On the Pole-Shifting Problem for Non-commutative Rings

James Brewer and Lee Klingler*
Department of Mathematics
Florida Atlantic University
Boca Raton, Florida 33431

Submitted by Robert M. Guralnick

INTRODUCTION

In this paper, we set down some thoughts on pole shifting for noncommutative rings. The inspiration for this work is the paper [5] of E. Sontag and Y. Wang. In that paper, the authors consider pole shifting over the noncommutative ring of continuous quaternionic-valued functions on a CW complex of dimension at most three. They prove a positive result for this ring and apply it, via a clever trick, to deduce a dynamic pole-shifting result for the (commutative) subring of the above ring consisting of all continuous *real*-valued functions.

The above work led us to consider the pole-shifting problem over arbitrary noncommutative rings. We have been able to extend to noncommutative rings one of the main theorems for commutative rings. We have also been able to show that several results from linear systems theory over commutative rings remain valid for noncommutative noetherian rings. In the process, we have seen that, in sharp contrast to the commutative case, an obstruction to pole shifting often occurs in the "dimension-one case." This can happen even for very nice rings. Before discussing the noncommutative version, we should first recall the problem in the commutative setting. In doing so, we may as well give the relevant definitions in their most general forms.

1. POSITIVE POLE-SHIFTING RESULTS

Let R be a ring, not necessarily commutative. An n-dimensional *system* over R is a pair (A, B), where A is an $n \times n$ matrix over R and B is an

*Research partially supported by a Humboldt Fellowship at the University of Essen.

655 Avenue of the Americas, New York, NY 10010 0024-3795/91/$3.50

$m \times n$ matrix over R, for some positive integers m and n. Two systems (A_1, B_1) and (A_2, B_2) are called *feedback equivalent* if there exists an invertible $m \times m$ matrix S, an invertible $n \times n$ matrix T, and an $n \times m$ matrix L such that

$$B_2 = S^{-1} B_1 T,$$

$$A_2 = T^{-1} A_1 T + L B_1 T.$$

REMARK. Note that if we take *left* free modules R^m and R^n, then we can view *right* multiplication by B as a map from R^m to R^n, and we can view *right* multiplication by A as a map from R^n to R^n. Indeed, in what follows, we always write maps on the right. Then the systems equivalence defined above is just a change of basis in R^n represented by the matrix T, a change of basis in R^m represented by the matrix S, and a *feedback* operation represented by the matrix L.

With this interpretation of the system (A, B), we call (A, B) *controllable* if

$$R^n = \mathrm{Im}(B) + \mathrm{Im}(BA) + \mathrm{Im}(BA^2) + \cdots,$$

where $\mathrm{Im}(X)$ denotes the image of the map X.

Finally, following Sontag and Wang, we shall call the n-dimensional system (A, B) *(arbitrarily) triangularizable* if, for each $r_1, r_2, \ldots, r_n \in R$, there exists a system (F, G) feedback equivalent to (A, B), such that

$$F = \begin{bmatrix} r_1 & 0 & \cdots & 0 \\ * & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & r_n \end{bmatrix}.$$

For a system over a commutative ring R, the pole-shifting problem is the following: Given an n-dimensional controllable system (A, B) and ring elements $r_1, r_2, \ldots, r_n \in R$, find a system (F, G) feedback equivalent to (A, B) such that the eigenvalues of F are $r_1, r_2, \ldots, r_n$. Clearly, if the system (A, B) is arbitrarily triangularizable, the pole-shifting problem has a positive solution. Consequently, over a noncommutative ring, the pole-shifting problem translates into the question of whether or not each controllable system can be arbitrarily triangularized.

For commutative rings, there is a condition, the GCU property, which implies that the pole-shifting problem has a positive solution. As originally formulated, that condition does not have an obvious translation in case the ring is noncommutative. However, implicit in [5] is a condition for noncommutative rings which, for commutative rings, turns out to be equivalent to the GCU property. We now define the GCU property and prove its equivalence to the condition of [5]. In the process, we shall find it convenient to introduce another equivalent form. An $m \times n$ matrix B over R is said to be *good* if there exists an $n \times n$ matrix A such that (A, B) is controllable. We say that the ring R has the *GCU property* if, for every good matrix B, there exists a vector ν such that νB is unimodular (cf. [1]).

