point in $\widehat{\mathbb{C}}$, if and only if $\mathrm{tr}(A)^2 = 4$;
(2) g is elliptic, i.e. g has two fixed points in $\widehat{\mathbb{C}}$ and fixes a geodesic in $\widehat{\mathbb{R}}^3$, if and only if $\mathrm{tr}(A)^2 \in [0, 4)$;
(3) g is hyperbolic, i.e. g has two fixed points in $\widehat{\mathbb{C}}$ and preserves an open disc in $\widehat{\mathbb{C}}$, if and only if $\mathrm{tr}(A)^2 \in (4, \infty)$.

This matrix formalism can be extended to all dimensions by using 2×2 matrices whose entries are Clifford numbers. This approach was originally introduced in 1901 by Vahlen in [17], but forgotten until it was rediscovered by Maass in 1949 [14], after which it was forgotten again and then re-rediscovered in the late 1980s by Ahlfors (see [1] for more details). This has been a very fruitful method of studying Möbius transformations, (e.g. [6]), but it is not only way to represent these transformations, and it is not actually the approach we will be using here.

Similar to the case where $n = 2$, if we restrict ourselves to $n = 4$, then we have a quaternionic formalism at our disposal, namely quaternionic linear fractionals:

$$h \mapsto (ah + b)(ch + d)^{-1}.$$

Apparently inspired by Coxeter's work with quaternions in [7], Wilker in [18] (and later Moussafir in [15]) proved that $\mathscr{M}_5$ (which acts on $\widehat{\mathbb{H}} = \widehat{\mathbb{R}}^4$) is 1-to-2 homomorphic with the matrix space $Sl(2, \mathbb{H})$, i.e. $\mathscr{M}_5 = PSl(2, \mathbb{H}) = Sl(2, \mathbb{H})/\pm I$.

In this paper, we use this representation of $\mathscr{M}_5$ to study the conjugacy classes and some conjugacy invariants of $PSl(2, \mathbb{H})$ with an eye towards replicating as much as possible the role that the trace function fulfills for $Sl(2, \mathbb{C})$. In particular, there are three conjugate invariants which allows for a distinction between several classes of transformations. However they will not allow us to characterize completely between elliptic and parabolic transformations. In order to do this, some work dealing with the solutions of quadratic equations in quaternionic variables needs to be explored. Many of the results contained herein are happy consequences of recent work regarding the solutions to quaternionic linear and quadratic equations [9–11].

2. Preliminary results regarding quaternions and quaternionic matrices

This section is mostly a review of the pertinent facts regarding quaternions and quaternionic matrices. For much more detail, see [19] or [8, Chapters 7 and 8].

A *quaternion* is an element of the vector space $\mathbb{H} = \langle 1, i, j, k\rangle_{\mathbb{R}}$. Multiplication is defined by specifying 1 as the multiplicative identity and

$$i^2 = j^2 = k^2 = -1, \quad ij = k = -ji.$$

For $h = x_1 + x_2 i + x_3 j + x_4 k \in \mathbb{H}$ with $x_1, x_2, x_3 \in \mathbb{R}$, the *real* part of h is x_1, which we denote by $\mathrm{Re}(h)$, and the *vectorial* part of h is $x_2 i + x_3 j + x_4 k$, which we denote by $v(h)$. The *conjugate* of h is given by $\bar{h} = \overline{\mathrm{Re}(h) + v(h)} = \mathrm{Re}(h) - v(h)$. We view $\mathbb{H}$ as $\mathbb{R}^4$ by identifying $\{1, i, j, k\}$ with the standard normal basis $\{e_1, e_2, e_3, e_4\}$, so that the Euclidean inner product on $\mathbb{H}$ is given by

$$\langle h_1, h_2\rangle = \mathrm{Re}(h_1 \bar{h}_2).$$

Just as we identify $\mathbb{H}$ with $\mathbb{R}^4$, we identify the vectors in $\mathbb{H}$ with $\mathbb{R}^3$. Using $\{i, j, k\}$ as the orthonormal basis we define the cross-product on the vectors in $\mathbb{H}$. Using this notation, we arrive at the following formula for the quaternionic product of two vectorial elements of $\mathbb{H}$.

