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The bipartite edge frustration of extension of splice and link graphs

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ABSTRACT

The smallest number of edges that have to be deleted from a graph G to obtain a bipartite spanning subgraph is called the bipartite edge frustration of G and denoted by $\varphi(G)$. In this paper we extend the splice and link for two graphs and determine their bipartite edge frustration. As an application, the bipartite edge frustration of a polybuckyball is computed.

1. Introduction

Erdös [1,2] and Edwards [3] proved that for any graph G there is a bipartite subgraph of G with at least $\frac{|E(G)|}{2} + \frac{|V(G)|-1}{4}$ edges. Those bounds were further improved for various classes of graphs; for example Staton [4] and Locke [5] proved that if G is a connected cubic graph and $G \neq K_4$ then G has a bipartite subgraph with at least $\frac{7|E(G)|}{9}$ edges and also the lower bound of $\frac{4}{5}|E(G)|$ was established for cubic triangle-free graphs [6].

Let G be a graph with the vertex and edge sets $V(G)$ and $E(G)$ respectively. The bipartite edge frustration of G is defined as the minimum number of edges that have to be deleted from G to obtain a bipartite spanning subgraph. We denote it by $\varphi(G)$.

It is easy to see, if G is bipartite then $\varphi(G) = 0$. It follows easily that $\varphi(G) \leq \frac{|E(G)|}{2}$ and that the complete graph on n vertices has the maximum possible bipartite edge frustration among all graphs on n vertices.

The quantity $\varphi(G)$ is, in general, difficult to compute; it is NP-hard for general graphs. Hence, it makes sense to search for classes of graphs that allow its efficient computation. Some results in this direction was reported in [7] for fullerenes and other polyhedral graphs and in [8] for some classes of nanotubes. In an earlier paper [9], the bipartite edge frustration was computed for some composite graphs. Splice and link are two important graph operations such that they have some application in chemistry. Splice of cycles serves as models of spirane molecules and models of complex molecules are built from simpler building block by iterating and/or combining the splice and link operation; see [10]. In this paper the notions of splice and link are extended and call them double splice and double link and then obtain their bipartite edge frustration. The paper is organized as follows. In the next section we recall some definitions and preliminaries about splice and link. Section 3 contains the main results about the bipartite edge frustration of double splice and double link.

2. Definitions and preliminaries

All graphs considered in this paper will be finite and simple. The notation we use is mostly standard and taken from standard graph theory textbooks, such as [11].

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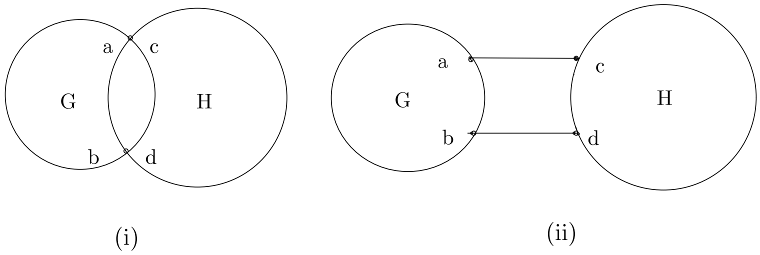

Fig. 1. The double splice and double link.

Definition 2.1. Let G and H be two simple and connected graphs with disjoint vertex sets. For given vertices $a \in V(G)$ and $b \in V(H)$, a splice of G and H is defined as the graph $(G.H)(a, b)$ obtained by identifying the vertices a and b. Similarly, a link of G and H is defined as the graph $(G \sim H)(a, b)$ obtained by joining a and b by an edge.

The following theorem immediately conclude.

Theorem 2.2. *Let G and H be two simple and connected graphs with disjoint vertex sets. For each $a \in V(G)$ and $b \in V(H)$, the bipartite edge frustration of splice and link of G and H are obtained as follows:*
(i) $\varphi\big((G.H)(a, b)\big) = \varphi(G) + \varphi(H)$,
(ii) $\varphi\big((G \sim H)(a, b)\big) = \varphi(G) + \varphi(H)$.

Proof. The proof is straightforward. $\square$

Now we extend the above operations, for splice of G and H by identifying two vertices and for link G and H by joining two vertices as the following definition.

