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Razumikhin-type exponential stability criteria of neutral stochastic functional differential equations

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ABSTRACT

The paper discusses both pth moment and almost sure exponential stability of solutions to neutral stochastic functional differential equations and neutral stochastic differential delay equations, by using the Razumikhin-type technique. The main goal is to find sufficient stability conditions that could be verified more easily then by using the usual method with Lyapunov functionals. The analysis is based on paper [X. Mao, Razumikhin-type theorems on exponential stability of neutral stochastic functional differential equations, SIAM J. Math. Anal. 28 (2) (1997) 389–401], referring to mean square and almost sure exponential stability.

1. Introduction and preliminary results

Deterministic neutral differential equations were introduced by Hale and Meyer [2] and discussed in Hale et al. (see references in [3]) and Kolmanovskii and Nosov (for details see also references in [6,7]), among others. Motivated by the chemical engineering systems in which the physical and chemical processes are distinguished by their complexity, as well as by the theory of aeroelasticity in which aeroelastic efforts present an interaction between aerodynamic, elastic and inertial forces, Kolmanovskii and Nosov [6] incorporated a Gaussian white noise excitation in deterministic cases by describing stochastic versions of deterministic neutral functional differential equations taking into account stochastic perturbations. Having in mind that the Gaussian white noise is mathematically described as a formal derivative of a Brownian motion process, mathematical models of such phenomena are represented by various types of neutral stochastic functional differential equations of the Ito type. Obviously, investigations of such stochastic equations are of great interest. However, since they describe dynamical systems with their past states, which make them more realistic but more complex, they cannot be effectively solved in all cases. The main interest in the field has often been directed to the existence, uniqueness and stability of the solutions, as well as to the study of their qualitative and quantitative properties. We refer the reader to papers [8,9] by X. Mao et al., to papers and books [11–15] by X. Mao, as well as to [6,16], among others.

It is well known that the classical and powerful technique applied in the study of stability is based on a stochastic version of the Lyapunov direct method. However, as Lyapunov functionals, now required instead of Lyapunov functions, can be difficult to find when applying the above method, some other more applicable criteria are needed in order to verify the required type of stability. Some of the criteria related to the pth moment exponential stability of the solutions to neutral stochastic functional differential equations are considered in [5,17]. Recall that the conditions guaranteeing this type of

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stability are very restrictive for the coefficients of the equations. In the present paper we apply a special technique – the Razumikhin technique – which is completely different from those used in [5,17]. Razumikhin developed this technique in [18,19] to study the stability of deterministic systems with a delay. There is a number of papers which apply this technique to various deterministic functional systems to solve some problems in applications, [1,10], for example. X. Mao incorporated Razumikhin's approach in stochastic functional differential equations [13] and in neutral stochastic functional differential equations [14]. Crucially, this approach requires that some of the conditions hold on a restriction of the considered function spaces instead on the whole function spaces. In the present paper we generalize the results from paper [14] by X. Mao referring to the exponential stability in mean square and to the almost sure exponential stability. In this sense, the assertions in Sections 2 and 3 are reduced for $p=2$ to the ones from paper [14]. It should be pointed out that this generalization is made possible by applying an elementary inequality basically different from the ones treating similar subjects.

The paper is organized as follows. In the remainder of this section we introduce some basic notions and notations, mainly from [5] and [14], and we present the neutral stochastic functional differential equation which will be the topic of our investigation. In Section 2 we present the main results, the conditions inspired by the Razumikhin's approach, under which the trivial solution is the pth moment exponentially stable. This approach also makes it possible the study of almost sure exponential stability, while the technique in [5,17] does not. The results from Section 2 are extended in Section 3 to neutral stochastic differential delay equations. Note also that the approach in [5,17] could not be applied to this type of equations, which points to the importance of the Razumikhin technique, both theoretically and in applications. We conclude the paper with some examples to illustrate the previous theory.

In general, we require that all random variables and processes are defined on a complete probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geqslant 0},P)$ with a natural filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ generated by the standard m-dimensional Brownian motion $w=\{w(t),t\geqslant 0\}$, $w(t)=(w_1(t),w_2(t),\dots,w_m(t))^T$, i.e. $\mathcal{F}_t=\sigma\{w(s),0\leqslant s\leqslant t\}$. Let the Euclidean norm be denoted by $|\cdot|$. For simplicity, let us take $trace[A^TA]=|A|^2$, where T stands for transpose of a matrix or vector. Let also $\|A\|$ be the operator norm of a matrix A, where $\|A\|=\sup\{|Ax|\colon\ |x|=1, x\in R^n\}$.