PROPOSITION 1. *Let R be a commutative ring. The following are equivalent*:

(i) *R has the GCU property.*

(ii) *For each good matrix B over R, there exists a vector ν and an invertible matrix P such that $\nu BP = [0\ 0\ \cdots\ 0\ 1]$.*

(iii) *For each good matrix B over R, there exists a matrix L and an invertible matrix P such that $LBP = [0 \mid I]$, where 0 is a block matrix of zeros and I is an identity matrix of the appropriate size* (*cf.* [5]).

Proof. Certainly (ii) implies (iii). If (iii) holds, then (ii) follows immediately by taking ν to be the last row of the matrix L. Thus, conditions (ii) and (iii) are equivalent.

(i) implies (ii): Let B be a good matrix over R. Since R has the GCU property, stably free R-modules are free [1, Lemma 1]. Thus, each unimodular row over R can be extended to an invertible matrix. Let $\nu B = [a_1\ a_2\ \cdots\ a_n]$ be a unimodular row in the image of B. We can find rows $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ such that the matrix

$$P^{-1} = \begin{bmatrix} & & \alpha_1 & \\ & & \alpha_2 & \\ & & \vdots & \\ & & \alpha_{n-1} & \\ a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

is invertible. It follows that $\nu BP = [0\ 0\ \cdots\ 0\ 1]$.

(ii) implies (i): Let B be a good matrix over R, and choose a vector ν and an invertible matrix P such that $\nu BP = [0 \ 0 \ \cdots \ 0 \ 1]$. Then $\nu BPP^{-1} = \nu B$ is the last row of P^{-1} and consequently is a unimodular row in the image of B. Thus, R has the GCU property. ∎

The key idea of the proof of the above proposition is Lemma 1 of [1], which says that, over a (commutative) GCU ring, stably free modules are free. One of the key ideas of the proof of that result is what is sometimes referred to as *Gabel's theorem*: If P is a stably free module over a commutative ring, then some power of P is free. Theorem 1 of [4] shows that Gabel's theorem is still valid if we relax the commutativity assumption but require the ascending-chain condition. This enables us to prove a noetherian, noncommutative version of Lemma 1 of [1]. For that, we need the notion of the GCU property for a (possibly) noncommutative ring. Since we are working with left modules, with maps written on the right, we say that the ring R has the *left GCU property* if, for every good matrix B, there exists a vector ν such that νB is unimodular, multiplying by scalars on the right. That is, if $\nu B = [a_1 \ \cdots \ a_n]$, then we require that there exist scalars $c_1, \ldots, c_n$ such that the *right* linear combination $a_1c_1 + \cdots + a_nc_n = 1$.

PROPOSITION 2. *Let R be a left-noetherian ring with the left GCU property. Then stably free left R-modules are free.*

Proof. Let P be a stably free left R-module. By [4, Theorem 1], there exist positive integers k and n such that P^k is isomorphic to R^n as left R-modules. We proceed to find a reachable system (A, B) such that the module generated by the rows of B is isomorphic to P.

Without loss of generality, we take R^n equal to P^k. Let $g: R^n \to R^n$ be the projection onto the first P-factor, and let $f: P^k \to P^k$ be defined as $f(p_1, \ldots, p_k) = (p_k, p_1, \ldots, p_{k-1})$. Then clearly

$$R^n = \operatorname{Im}(g) + \operatorname{Im}(gf) + \cdots + \operatorname{Im}(gf^{k-1}).$$

Thus, if A and B are any matrices representing f and g, respectively, it follows that (A, B) is a reachable system, where the left module generated by the rows of B is isomorphic to P. Since R has the left GCU property, there exists a vector ν such that $\nu B = [a_1 \ \cdots \ a_n]$ is unimodular. In particular, there exists scalars $c_1, \ldots, c_n$ such that the right linear combination

$a_1c_1 + \cdots + a_nc_n$ is equal to 1. Consequently, right multiplication by the column vector

$$\mu = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is a surjection from R^n onto R, in fact mapping the free left module $R \cdot [a_1 \ \cdots \ a_n]$ onto R. Since R is projective, this yields a direct-sum decomposition $R^n = R\cdot[a_1 \ \cdots \ a_n] \oplus \ker(\mu)$.