Definition 2.3. Let G and H be two simple and connected graphs with disjoint vertex sets. For given vertices $a, b \in V(G)$ and $c, d \in V(H)$, a double splice of G and H is defined as the graph $(G : H)(a, b : c, d)$ obtained by identifying the vertices a and c and vertices b and d. Similarly, a double link of G and H is defined as the graph $(G \approx H)(a, b : c, d)$ obtained by joining a and c by an edge and b and d by another edge. A double splice and double link of two graphs are shown schematically in Fig. 1.

Let us mention to the bipartite edge frustration of double splice and double link of graphs. At first we define a concept that is used for proving the next theorems.

Definition 2.4. Let G be a graph. For $a, b \in G$, $\varphi_{a,b}(G)$ is the smallest number of edges that have to be deleted from a graph G to obtain a bipartite spanning subgraph such that a, b are occurred in the same partition. Similarly, we define $\varphi'_{a,b}(G)$, for each $a, b \in G$, as the smallest number of edges that have to be deleted from a graph G to obtain a bipartite spanning subgraph such that a, b are occurred in the different partitions.

It is easy to show that $\varphi(G) = \min\{\varphi'_{a,b}(G), \varphi_{a,b}(G)\}$.

Example. (i) $\varphi_{a,b}(P_n) = \begin{cases} 0 & 2|d(a,b) \\ 1 & 2 \nmid d(a,b) \end{cases}$, $\quad \varphi'_{a,b}(P_n) = \begin{cases} 1 & 2|d(a,b) \\ 0 & 2 \nmid d(a,b), \end{cases}$
(ii) $\varphi_{a,b}(C_{2n}) = \begin{cases} 0 & 2|d(a,b) \\ 1 & 2 \nmid d(a,b) \end{cases}$, $\quad \varphi'_{a,b}(C_{2n}) = \begin{cases} 1 & 2|d(a,b) \\ 0 & 2 \nmid d(a,b), \end{cases}$
(iii) $\varphi_{a,b}(C_{2n+1}) = \varphi'_{a,b}(C_{2n+1}) = \varphi(C_{2n+1})$, for each $a, b \in V(C_{2n+1})$,
(iv) $\varphi_{a,b}(K_n) = \varphi'_{a,b}(K_n) = \varphi(K_n)$, for each $a, b \in V(K_n)$.

Remark. Let G and H be two graphs. If $ab \in E(G)$ and $cd \in E(H)$, then these edges are identified in double splice graph $(G : H)(a, b : c, d)$. In this case $|E\big((G : H)(a, b : c, d)\big)| = |E(G)| + |E(H)| - 1$. Otherwise $|E\big((G : H)(a, b : c, d)\big)| = |E(G)| + |E(H)|$. In double splice graph when the vertices a and c are identified, we can certainly assume that $u = a = b$, by similar argument we assume $v = b = d$. Indeed we can assume that, $u, v \in V(G) \cap V(H)$. We abbreviate the notation to $(G : H)$ when the vertices $u, v \in V(G) \cap V(H)$ are clear from context.

3. Main results

In this section formulas for the bipartite edge frustration of double splice and double link of two graphs are computed.

Theorem 3.1. *Let G and H be two graphs. For each $u, v \in V(G) \cap V(H)$ such that $|E\big((G : H)\big)| = |E(G)| + |E(H)|$, we have*

$$\varphi\big((G : H)\big) = \min\{\varphi_{u,v}(G) + \varphi_{u,v}(H), \varphi'_{u,v}(G) + \varphi'_{u,v}(H)\}.$$

Proof. Suppose that G_0 is a bipartite subgraph of G by removing $\varphi_{u,v}(G)$ edges, such that u, v are in the same partition. Similarly, assume H_0 is a bipartite subgraph of H by removing $\varphi_{u,v}(H)$ edges, such that u, v are in the same partition. It follows immediately that $(G_0 : H_0)$ is a bipartite subgraph of $(G : H)$. Then by deleting $\varphi_{u,v}(G) + \varphi_{u,v}(H)$ edges of

$(G:H)$, we can obtain a bipartite subgraph. So $\varphi\big((G:H)\big) \le \varphi_{u,v}(G) + \varphi_{u,v}(H)$. By similar argument, we can show that $\varphi\big((G:H)\big) \le \varphi'_{u,v}(G) + \varphi'_{u,v}(H)$ and then

$$\varphi\big((G:H)\big) \le \min\{\varphi_{u,v}(G) + \varphi_{u,v}(H), \varphi'_{u,v}(G) + \varphi'_{u,v}(H)\}.$$

Conversely, suppose that by removing $\varphi\big((G:H)\big)$ edges of $(G:H)$, the bipartite subgraph K is obtained from $(G:H)$. Set $\varphi\big((G:H)\big) = r + s$, such that r and s are the number of edges of G and H that are removed from $(G:H)$, respectively.