Let $C([-\tau,0];R^n)$, where $\tau=\text{const}>0$, be the family of continuous functions φ from $[-\tau,0]$ to R^n with the norm $\|\varphi\|=\sup_{-\tau\leqslant\theta\leqslant 0}|\varphi(\theta)|$. Let also $L^p_{\mathcal{F}}([-\tau,0];R^n)$ be the family of all $\mathcal{F}$-measurable $C([-\tau,0],R^n)$-valued random variables $\phi=\{\phi(\theta),-\tau\leqslant\theta\leqslant 0\}$ such that $\sup_{-\tau\leqslant\theta\leqslant 0}E|\phi(\theta)|^p<\infty$. In that manner, $C([-\tau,0];R^n)\subset L^p_{\mathcal{F}}([-\tau,0];R^n)$ is valid.

In this paper we study the following n-dimensional neutral stochastic functional differential equation

$$d\big[x(t)-G(x_t)\big]=f(t,x_t)\,dt+g(t,x_t)\,dw(t),\quad t\geqslant 0 \tag{1}$$

with initial data $x_0=\xi=\{\xi(\theta),-\tau\leqslant\theta\leqslant 0\}\in L^p_{\mathcal{F}_0}([-\tau,0];R^n)$, where $\mathcal{F}_t=\mathcal{F}_0$, $-\tau\leqslant t\leqslant 0$. The coefficients of this equation,

$$G:C\big([-\tau,0];R^n\big)\to R^n,$$
$$f:R_+\times C\big([-\tau,0];R^n\big)\to R^n,\qquad g:R_+\times C\big([-\tau,0];R^n\big)\to R^{n\times m}$$

are continuous functionals and $x_t=\{x(t+\theta),-\tau\leqslant\theta\leqslant 0\}$ is a $C([-\tau,0];R^n)$-valued stochastic process.

An $\mathcal{F}_t$-adapted process $x=\{x(t),-\tau\leqslant t<\infty\}$ is said to be the solution to Eq. (1) if it satisfies the initial condition and the corresponding integral equation holds a.s., that is, for every $t\geqslant 0$,

$$x(t)-G(x_t)=\xi(0)-G(\xi)+\int_0^t f(s,x_s)\,ds+\int_0^t g(s,x_s)\,dw(s)\quad\text{a.s.} \tag{2}$$

Kolmanovskii and Nosov [6] proved the basic existence-and-uniqueness theorem under the following conditions:

- For a constant $k\in(0,1)$ and for all $\phi_1,\phi_2\in L^2_{\mathcal{F}}([-\tau,0];R^n)$, let

$$E\big|G(\phi_1)-G(\phi_2)\big|^2\leqslant k\sup_{-\tau\leqslant\theta\leqslant 0}E\big|\phi_1(\theta)-\phi_2(\theta)\big|^2.$$

- The functionals f and g are uniformly Lipschitz continuous in the second argument, or they are locally Lipschitz continuous and satisfy the linear-growth condition (for more details see [6,15,16], among others).

Moreover, if $\xi\in L^p_{\mathcal{F}_0}([-\tau,0];R^n)$, then there exists the pth moment of the solution $x(t;\xi)$. In the sequel, we assume, with no emphasis on conditions, that there exists a unique solution to Eq. (1) satisfying $\sup_{-\tau\leqslant t\leqslant T}E|x(t;\xi)|^p<\infty$ for all $T>0$, and that all the Lebesgue and Ito integrals further employed are well defined.

As we mentioned above, the following inequalities will be used in our investigation: Let $p\geqslant 1$, $x,y\in R^n$. Then:

For $\alpha>0$,

$$|x+y|^p\leqslant(1+\alpha)^{p-1}\left(|x|^p+\frac{|y|^p}{\alpha^{p-1}}\right). \tag{3}$$

For $0 < \varepsilon < 1$,

$$|x+y|^p \leqslant \frac{|x|^p}{(1-\varepsilon)^{p-1}} + \frac{|y|^p}{\varepsilon^{p-1}}. \tag{4}$$

The inequality (3) can be found in Mao [15, Lemma 4.1], while (4) is obtained from (3) by putting $\alpha = \varepsilon/(1-\varepsilon)$.