But $R\cdot[a_1 \ \cdots \ a_n]$ is a submodule of $P = \mathrm{Im}(g)$, the left R-module generated by the rows of B, from which it follows that P contains a rank-one free summand, and we can decompose $P \simeq R \oplus P_1$. Since P is stably free, the summand P_1 is as well. By the same argument as above, P_1 contains a rank-one free summand, and so on. Since P is a noetherian left R-module, this must eventually exhaust all of P, and hence P is free. ■

As a corollary to Proposition 2, we note that the three equivalent conditions of Proposition 1 remain equivalent if we assume that the ring R is (left) noetherian rather than commutative.

COROLLARY 1. *Let R be a left noetherian ring. The following are equivalent*:

(i) *R has the left GCU property.*

(ii) *For each good matrix B over R, there exists a vector v and an invertible matrix P such that $vBP = [0 \ 0 \ \cdots \ 0 \ 1]$.*

(iii) *For each good matrix B over R, there exists a matrix L and an invertible matrix P such that $LBP = [0 \mid I]$, where 0 is a block matrix of zeros and I is an identity matrix of the appropriate size.*

Proof. The proof is the same as that of Proposition 1, using Proposition 2 in place of [1, Lemma 1]. ■

The following result removes the commutativity assumption of [1, Theorem 1].

THEOREM 1. *Let R be a ring satisfying property* (ii) *of Proposition* 1. *Then each controllable system over R is arbitrarily triangularizable.*

Proof. Let (A, B) be an n-dimensional controllable system over R with $r_1, r_2, \ldots, r_n$ elements of R. We induct on n. For $n = 1$, suppose that the one-dimensional system

$$R^m \xrightarrow{B = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}} R \xrightarrow{a} R \tag{1}$$

is controllable. By (ii) of Proposition 1, there exist a vector $\nu = [v_1 \ \cdots \ v_m]$ and a unit u such that $\nu B u = v_1 b_1 u + \cdots + v_m b_m u = 1$, so that $u \nu B = 1$. Then, given any element $r \in R$, we can write $r = a + (r - a)u\nu B$, and hence the system (1) is arbitrarily triangularizable.

Suppose that the result is true for controllable systems of dimension less than n, and let (A, B) be a controllable system of dimension n. By a change of basis in the state space, we may harmlessly replace B by BP and A by $P^{-1}AP$ and assume that there exists a vector ν such that $\nu B = [0 \ 0 \ \cdots \ 0 \ 1]$. Then we can write

$$A = \left[\begin{array}{c|c} A_{11} & A_{1n} \\ \hline A_{n1} & a_{nn} \end{array}\right] \quad \text{and} \quad B = [\, B_1 \mid B_2 \,],$$

where A_{11} is $(n-1) \times (n-1)$ [so that A_{n1} is $1 \times (n-1)$ and A_{1n} is $(n-1) \times 1$] and B_2 is $m \times 1$ [so that B_1 is $m \times (n-1)$]. Let

$$C_1 = \begin{bmatrix} A_{21} \\ B_1 \end{bmatrix}.$$

As the following lemma shows, the system (A_{11}, C_1) is also controllable.

LEMMA (Eising). *Suppose that (A, B) can be partitioned as*

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \quad \textit{and} \quad B = [\, B_1 \mid B_2 \,].$$

Set

$$C_1 = \begin{bmatrix} A_{21} \\ B_1 \end{bmatrix}.$$

If (A, B) is controllable, then so is the system (A_{11}, C_1).

Proof of Lemma. We shall only sketch the idea. We have that

$$R^n = \mathrm{Im}(B) + \mathrm{Im}(BA) + \mathrm{Im}(BA^2) + \cdots,$$

so it follows that R^n is the span of the rows of the following matrices:

$$\left[B_1 \mid B_2 \right], \qquad \left[B_1A_{11} + B_2A_{21} \mid B_1A_{12} + B_2A_{22} \right],$$

$$\left[\begin{array}{c|c} B_1A_{11}^2 + B_2A_{21}A_{11} + B_1A_{12}A_{21} + B_2A_{22}A_{21} & \\ B_1A_{11}A_{12} + B_2A_{21}A_{12} + B_1A_{12}A_{22} + B_2A_{22}^2 & \end{array} \right], \ldots . \tag{2}$$