If u and v occur in the same partition of K, then $r \ge \varphi_{u,v}(G)$ and $s \ge \varphi_{u,v}(H)$. Hence $\varphi\big((G:H)\big) = r + s \ge \varphi_{u,v}(G) + \varphi_{u,v}(H)$. By the same argument, if u, v occur in different partitions of graph K, then $\varphi\big((G:H)\big) = r + s \ge \varphi'_{u,v}(G) + \varphi'_{u,v}(H)$. Therefore

$$\varphi\big((G:H)\big) \ge \min\{\varphi_{u,v}(G) + \varphi_{u,v}(H), \varphi'_{u,v}(G) + \varphi'_{u,v}(H)\}.$$

This completes the proof. □

Theorem 3.2. *Let G and H be two graphs. For each $u, v \in V(G) \cap V(H)$ such that $|E\big((G:H)\big)| = |E(G)| + |E(H)| - 1$, we have*

$$\varphi\big((G:H)\big) = \min\{\varphi_{u,v}(G) + \varphi_{u,v}(H) - 1, \varphi'_{u,v}(G) + \varphi'_{u,v}(H)\}.$$

Proof. We can proceed analogously to the proof of Theorem 3.1. □

Lemma 3.3. *Let G be a connected graph. If G_0 be a bipartite subgraph of G by deleting $\varphi(G)$ edges, then G_0 is connected.*

Proof. Suppose there exists a bipartite diconnected subgraph G_0 of G by removing $\varphi(G)$ edges. The subgraph G_0 has at least two components, say G_1 and G_2, such that there exists an edge e of G connecting G_1 and G_2. Notice that G_1 and G_2 are bipartite. Therefore $G_0 + e$ is bipartite, contradict by maximality of G_0. □

Theorem 3.4. *Let G and H be two graphs and $a, b \in V(G)$, $c, d \in V(H)$. Then*

(i) *If $\big(\varphi_{a,b}(G) = \varphi(G), \varphi_{c,d}(H) = \varphi(H)\big)$ or $\big(\varphi'_{a,b}(G) = \varphi(G), \varphi'_{c,d}(H) = \varphi(H)\big)$, then*

$$\varphi\big((G \approx H)(a, b : c, d)\big) = \varphi(G) + \varphi(H).$$

(ii) *If $\big(\varphi_{a,b}(G) = \varphi(G), \varphi'_{c,d}(H) = \varphi(H)\big)$ or $\big(\varphi'_{a,b}(G) = \varphi(G), \varphi_{c,d}(H) = \varphi(H)\big)$, then*

$$\varphi\big((G \approx H)(a, b : c, d)\big) = \varphi(G) + \varphi(H) + 1.$$

Proof. (i) Suppose that $\varphi_{a,b}(G) = \varphi(G), \varphi_{c,d}(H) = \varphi(H)$, then there is a bipartite subgraph G_0 of G (by deleting $\varphi_{a,b}(G) = \varphi(G)$ edges) such that a, b lie in the same partition and similarly there is a bipartite subgraph H_0 of H (by deleting $\varphi_{c,d}(H) = \varphi(H)$ edges) such that c, d lie in the same partition. By Lemma 3.3 G_0 and H_0 are connected. Notice that distances between a, b and c, d are even. Then there are paths P_1 and P_2 with even lengths as follows $P_1 : a = x_0, x_1, \dots, x_{2n} = b$ and $P_2 : d = y_1, y_2, \dots, y_{2m} = c$. The cycle

$$C_1 : a = x_0, x_1, \dots, x_{2n} = b, \qquad d = y_1, y_2, \dots, y_{2m} = c, a,$$

is even cycle. Then any new odd cycle is not added in $(G_0 \approx H_0)(a, b : c, d)$. Hence the graph $(G_0 \approx H_0)(a, b : c, d)$ does not have odd cycle. Then