Lemma 2.1. *Let* $\alpha, \beta \in \mathbb{H}$ *such that* $\mathrm{Re}(\alpha) = \mathrm{Re}(\beta) = 0$. *Then*

$$\alpha\beta = -\langle \alpha, \beta\rangle 1 + \alpha \times \beta.$$

The following technical lemma will be needed in the following section.

Lemma 2.2. *Let* $\alpha \in \mathbb{H}$. *If* $v(\alpha) \neq 0$, *then* $h \in \mathbb{H}$ *satisfies* $h\bar{\alpha} = \alpha h$ *if and only if* $\mathrm{Re}(h) = 0$ *and* $h = v(h) \perp v(\alpha)$.

Proof.

$$\begin{aligned}(\mathrm{Re}(h) + v(h))(\mathrm{Re}(\alpha) - v(\alpha)) &= (\mathrm{Re}(\alpha) + v(\alpha))(\mathrm{Re}(h) + v(h)),\\ -\mathrm{Re}(h)v(\alpha) + \mathrm{Re}(\alpha)v(h) - v(h)v(\alpha) &= \mathrm{Re}(h)v(\alpha) + \mathrm{Re}(\alpha)v(h) + v(\alpha)v(h).\end{aligned}$$

So,

$$\begin{aligned}2\mathrm{Re}(h)v(\alpha) &= -v(h)v(\alpha) - v(\alpha)v(h)\\ &= \langle v(h), v(\alpha)\rangle 1 + \langle v(\alpha), v(h)\rangle 1 - v(h) \times v(\alpha) - v(\alpha) \times v(h)\\ &= 2\langle v(h), v(\alpha)\rangle 1.\end{aligned}$$

$\mathrm{Re}(h)v(\alpha)$ is vectorial, whereas $\langle v(h), v(\alpha)\rangle 1$ is real. Thus, both are equal to 0, and this proves the lemma.

Two quaternions h_1 and h_2 are called *similar*, or $h_1 \sim h_2$, if there exists $q \in \mathbb{H}$ such that $h_2 = qh_1q^{-1}$. It is well known that $h_1 \sim h_2$ if and only if $\mathrm{Re}(h_1) = \mathrm{Re}(h_2)$ and $|v(h_1)| = |v(h_2)|$ (see [19, Theorem 2.2]).

Let $M_2(\mathbb{H})$ be the space of 2×2 quaternionic matrices. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{H})$, we define $\det(A) = |ad - aca^{-1}b|$. The matrix A is invertible if and only if $\det(A) \neq 0$ (see [2] or [19]). Furthermore, $\det(AB) = \det(A)\det(B)$ for $A, B \in M_2(\mathbb{H})$. Let $Sl(2, \mathbb{H})$ be the space of all elements in $M_2(\mathbb{H})$ with determinant 1.

As mentioned in the introduction, Moussafir and Wilker in [15] and [18], respectively, show that each quaternionic linear functional $h \mapsto (ah+b)(ch+d)^{-1}$ represents a unique element of $\mathscr{M}_5$ and vice versa. Furthermore, a given invertible element of $M_2(\mathbb{H})$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, determines a quaternionic linear fractional, namely $\phi : h \mapsto (ah+b)(ch+d)^{-1}$. This mapping $Gl(2, \mathbb{H}) \to \mathscr{M}_5$ is a group homomorphism.

Two invertible elements of $M_2(\mathbb{H})$, A and B, represent the same Möbius transformation if and only if $B = xA$ for some nonzero $x \in \mathbb{R}$. This gives a 2-to-1 group homomorphism $Sl(2, \mathbb{H}) \to \mathscr{M}_5$, which we can make 1-to-1 by reverting to $PSl(2, \mathbb{H}) = Sl(2, \mathbb{H})/\pm I$:

$$\mathscr{M}_5 = PSl(2, \mathbb{H}), \text{ as groups.} \qquad \square$$

3. Conjugacy classes of $PSl(2, \mathbb{H})$

In order to describe the conjugacy classes of $PSl(2, \mathbb{H})$, we restrict ourselves to upper triangular matrices. It is known that every invertible $n \times n$ quaternionic matrix is triangularizable through purely algebraic means (see [5], also [19, Section 6]). However, it should be noted that, in direct analogy to the situation with $PSl(2, \mathbb{C})$, there is a geometric proof for the 2×2 case, namely, by reminding ourselves that every Möbius transformation on $\widehat{\mathbb{H}}$ is conjugate to a transformation that fixes ∞.