We are interested in the span of the rows of the matrices

$$\begin{bmatrix} A_{21} \\ B_1 \end{bmatrix}, \quad \begin{bmatrix} A_{21}A_{11} \\ B_1A_{11} \end{bmatrix}, \quad \begin{bmatrix} A_{21}A_{11}^2 \\ B_1A_{11}^2 \end{bmatrix}, \quad \begin{bmatrix} A_{21}A_{11}^3 \\ B_1A_{11}^3 \end{bmatrix}, \ldots . \tag{3}$$

Suppose that the notation is such that A_{11} is $n_1 \times n_1$, so that B_1 is $m \times n_1$. We need to show that the rows of the matrices in (3) span R^{n_1}. Focusing on the left-hand side of the matrices occurring in (2), which do span R^{n_1}, we have to see that any row which is a combination of such rows occurs in the span of the rows of the matrices in (3). This is easily verified by inspection. For example, B_1 is present in both. Now, B_1A_{11} belongs to (3), and, since A_{21} belongs to (3), the rows of B_2A_{21} are in the span of the rows. Hence, the rows of $B_1A_{11} + B_2A_{21}$ belong to the span of the rows of the matrices in (3), etc. ■

Continuing the proof of the theorem, by induction, there exists an invertible matrix P_1 and a matrix K such that

$$P_1^{-1}A_{11}P_1 + KC_1P_1 = \begin{bmatrix} r_1 & 0 & \cdots & 0 \\ * & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & r_{n-1} \end{bmatrix}.$$

Write K as $K = [K_1 \mid K_2]$, so that

$$P_1^{-1}A_{11}P_1 + KC_1P_1 = P_1^{-1}A_{11}P_1 + K_1A_{21}P_1 + K_2B_1P_1.$$

Replace A by

$$A' = \left[\begin{array}{ccc|c} & P_1^{-1} & & \begin{matrix}0\\ \vdots \\ 0\end{matrix} \\ \hline 0 & \cdots & 0 & 1 \end{array}\right] \cdot A \cdot \left[\begin{array}{ccc|c} & P_1 & & \begin{matrix}0\\ \vdots \\ 0\end{matrix} \\ \hline 0 & \cdots & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} P_1^{-1} & A_{11} & P_1 & \begin{matrix}*\\ \vdots \\ *\end{matrix} \\ \hline & A_{21} & P_1 & * \end{array}\right],$$

and B by

$$B' = [B_1 \mid B_2] \left[\begin{array}{ccc|c} & P_1 & & \begin{matrix}0\\ \vdots \\ 0\end{matrix} \\ \hline 0 & \cdots & 0 & 1 \end{array}\right] = [B_1 P_1 \mid B_2].$$

Let

$$Q = \left[\begin{array}{ccc|c} & I_{n-1} & & K_1 \\ \hline 0 & \cdots & 0 & 1 \end{array}\right], \qquad \text{so that} \quad Q^{-1} = \left[\begin{array}{ccc|c} & I_{n-1} & & -K_1 \\ \hline 0 & \cdots & 0 & 1 \end{array}\right].$$

Then replace A' by

$$A'' = QA'Q^{-1} = \left[\begin{array}{c|c} P_1^{-1}A_{11}P_1 + K_1 A_{21} P_1 & \begin{matrix}*\\ \vdots \\ *\end{matrix} \\ \hline A_{21}P_1 & * \end{array}\right],$$

and B' by

$$B'' = B' \cdot Q^{-1} = [B_1 P_1 \mid B_2] \cdot \left[\begin{array}{ccc|c} & I_{n-1} & & -K_1 \\ \hline 0 & \cdots & 0 & 1 \end{array}\right]$$

$$= [B_1 P_1 \mid B_2 - B_1 P_1 K_1].$$

Now replace A'' by

$$A''' = A'' + \left[\begin{array}{ccc} & K_2 & \\ \hline 0 & \cdots & 0 \end{array}\right] \cdot B''$$

$$= \left[\begin{array}{c|c} P_1^{-1}A_{11}P_1 + K_1A_{21}P_1 & \begin{matrix} * \\ \vdots \\ * \end{matrix} \\ \hline A_{21}P_1 & * \end{array}\right] + \left[\begin{array}{c|c} K_2B_1P_1 & \begin{matrix} * \\ \vdots \\ * \end{matrix} \\ \hline 0 \quad \cdots \quad 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} r_1 & \cdots & 0 & x_1 \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & r_{n-1} & x_{n-1} \\ \hline & A_{21}P_1 & & x_n \end{array}\right],$$