2. Main results

As usual, let $G(0) = 0$, $f(t,0) \equiv 0$, $g(t,0) \equiv 0$, which yields that Eq. (1) admits a trivial solution $x(t;0) \equiv 0$. Recall that the trivial solution to Eq. (1) is said to be the pth moment exponentially stable if there exists a pair of positive constants γ and M such that

$$E\left|x(t;\xi)\right|^p \leqslant Me^{-\gamma t} \sup_{-\tau\leqslant\theta\leqslant 0} E\left|\xi(\theta)\right|^p, \quad t \geqslant 0, \tag{5}$$

or equivalently,

$$\limsup_{t\to\infty} \frac{1}{t} \ln E\left|x(t;\xi)\right|^p \leqslant -\gamma \tag{6}$$

for all $\xi \in L^p_{\mathcal{F}_0}([-\tau,0];R^n)$. Likewise, the trivial solution to Eq. (1) is said to be almost surely exponentially stable if there exists a positive constant γ such that

$$\limsup_{t\to\infty} \frac{1}{t} \ln\left|x(t;\xi)\right| \leqslant -\gamma \quad \text{a.s.} \tag{7}$$

for all $\xi \in L^p_{\mathcal{F}_0}([-\tau,0];R^n)$.

In this section, we will prove the main result of the present paper, that is, the Razumikhin-type theorem, showing that the trivial solution to Eq. (1) is the pth moment exponentially stable, where $p \geqslant 2$. Remember that the pth moment exponential stability and almost sure exponential stability do not imply each other in general. However, our main result makes it possible to prove that, under some conditions, the pth moment exponential stability implies almost sure exponential stability.

Theorem 1. *Let there exist a constant $k \in (0,1)$ such that*

$$E\left|G(\phi)\right|^p \leqslant k \sup_{-\tau\leqslant\theta\leqslant 0} E\left|\phi(\theta)\right|^p, \tag{8}$$

for all $\phi \in L^p_{\mathcal{F}}([-\tau,0];R^n)$. Let $q > (1-k^{\frac{1}{p}})^{-p}$ and let there exist $\lambda > 0$ so that

$$\begin{aligned} &E\left\{\frac{p}{2}\left|\phi(0)-G(\phi)\right|^{p-4}\left(\left|\phi(0)-G(\phi)\right|^2\left[2\left(\phi(0)-G(\phi)\right)^T f(t,\phi) + \left|g(t,\phi)\right|^2\right] + (p-2)\left|\left(\phi(0)-G(\phi)\right)^T g(t,\phi)\right|^2\right)\right\} \\ &\quad \leqslant -\lambda E\left|\phi(0)-G(\phi)\right|^p \end{aligned} \tag{9}$$

for all $t \geqslant 0$ and those $\phi \in L^p_{\mathcal{F}}([-\tau,0];R^n)$ satisfying

$$E\left|\phi(\theta)\right|^p < qE\left|\phi(0)-G(\phi)\right|^p, \quad -\tau \leqslant \theta \leqslant 0. \tag{10}$$

Then, for every $\xi \in L^p_{\mathcal{F}_0}([-\tau,0];R^n)$ the solution $x(t;\xi)$ of Eq. (1) satisfies

$$E\left|x(t;\xi)\right|^p \leqslant q\left(1+k^{\frac{1}{p}}\right)^p e^{-\bar{\gamma}t} \sup_{-\tau\leqslant\theta\leqslant 0} E\left|\xi(\theta)\right|^p, \quad t \geqslant 0, \tag{11}$$

where

$$\bar{\gamma} = \min\left\{\lambda, \frac{1}{\tau}\ln\frac{q}{(1+(kq)^{\frac{1}{p}})^p}\right\} > 0, \tag{12}$$

that is, the trivial solution to Eq. (1) is the pth moment exponentially stable.

Theorem 1 points to the difference between the usual technique in investigating sufficient conditions on the pth mean stability. In fact, the essence of the Razumikhin technique is to weaken the condition (9) in the sense that it can be valid not for all $\phi \in L^p_{\mathcal{F}}([-\tau,0);R^n)$, as before, but only for those satisfying (10).