Proposition 3.1. *For $A \in PSl(2, \mathbb{H})$, there are $\alpha, \beta, \delta \in \mathbb{H}$ such that $|\alpha\delta| = 1$ and*

$$A \sim \begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix}.$$

To proceed with the classification of the conjugacy classes in $PSl(2, \mathbb{H})$, we will need to find the solutions to the linear equation

$$ha = bh + c$$

for given coefficients $a, b, c \in \mathbb{H}$. Solutions to this equation have been studied for quite some time, e.g. [12] from 1944. For a brief overview of this subject, see [19, Section 4]. In 2001, Gross, Trenkler and Troschke published a complete description of all of the possible solutions to this equation in [9], which we now give in modified form.

Theorem 3.2 (Gross, Trenkler and Troschke). *Let $\alpha, \beta, \delta \in \mathbb{H}$. Then the equation $h\delta = \alpha h + \beta$ is uniquely solvable (in $\mathbb{H}$) if and only if $\alpha \not\sim \delta$. If $\bar{\delta} = \delta \sim \alpha$ (i.e. both δ and α are both real and equal), then the equation is solvable if and only if $\beta = 0$, in which case there are infinitely many solutions. If $\bar{\delta} \neq \delta \sim \alpha$ (i.e. δ and α are conjugate, but not real), then the equation is solvable if and only if $\beta\bar{\delta} = \alpha\beta$.*

The above theorem immediately gives us the following corollary.

Corollary 3.3. *Let* $A = \left[\begin{smallmatrix} \alpha & \beta \\ 0 & \delta \end{smallmatrix}\right] \in PSl(2, \mathbb{H})$. *Then, as an element of* $\mathscr{M}_5$, A *has exactly* 1, 2, *or infinitely many fixed points in* $\widehat{\mathbb{H}} = \mathbb{H} \cup \{\infty\} = \partial H^5$. *Furthermore, A has*:

(1) *Exactly* 1 *fixed point in* $\widehat{\mathbb{H}}$ (*i.e.*∞) *if and only if either* $\alpha \notin \mathbb{R}$, $\alpha \sim \delta$ *and* $\beta\overline{\delta} \neq \alpha\beta$ *or* $\alpha \in \mathbb{R}$, $\alpha \sim \delta$ *and* $\beta \neq 0$.
(2) *Exactly* 2 *fixed points in* $\widehat{\mathbb{H}}$ *if and only if* $\alpha \not\sim \delta$.
(3) *Infinitely many fixed points in* $\widehat{\mathbb{H}}$ *if and only if either* $\alpha \notin \mathbb{R}$, $\alpha \sim \delta$ *and* $\beta\bar{\delta} = \alpha\beta$ *or* $\alpha \in \mathbb{R}$, $\alpha \sim \delta$ *and* $\beta = 0$.

Corresponding to the three categories of the previous corollary, we have the usual subcategories of Möbius transformations on $\widehat{\mathbb{H}}$.

Definition. Let g be an element of $\mathscr{M}_5$. Then g is:

(1) *parabolic* if g has exactly one fixed point in $\widehat{\mathbb{H}}$;
(2) *loxodromic* if g has exactly two fixed points in $\widehat{\mathbb{H}}$;
  (a) *hyperbolic*, if g is loxodromic and conjugate to the transformation given by $h \mapsto k^2 h$ for some non-zero $k \in \mathbb{R}$;
(3) *elliptic* if g has infinitely many fixed points in $\widehat{\mathbb{H}}$.

Proposition 3.4. *The conjugacy classes of* $PSl(2, \mathbb{H})$ *are given by*:

(1) (*Parabolic classes*)

$$\left\{ \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} : |a| = 1 \right\}$$

with uniqueness up to the similarity class of a in $\mathbb{H}$.

(2) (*Loxodromic classes*)

$$\left\{ \begin{bmatrix} ka & 0 \\ 0 & k^{-1}d \end{bmatrix} : k \geqslant 1, |a| = |d| = 1, ka \not\sim k^{-1}d \right\}$$

with uniqueness up to the similarity classes of ka *and* $k^{-1}d$ *in* $\mathbb{H}$ *and order of the diagonal entries*.