and we have only to show that the system (A''', B'') can be put into the proper form. There exists a vector ν such that $\nu B = [0 \ 0 \ \cdots \ 0 \ 1]$, and it follows easily that $\nu B'' = [0 \ 0 \ \cdots \ 0 \ 1]$. Hence,

$$A''' + \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n + r_n \end{bmatrix} \cdot \nu B'' = \begin{bmatrix} r_1 & 0 & \cdots & 0 \\ * & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & r_n \end{bmatrix},$$

as intended. This completes the proof. ∎

Theorem 1 has a number of interesting corollaries. The first of these was known for commutative rings, but unknown in general.

COROLLARY 2. *If R is a local ring, then each controllable system over R is arbitrarily triangularizable.*

Proof. If (A, B) is controllable, then the two-sided ideal of R generated by the entries of B is R. Therefore, some entry of B is a unit. By performing row and column operations on B, we can arrange to have the last row in the form $[0 \ 0 \ \cdots \ 0 \ 1]$. This amounts to finding an invertible matrix P and a vector ν such that $\nu BP = [0 \ 0 \ \cdots \ 0 \ 1]$. ∎

COROLLARY 3 [2, Proposition 3.7]. *If R is a (commutative) elementary divisor ring, then each controllable system over R is arbitrarily triangularizable.*

Proof. If (A, B) is controllable, then the ideal generated by the entries of B is R. There exist invertible matrices Q and P such that

$$PBQ = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

If ν denotes the last row of P, then $\nu BQ = [0 \ 0 \ \cdots \ 0 \ 1]$. ■

COROLLARY 4 [1, Theorem 1]. *If R is a GCU ring, then each controllable system is arbitrarily triangularizable.*

Proof. The statement follows immediately from Proposition 1 and Theorem 1. ■

As most of the work in [5] goes toward showing that the ring of continuous quaternionic-valued functions on a CW complex of dimension at most three satisfies Proposition 1, Theorem 3.7 of [5] may be recaptured as follows.

COROLLARY 5 [5, Theorem 3.7]. *Let R be the ring of continuous quaternionic-valued functions on a CW complex of dimension at most three. Then each controllable system over R is arbitrarily triangularizable.*

2. PATHOLOGY

In this section, we shall give some indication of the kinds of difficulties one can encounter when trying to consider pole shifting over noncommutative rings. We begin by showing that some very nice rings may fail to have the arbitrary-triangularization property.

Consider a ring R with elements $a, b \in R$, and look at the system

$$R \xrightarrow{b} R \xrightarrow{a} R, \tag{4}$$

where the maps are right multiplication by the elements b and a, respectively. This system is controllable if and only if

$$R = \operatorname{Im}(b) + \operatorname{Im}(ba) + \operatorname{Im}(ba^2) + \cdots$$
$$= Rb + Rba + Rba^2 + \cdots ,$$

where Rx denotes the left ideal of R generated by x. In particular, if there exist elements $r, s \in R$ such that

$$1 = rb + sba, \tag{5}$$

then the system in (4) is controllable.

Similarly, arbitrary triangularization of the system in (4) means the following: Given an element $x \in R$, there must be a unit $u \in R$ and an element $k \in R$ such that

$$x = u^{-1}(a + kb)u.$$

We can now give two examples of the type mentioned above.

(a) *An example of a noncommutative principal-ideal domain and a one-dimensional system over it which is not arbitrarily triangularizable.* Let F be a field, with t an indeterminate, and denote by $F(t)$ the ring of algebraic functions over F in the variable t. Let R be the skew polynomial ring $F(t)[\theta, \delta]$, where θ is an indeterminate and δ is the ordinary formal derivative (as applied to θ). By [3, Theorem 1.11], the ring R is a left and right principal-ideal domain. The multiplication in R is determined as follows: If $c \in F(t)$, then $\theta c = c\theta + \delta(c)$. Therefore, $\theta t = t\theta + 1$, so that $1 = (-t)\theta + 1 \cdot \theta t$. By (5) above, the system in (4), with $b = \theta$ and $a = t$, is controllable. However, it is not arbitrarily triangularizable. To see this, notice that taking $r_1 = 0$, we would have to be able to find an element $k \in R$ with $0 = t + k\theta$, which is a contradiction to the fact that $\{1, \theta, \theta^2, \dots\}$ is a basis for R over $F(t)$.