$$\varphi\big((G \approx H)(a, b : c, d)\big) = \varphi(G) + \varphi(H).$$

Now if $\varphi'_{a,b}(G) = \varphi(G)$ and $\varphi'_{c,d}(H) = \varphi(H)$, then by similar argument we can see that there is a bipartite subgraph G_0 of G (by deleting $\varphi'_{a,b}(G) = \varphi(G)$ edges) such that a, b belong in the different partitions and similarly there is a bipartite subgraph H_0 of H (by deleting $\varphi'_{c,d}(H) = \varphi(H)$ edges) such that c, d lie in the different partitions. Suppose that $d_{G_0}(a, b) = 2n - 1$ and $d_{H_0}(c, d) = 2m - 1$, then the cycle

$$C_2 : a = x_0, x_1, \dots, x_{2n-1} = b, \qquad d = y_1, y_2, \dots, y_{2m-1} = c, a,$$

has even length. Except the odd cycles in the copies of G and H, there is no odd cycle in $(G \approx H)(a, b : c, d)$. Therefore

$$\varphi\big((G \approx H)(a, b : c, d)\big) = \varphi(G) + \varphi(H).$$

(ii) Suppose $\varphi_{a,b}(G) = \varphi(G)$ and $\varphi'_{c,d}(H) = \varphi(H)$, then there exists a bipartite subgraph G_0 of G (by deleting $\varphi_{a,b}(G) = \varphi(G)$ edges) such that a, b belong in the same partition and a bipartite subgraph H_0 of H (by deleting $\varphi'_{c,d}(H) = \varphi(H)$ edges) such that c, d lie in the different partition. Since the distance between a, b in G_0 is even and the distance between c, d in H_0 is odd, we assume $d_{G_0}(a, b) = 2n$ and $d_{H_0}(c, d) = 2m - 1$. Then the cycle

$$C_3 : a = x_0, x_1, \dots, x_{2n} = b, \qquad d = y_1, y_2, \dots, y_{2m-1} = c, a,$$

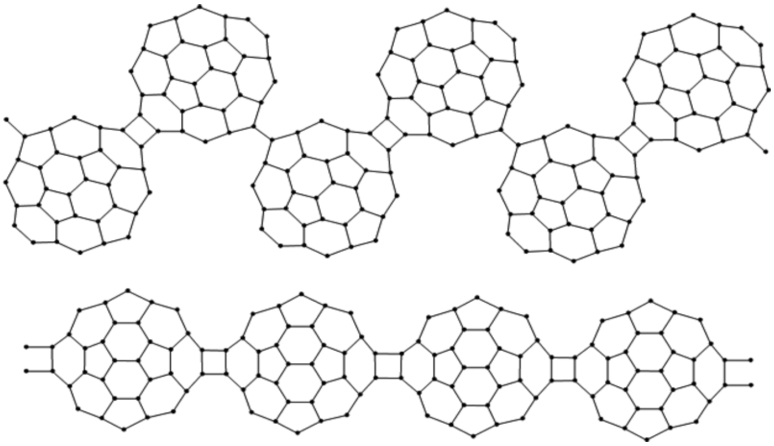

Fig. 2. The molecular graph of a polybuckyball.

has odd length. Hence except deleted edges ($\varphi(G) + \varphi(H)$ edges), we must delete an edge of C_3 for obtaining a bipartite subgraph of $(G \approx H)(a, b : c, d)$. Then

$$\varphi\big((G \approx H)(a, b : c, d)\big) = \varphi(G) + \varphi(H) + 1.$$

In the same manner, we can see that if $\big(\varphi'_{a,b}(G) = \varphi(G), \varphi_{c,d}(H) = \varphi(H)\big)$, then

$$\varphi\big((G \approx H)(a, b : c, d)\big) = \varphi(G) + \varphi(H) + 1. \quad \square$$

For the sake of completeness, we mention here a theorem of Došlić and Vukičevićas follows:

Theorem 3.5. *Let C_n be an icosahedral fullerene on $n = 20(i^2 + ij + j^2)$ vertices. Then $\varphi(C_n) = 6(i + j)$.*

For fullerene C_{60}, it is easy to see that $i = j = 1$, then $\varphi(C_{60}) = 12$.