  (*a*) (*Hyperbolic classes*)

$$\left\{ \begin{bmatrix} k & 0 \\ 0 & k^{-1} \end{bmatrix} : k \geqslant 1 \right\},$$

  with uniqueness up to order of the diagonal entries.

(3) (*Elliptic classes*)

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : |a| = 1 \right\},$$

with uniqueness up to the similarity class of a in $\mathbb{H}$.

Proof. Let $f \in PSl(2, \mathbb{H})$.

Part 1: Suppose f is parabolic.

f has no fixed points in $\mathbb{H}$. By Proposition 3.1 and Corollary 3.3 we know that $f \sim \left[\begin{smallmatrix}\alpha & \beta\\ 0 & \delta\end{smallmatrix}\right]$ such that $\alpha \sim \delta$ and $\beta\bar{\delta} \neq \alpha\beta$. Since $\alpha \sim \delta$, we can easily adjust this so that we have $f \sim \left[\begin{smallmatrix}\alpha & \beta\\ 0 & \alpha\end{smallmatrix}\right]$ such that $\beta\bar{\alpha} \neq \alpha\beta$, which, by Lemma 2.2, is to say that either $\mathrm{Re}(\beta) \neq 0$ or $v(\beta) \not\perp v(\alpha)$. To proceed, we consider two mutually exclusive cases.

Case 1: $v(\alpha) \neq 0$.

We split β into three parts: $\beta = \mathrm{Re}(\beta) + kv(\alpha) + \hat{\beta}$, where $k \in \mathbb{R}$ and $\hat{\beta}$ is a vector in $\mathbb{H}$ such that $v(\alpha) \perp \hat{\beta}$. Note that, while $\hat{\beta}$ may very well be zero, $\beta - \hat{\beta} \neq 0$. Furthermore, note that $\beta - \hat{\beta}$ commutes with α.

Then

$$\begin{aligned}\begin{pmatrix}\alpha & \beta\\ 0 & \alpha\end{pmatrix} &\sim \begin{pmatrix}1 & \frac{1}{2}v(\alpha)^{-1}\hat{\beta}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ 0 & \alpha\end{pmatrix}\begin{pmatrix}1 & -\frac{1}{2}v(\alpha)^{-1}\hat{\beta}\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\alpha & \beta-\hat{\beta}\\ 0 & \alpha\end{pmatrix}\\ &\sim \begin{pmatrix}(\beta-\hat{\beta})^{-1} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & \beta-\hat{\beta}\\ 0 & \alpha\end{pmatrix}\begin{pmatrix}(\beta-\hat{\beta}) & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix}.\end{aligned}$$

Case 2: $v(\alpha) = 0$.

In this case, we may assume that $\alpha = 1$. Then

$$\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix} \sim \begin{pmatrix}\beta^{-1} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}\begin{pmatrix}\beta & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}.$$

For the uniqueness of conjugacy class, suppose $\alpha, \alpha' \in \mathbb{H}$ such that $\alpha = A\alpha' A^{-1}$ for some $A \in \mathbb{H}$. Then

$$\begin{pmatrix}\alpha' & 1\\ 0 & \alpha'\end{pmatrix} \sim \begin{pmatrix}A & A\\ 0 & A\end{pmatrix}\begin{pmatrix}\alpha' & 1\\ 0 & \alpha'\end{pmatrix}\begin{pmatrix}A^{-1} & A^{-1}\\ 0 & A^{-1}\end{pmatrix} = \begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix}.$$

Thus, $\alpha \sim \alpha'$ if and only if $\left(\begin{smallmatrix}\alpha' & 1\\ 0 & \alpha'\end{smallmatrix}\right) \sim \left(\begin{smallmatrix}\alpha & 1\\ 0 & \alpha\end{smallmatrix}\right)$. This proves Part 1 of the theorem.

Part 2: Suppose that f is elliptic.

In this case, f has infinitely many fixed points in $\mathbb{H}$. Again using Proposition 3.1 and Corollary 3.3, we know that $f \sim \left[\begin{smallmatrix}\alpha & \beta\\ 0 & \delta\end{smallmatrix}\right]$ such that $\alpha \sim \delta$ and $\beta\bar{\delta} = \alpha\beta$. As before, we can adjust the entries so that $f \sim \left[\begin{smallmatrix}\alpha & \beta\\ 0 & \alpha\end{smallmatrix}\right]$ such that $|\alpha| = 1$ and $\beta\bar{\alpha} = \alpha\beta$. Lemma 2.2 tells us that $\mathrm{Re}(\beta) = 0$ and $v(\beta) \perp v(\alpha)$. In particular, β anti-commutes with $v(\alpha)$.