(b) *An example of a simple artinian ring and a one-dimensional system over it which is not arbitrarily triangularizable.* Let F be a field, and denote by $M_2(F)$ the ring of all 2×2 matrices over F. It is well known that R is a simple artinian ring, and we now give an example of a one-dimensional controllable system over R which fails to be arbitrarily triangularizable. In this case we let

$$b = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

so that $1 = ab + ba$. By (5) above, the system in (4) is controllable. As in the first example, if we take $r_1 = 0$, we would have to be able to find a matrix $k \in M_2(F)$ with $0 = a + kb$, which is clearly impossible.

Note that this example can be made to work with matrices of any size over any ring S.

It might be worthwhile at this point to justify our statement in the introduction to the effect that, for commutative rings, pole shifting for one-dimensional systems is always possible. Specifically, if R is a commutative ring and if

$$R^m \xrightarrow{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}} R \xrightarrow{a} R \tag{6}$$

is a controllable, one-dimensional system over R, then

$$1 = \sum_{i,j} r_{ij} b_i a^j$$

for some elements $r_{ij} \in R$. Since R is commutative, we can interchange and relabel and obtain the equation

$$1 = s_1 b_1 + \cdots + s_m b_m$$

for some elements $s_1, \dots, s_m \in R$. Thus, given an element $c \in R$, since

$$c - a = (s_1 c - s_1 a) b_1 + \cdots + (s_m c - s_m a) b_m$$

it is evident that the system (6) is arbitrarily triangularizable.

Recall that there exists rings R for which $R \oplus R \simeq R$ as R-modules. For such rings, $R \simeq R^2 \simeq R^3 \simeq \cdots$. Obviously, this wreaks havoc with trying to take the standard approach of defining the dimension of a system to be the dimension (i.e., rank) of the state module. The observant reader will notice that we finessed that problem by defining a system as a pair (A, B) of matrices, and taking as the dimension of the system the size of the (square) matrix A. If, say, A is an $n \times n$ matrix and B is an $m \times n$ matrix, then A induces a mapping φ on R^n by right multiplication, and B induces a mapping ψ from R^m to R^n by right multiplication. If $R^n \simeq R^t$, then φ is also a mapping on R^t, which may be given by a $t \times t$ matrix A_1, and ψ is a mapping from R^m to R^t, which may be given by an $m \times t$ matrix B_1. If R

has the property of Theorem 1, then not only is (A, B) arbitrarily triangularizable, but so is (A_1, B_1). After all, (A, B) is controllable if and only if (A_1, B_1) is controllable! This may seem rather strange at first glance.

There is a related remark. A ring R is said to be an *IBN ring* (for *invariant basis number*) if $R^h \simeq R^k$ implies that $h = k$. For such rings, we would be able to take the standard approach to defining the dimension of a system. For example, commutative rings are IBN rings. By a variant of the argument of the second example above, one can show that, if R is not an IBN ring, then there is a controllable system over R which is not arbitrarily triangularizable. It follows from this that a ring satisfying the hypothesis of Theorem 1 must be an IBN ring. When we first proved Theorem 1, we were fearful that we would have to avoid all of the aforementioned unpleasantness by including the additional assumption that the ring is an IBN ring.

REFERENCES

1 J. Brewer, D. Katz, and W. Ullery, On the pole assignability property over commutative rings, *J. Pure Appl. Algebra* 48:1–7 (1987).

2 R. Bumby, E. Sontag, H. Sussman, and W. Vasconcelos, Remarks on the pole-shifting problem over rings, *J. Pure Appl. Algebra* 20:113–127 (1981).

3 K. Goodearl and R. Warfield, Jr., *An Introduction to Noncommutative Noetherian Rings*, Cambridge U.P., 1989.

4 Y. T. Lam, Series summation of stably free modules, *Quart. J. Math. Oxford* 27:37–46 (1976).

5 E. Sontag and Y. Wang, Pole shifting for families of linear systems depending on at most three parameters, *Linear Algebra Appl.* 137/138:3–38 (1990).

Received 10 January 1991; final manuscript accepted 11 April 1991