Example 3.6. In this example the bipartite edge frustration of a polybuckyball is computed, Fig. 2. The molecular graph of a polybuckyball is instructed by operations link or double link on the same IPR fullerene graphs on 60 vertices. We can obtain the bipartite edge frustration of polybuckyball by using Theorems 3.4 and 3.5. The bipartite edge frustration of polybuckyball, that is made by n copies of C_{60} by operations link or double link is equal to $\varphi(C_{60}) + \cdots + \varphi(C_{60}) = n\varphi(C_{60}) = 12n$.

Definition 3.7. Let G and H be two connected graphs on disjoint vertex sets, and let $a \in V(G)$ and $b \in V(H)$. An n-link of G and H is a graph obtained by connecting the vertices a and b by a path of length n so that each of these vertices is identified with one of the terminal vertices of P_n. We denote n-link of G and H by $(G{\sim_n}H)(a, b)$.

Theorem 3.8. *Let G and H be two connected graphs with disjoint vertex sets. For each $a \in V(G)$ and $b \in V(H)$, the bipartite edge frustration of n-link of G and H are obtained as follows:*

$$\varphi\big((G{\sim_n}H)(a, b)\big) = \varphi(G) + \varphi(H).$$

Definition 3.9. Let G and H be two simple and connected graphs with disjoint vertex sets. For given vertices $a, b \in V(G)$ and $c, d \in V(H)$, a (m, n)-link of G and H is defined as the graph $(G{\approx_{m,n}}H)(a, b : c, d)$ obtained by joining a and c by a path of length m and b and d by another path of length n, see Fig. 3.

The following theorem can be proved in much the same way as Theorem 3.4. So the proof of next theorem is left for the reader.

Theorem 3.10. *Let G and H be two graphs and $a, b \in V(G)$, $c, d \in V(H)$. Then*

(i) *If $\big(\varphi_{a,b}(G) = \varphi(G), \varphi_{c,d}(H) = \varphi(H)\big)$ or $\big(\varphi'_{a,b}(G) = \varphi(G), \varphi'_{c,d}(H) = \varphi(H)\big)$ and $m + n$ be an even number, then*

$$\varphi\big((G{\approx_{m,n}}H)(a, b : c, d)\big) = \varphi(G) + \varphi(H).$$

(ii) *If $\big(\varphi_{a,b}(G) = \varphi(G), \varphi_{c,d}(H) = \varphi(H)\big)$ or $\big(\varphi'_{a,b}(G) = \varphi(G), \varphi'_{c,d}(H) = \varphi(H)\big)$ and $m + n$ be an odd number, then*

$$\varphi\big((G{\approx_{m,n}}H)(a, b : c, d)\big) = \varphi(G) + \varphi(H) + 1.$$

(iii) *If $\big(\varphi_{a,b}(G) = \varphi(G), \varphi'_{c,d}(H) = \varphi(H)\big)$ or $\big(\varphi'_{a,b}(G) = \varphi(G), \varphi_{c,d}(H) = \varphi(H)\big)$ and $m + n$ be an even number, then*

$$\varphi\big((G{\approx_{m,n}}H)(a, b : c, d)\big) = \varphi(G) + \varphi(H) + 1.$$

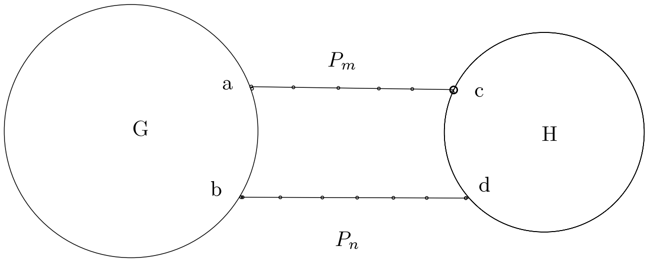

Fig. 3. The (m, n)-double link.

(iv) *If* $\big(\varphi_{a,b}(G) = \varphi(G), \varphi'_{c,d}(H) = \varphi(H)\big)$ *or* $\big(\varphi'_{a,b}(G) = \varphi(G), \varphi_{c,d}(H) = \varphi(H)\big)$*, and* $m + n$ *be an odd number, then*

$$\varphi\big((G \approx_{m,n} H)(a, b : c, d)\big) = \varphi(G) + \varphi(H).$$

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