If $\alpha \in \mathbb{R}$, then $\alpha = 1$ and β is necessarily 0; and we are done. Suppose $v(\alpha) \neq 0$. Then

$$\begin{pmatrix}\alpha & \beta\\ 0 & \alpha\end{pmatrix} \sim \begin{pmatrix}1 & \frac{1}{2}v(\alpha)^{-1}\beta\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ 0 & \alpha\end{pmatrix}\begin{pmatrix}1 & -\frac{1}{2}v(\alpha)^{-1}\beta\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}.$$

It is easily seen that $\alpha \sim \alpha'$ if and only if $\left(\begin{smallmatrix}\alpha' & 0\\ 0 & \alpha'\end{smallmatrix}\right) \sim \left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha\end{smallmatrix}\right)$. This proves the second part of the theorem.

Part 3: Suppose f is loxodromic.

In this case, f has exactly one fixed point in $\mathbb{H}$. Furthermore, $f \sim \left[\begin{smallmatrix}\alpha & \beta\\ 0 & \delta\end{smallmatrix}\right]$ such that $\alpha \not\sim \delta$. Let x be the fixed point for the transformation represented by $\left[\begin{smallmatrix}\alpha & \beta\\ 0 & \delta\end{smallmatrix}\right]$. Then $\beta = -\alpha x + x\delta$, and

$$f \sim \begin{pmatrix}1 & -x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & -\alpha x + x\delta\\ 0 & \delta\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\alpha & 0\\ 0 & \delta\end{pmatrix}.$$

Regarding the uniqueness of this representation, it is clear that, for quaternions α, δ, α', and δ', $\alpha \sim \alpha'$ and $\delta \sim \delta'$, if and only if $\left(\begin{smallmatrix}\alpha' & 0\\ 0 & \delta'\end{smallmatrix}\right) \sim \left(\begin{smallmatrix}\alpha & 0\\ 0 & \delta\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}\alpha' & 0\\ 0 & \delta'\end{smallmatrix}\right) \sim \left(\begin{smallmatrix}\delta & 0\\ 0 & \alpha\end{smallmatrix}\right)$.

The result regarding the hyperbolic transformations follows directly from the definition. This proves the theorem. □

Corollary 3.5. *An element $A \in Sl(2, \mathbb{H})$ is conjugate to a diagonalizable matrix if and only if it represents a loxodromic or elliptic transformation in $\mathscr{M}_5$.*

4. Conjugate invariants

We now introduce some conjugate invariants on $Sl(2, \mathbb{H})$. Of all of these invariants, I believe that F_3, often called the *quaternionic trace function*, is the most well known.

Theorem 4.1. *The following functions on $Sl(2, \mathbb{H})$ are conjugate invariant*:

$$F_1\begin{pmatrix}a & b\\ c & d\end{pmatrix} = |d|^2\mathrm{Re}(a) + |a|^2\mathrm{Re}(d) - \mathrm{Re}(\bar{a}bc) - \mathrm{Re}(bc\bar{d}),$$

$$F_2\begin{pmatrix}a & b\\ c & d\end{pmatrix} = |a|^2 + |d|^2 + 4\mathrm{Re}(a)\mathrm{Re}(d) - 2\mathrm{Re}(bc),$$

$$F_3\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \mathrm{Re}(a) + \mathrm{Re}(d).$$

Proof. We embed $Sl(2, \mathbb{H})$ into $Sl(4, \mathbb{C})$ by the group homomorphism (cf. [2]):

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} \mapsto \left(\begin{array}{cc|cc}a_1 & a_2 & b_1 & b_2\\ -\bar{a}_2 & \bar{a}_1 & -\bar{b}_2 & \bar{b}_1\\ \hline c_1 & c_2 & d_1 & d_2\\ -\bar{c}_2 & \bar{c}_1 & -\bar{d}_2 & \bar{d}_1\end{array}\right),$$

where $a = a_1 + a_2 j$ such that $a_1, a_2 \in \mathbb{C}$, etc.

The characteristic polynomial of this matrix in $Sl(4,\mathbb{C})$ will have conjugate invariant functions in the variables $a_1, a_2, \ldots, d_1, d_2$. Computing this polynomial, we find:

$$\begin{aligned}
&\left|\begin{array}{cc|cc} X-a_1 & -a_2 & -b_1 & -b_2 \\ \bar{a}_2 & X-\bar{a}_1 & \bar{b}_2 & -\bar{b}_1 \\ \hline -c_1 & -c_2 & X-d_1 & -d_2 \\ \bar{c}_2 & -\bar{c}_1 & \bar{d}_2 & X-\bar{d}_1 \end{array}\right| \\
&\quad = X^4 - 2[(\operatorname{Re} a) + (\operatorname{Re} d)]X^3 \\
&\qquad + \left[|a|^2 + |d|^2 + 4(\operatorname{Re} a)(\operatorname{Re} d) - 2(\operatorname{Re} bc)\right]X^2 \\
&\qquad - 2\left[|d|^2\operatorname{Re}(a) + |a|^2\operatorname{Re}(d) - \operatorname{Re}(\bar{a}bc) - \operatorname{Re}(bc\bar{d})\right]X + |ad - aca^{-1}b| \\
&\quad = X^4 - 2F_3\begin{pmatrix} a & b \\ c & d \end{pmatrix} X^3 + F_2\begin{pmatrix} a & b \\ c & d \end{pmatrix} X^2 \\
&\qquad - 2F_1\begin{pmatrix} a & b \\ c & d \end{pmatrix} X + det_{\mathbb{H}}\begin{pmatrix} a & b \\ c & d \end{pmatrix}
\end{aligned}$$

This proves the proposition. □

We now show how to use these invariants to determine (at least partially) the conjugacy class of a given element in $PSl(2,\mathbb{H})$. Note that, for $A \in Sl(2,\mathbb{H})$, $F_1(-A) = -F_1(A)$, $F_2(-A) = F_2(A)$, and $F_3(-A) = -F_3(A)$.

Proposition 4.2. *Let* $f = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in PSl(2,\mathbb{H})$. *Set* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Sl(2,\mathbb{H})$. *Then f is elliptic, hyperbolic or parabolic if and only if*

$$F_1(A) = F_3(A), \tag{1}$$
$$F_2(A) = F_1(A)^2 + 2. \tag{2}$$

Furthermore, suppose A satisfies Eqs. (1) *and* (2). *Then*

(1) *f is hyperbolic if and only if* $|F_1(A)| > 2$.
(2) *f is elliptic, parabolic, or the identity if and only if* $-2 \leqslant |F_1(A)| \leqslant 2$.

Proof. It is easy to show, using the conjugate-equivalent representations given in Proposition 3.4, that elliptic, hyperbolic, and parabolic transformations all satisfy the two given equations. Thus, we need only to show that a loxodromic, non-hyperbolic transformation will not satisfy these equations.

Let $f \in PSl(2,\mathbb{H})$ be a loxodromic, non-hyperbolic transformation. Then f is conjugate to a transformation represented by a matrix $\left(\begin{smallmatrix} ka & 0 \\ 0 & \frac{1}{k}d \end{smallmatrix}\right)$, such that $k \geqslant 1$ and $a, d \in \mathbb{H}$ with $ka \not\sim \frac{1}{k}d$ and $|a| = |d| = 1$.

Suppose $F_1(A) = F_3(A)$ and $F_2(A) = F_1(A)^2 + 2$. Then

$$\begin{aligned}|F_1(A)| &= \frac{1}{k}(\mathrm{Re}\, a) + k(\mathrm{Re}\, d)\\ &= |F_3(A)|\\ &= k(\mathrm{Re}\, a) + \frac{1}{k}(\mathrm{Re}\, d).\end{aligned}$$

Thus, either $k = 1$ and $\mathrm{Re}\, a \neq \mathrm{Re}\, d$ or $k > 1$ and $\mathrm{Re}\, a = \mathrm{Re}\, d$.

Case 1: $k = 1$ *and* $\mathrm{Re}\, a \neq \mathrm{Re}\, d$

In this case, $F_2(A) = 2 + 4(\mathrm{Re}\, a)(\mathrm{Re}\, d)$ and $|F_1(A)| = (\mathrm{Re}\, a) + (\mathrm{Re}\, d)$. Thus, $F_2 = 2 + {F_1}^2$ implies that $0 = ((\mathrm{Re}\, a) - (\mathrm{Re}\, d))^2$, i.e. $\mathrm{Re}\, a = \mathrm{Re}\, d$, a contradiction.

Case 2: $k > 1$ *and* $\mathrm{Re}\, a = \mathrm{Re}\, d$

In this case, $F_2(A) = k^2 + \frac{1}{k^2} + 4(\mathrm{Re}\, a)^2$ and $|F_1(A)| = \left(k + \frac{1}{k}\right)(\mathrm{Re}\, a)$. So, $F_2 = 2 + {F_1}^2$ implies that

$$k^2(1 - (\mathrm{Re}\, a)^2) + \frac{1}{k^2}(1 - (\mathrm{Re}\, a)^2) = 2(1 - (\mathrm{Re}\, a)^2).$$

Since $0 \leqslant (1 - (\mathrm{Re}\, a)^2) \leqslant 1$ and $k^2 + \frac{1}{k^2} = 2$ if and only if $k = 1, 0 = 1 - (\mathrm{Re}\, a)^2$. This contradicts the assumption that f is not hyperbolic.

To prove the last statement of the proposition, suppose A satisfies equations (1) and (2). If f is hyperbolic, then it is conjugate to the transformation represented by the matrix $\left(\begin{smallmatrix} k & 0 \\ 0 & k^{-1}\end{smallmatrix}\right)$ for some $k > 1$. Then $|F_1(A)| = k + k^{-1} > 2$. Similarly, if f is elliptic, parabolic or the identity, then it is conjugate to a transformation represented by a triangular matrix with diagonal entries both α for some $\alpha \in S^1 \subset \mathbb{C} \subset \mathbb{H}$. So, $-2 \leqslant |F_1(A)| = 2\mathrm{Re}(\alpha) \leqslant 2$. This completes the proof of the proposition. $\square$

5. Distinguishing between elliptic and parabolic transformations

A careful study of the invariants described in the previous section will reveal that elliptic and parabolic transformations are indistinguishable with regard to the functions F_1, F_2 and F_3. In particular, if

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad B = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$$

for $\alpha \in S^1 \subset \mathbb{H}$, then $F_1(A) = F_1(B)$, $F_2(A) = F_2(B)$, and $F_3(A) = F_3(B)$. So, we must use another technique in order to distinguish an elliptic transformation from a parabolic one. The resulting technique is rather less elegant than the use of the three invariants, but it gets the job done.

In [11], Huang and So describe all of the possible quaternionic solutions to the quadratic equation

$$h^2 + Bh + C = 0 \quad \text{for } B, C \in \mathbb{H}.$$

A corollary of their main theorem (Corollary 2.4 in the paper) is of most interest to us here:

Theorem 5.1 (Huang–So). *The quadratic equation* $h^2 + Bh + C = 0$ *has infinitely many solutions if and only if* $B, C \in \mathbb{R}$ *and* $B^2 - 4C < 0$.

It is interesting to note that the proof of the following theorem solves a purely algebraic problem (the solving of a certain quaternionic quadratic equation) by simplifying it using a geometric method.

Theorem 5.2. *Let g be an element of* $\mathscr{M}_5$ *with quaternionic matrix representation* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Sl(2, \mathbb{H})$ *such that g is either parabolic or elliptic, i.e. A satisfies Eqs.* (1) *and* (2) *from Proposition* 4.2 *and* $-2 \leqslant |F_1(A)| \leqslant 2$.

(1) *If* $b = 0$, *then g is elliptic if and only if either* $c\bar{a} = dc$ *or* $a \in \mathbb{R}$ *and* $c = 0$.
(2) *If* $b \neq 0$, *then g is elliptic if and only if*
  (a) $a + bdb^{-1}$, $(c - db^{-1}a)b \in \mathbb{R}$, *and*
  (b) $(a + bdb^{-1})^2 + 4b(c - db^{-1}a) < 0$.

Proof. The result for when $b = 0$ is the proof of Part 3 in 3.3 (Note that in this case $a \sim d$ a priori). So, we may proceed to the case where $b \neq 0$.

Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ -db^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ db^{-1} & 1 \end{pmatrix} = \begin{pmatrix} a + bdb^{-1} & b \\ c - db^{-1}a & 0 \end{pmatrix}.$$

Thus, the number of fixed points of g is equal to the number of fixed points of the transformation $\tilde{g}$ represented by $\left(\begin{smallmatrix} \Gamma & b \\ \Delta & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a+bdb^{-1} & b \\ c-db^{-1}a & 0 \end{smallmatrix}\right)$.

Note that both $\Gamma \neq 0$ and $\Delta \neq 0$. The fixed points of $\tilde{g}$ satisfy the equation:

$$h\Delta h = \Gamma h + b$$

If $h \in \mathbb{H}$ satisfies the above equation, then $\tilde{h} = \Delta^{-1}h$ satisfies

$$\Delta^{-1}\tilde{h}^2 - \Gamma\Delta^{-1}\tilde{h} - b = 0,$$

i.e.

$$\tilde{h}^2 - \Delta\Gamma\Delta^{-1}\tilde{h} - \Delta b = 0.$$

According to the Huang–So Corollary, this quadratic equation has infinitely many solutions if and only if

(1) $\Delta\Gamma\Delta^{-1}$, $\Delta b \in \mathbb{R}$, and
(2) $\Delta\Gamma^2\Delta^{-1} + 4\Delta b < 0$.

Furthermore, the quantity $\Delta\Gamma\Delta^{-1}$ is real if and only if Γ is a real number. Substituting a, b, c and d back into Γ and Δ and simplifying gives the result. $\square$

References

[1] L.V. Ahlfors, Möbius Transformations and Clifford Numbers from Differential Geometry and Complex Analysis, Springer, Berlin, 1985, pp. 65–73.
[2] H. Aslaksen, Quaternionic determinants, Math. Intelligencer 18 (1996) 57–65.
[3] A. Beardon, The Geometry of Discrete Groups, Springer, New York, 1983.
[4] L. Bieberbach, Conformal Mapping (F. Steinhardt, Trans.), Chelsea, New York, 1953.
[5] J.L. Brenner, Matrices of quaternions, Pacific J. Math. 1 (1951) 329–335.
[6] C. Cao, P.L. Waterman, Conjugacy Invariants of Möbius Groups from Quasiconformal Mappings and Analysis (Ann Arbor, MI, 1995), Springer, New York, 1998, pp. 109–139.
[7] H.S.M. Coxeter, Quaternions and reflections, Amer. Math. Monthly 53 (1946) 136–146.
[8] H.-D. Ebbinghaus, H. Hermes, F. Hirzebruch, M. Koechner, K. Mainzer, J. Neukirch, A. Prestl, R. Remmert, Numbers, Spring-Verlag, Berlin, 1990.
[9] J. Gross, G. Trenkler, S.-O. Troschke, Quaternions: further contributions to a matrix oriented approach, Linear Algebra Appl. 326 (2001) 205–213.
[10] R. Heidrich, G. Jank, On the iteration of quaternionic Moebius transformations, Complex Variables 29 (1996) 313–318.
[11] L. Huang, W. So, Quadratic formulas for quaternions, Appl. Math. Lett. 15 (2002) 533–540.
[12] R.E. Johnson, On the equation $\chi\alpha = \gamma\chi + \beta$ over an algebraic division ring, Bull. Amer. Math. Soc. 50 (1944) 202–207.
[13] G.A. Jones, D. Singerman, Complex Functions: An Algebraic and Geometric Viewpoint, Cambridge University Press, Cambridge, 1991.
[14] H. Maass, Automorphe Funktioned von mehreren Veränderlichen und Dirichletsche Reihen, Hamburg Abh. 16 (1949).
[15] J.-O. Moussafir, Quaternionic linear fractional transformations and direct isometries of H^5, J. Geom. Phys. 37 (3) (2001) 183–189.
[16] J.G. Ratcliffe, Foundations of Hyperbolic Manifolds, Springer, New York, 1994.
[17] K.T. Vahlen, Über Bewegungen und komplexe Zahlen, Math. Ann. 55 (1902).
[18] J.B. Wilker, The quaternion formalism for Möbius groups in four or fewer dimensions, Linear Algebra Appl. 190 (1993) 99–136.
[19] F. Zhang, Quaternions and matrices of quaternions, Linear Algebra Appl. 251 (1997) 21–57.