For simplicity, this theorem will be proved gradually, with the help of the forthcoming lemmas. We shall also use the notation $x(t)$ instead of $x(t;\xi)$ to denote the solution to Eq. (1) for a given initial data $\xi \in L^p_{\mathcal{F}_0}([-\tau,0];R^n)$.

The first lemma is an auxiliary result and it is of an independent interest, but it will be important in our analysis.

Lemma 1. *Let the condition* (8) *be satisfied for some* $k \in (0,1)$. *Then,*

$$E\left|\phi(0) - G(\phi)\right|^p \leqslant \left(1 + k^{\frac{1}{p}}\right)^p \sup_{-\tau \leqslant \theta \leqslant 0} E\left|\phi(\theta)\right|^p$$

for all $\phi \in L^p_{\mathcal{F}}([-\tau, 0]; R^n)$.

Proof. The proof holds straightforwardly by using (8) and by applying the inequality (3) for $\alpha = k^{\frac{1}{p}}$. Then,

$$\begin{aligned} E\left|\phi(0) - G(\phi)\right|^p &\leqslant \left(1 + k^{\frac{1}{p}}\right)^{p-1}\left(E\left|\phi(0)\right|^p + k^{-\frac{p-1}{p}} E\left|G(\phi)\right|^p\right) \\ &\leqslant \left(1 + k^{\frac{1}{p}}\right)^{p-1}\left(E\left|\phi(0)\right|^p + k \cdot k^{-\frac{p-1}{p}} \sup_{-\tau \leqslant \theta \leqslant 0} E\left|\phi(\theta)\right|^p\right) \\ &\leqslant \left(1 + k^{\frac{1}{p}}\right)^p \sup_{-\tau \leqslant \theta \leqslant 0} E\left|\phi(\theta)\right|^p. \qquad \square \end{aligned}$$

Lemma 2. *Let the condition (8) be satisfied for some* $k \in (0,1)$. *Let* $x(t)$ *be a solution to Eq.* (1) *and* $\rho \geqslant 0$, $0 < \gamma < -\frac{1}{\tau} \ln k$ *so that, for all* $0 \leqslant t \leqslant \rho$,

$$e^{\gamma t} E\left|x(t) - G(x_t)\right|^p \leqslant \left(1 + k^{\frac{1}{p}}\right)^p \sup_{-\tau \leqslant \theta \leqslant 0} E\left|x(\theta)\right|^p. \tag{13}$$

Then,

$$e^{\gamma t} E\left|x(t)\right|^p \leqslant \frac{(1 + k^{\frac{1}{p}})^p}{(1 - (ke^{\gamma\tau})^{\frac{1}{p}})^p} \sup_{-\tau \leqslant \theta \leqslant 0} E\left|x(\theta)\right|^p, \quad 0 \leqslant t \leqslant \rho. \tag{14}$$

Proof. Let $\varepsilon \in (0,1)$ and $0 \leqslant t \leqslant \rho$. By applying the inequality (4) and the assumption (8), one can derive

$$E\left|x(t)\right|^p \leqslant \frac{1}{(1-\varepsilon)^{p-1}} E\left|x(t) - G(x_t)\right|^p + \frac{k}{\varepsilon^{p-1}} \sup_{-\tau \leqslant \theta \leqslant 0} E\left|x(t+\theta)\right|^p.$$

Then, by using (13) it follows, for all $0 \leqslant t \leqslant \rho$, that

$$\begin{aligned} e^{\gamma t} E\left|x(t)\right|^p &\leqslant \frac{1}{(1-\varepsilon)^{p-1}} \sup_{0 \leqslant t \leqslant \rho}\left[e^{\gamma t} E\left|x(t) - G(x_t)\right|^p\right] + \frac{k}{\varepsilon^{p-1}} \sup_{0 \leqslant t \leqslant \rho}\left[e^{\gamma t} \sup_{-\tau \leqslant \theta \leqslant 0} E\left|x(t+\theta)\right|^p\right] \\ &\leqslant \frac{(1 + k^{\frac{1}{p}})^p}{(1-\varepsilon)^{p-1}} \sup_{-\tau \leqslant \theta \leqslant 0} E\left|x(\theta)\right|^p + \frac{ke^{\gamma\tau}}{\varepsilon^{p-1}} \sup_{-\tau \leqslant t \leqslant \rho}\left[e^{\gamma t} E\left|x(t)\right|^p\right]. \end{aligned}$$

This inequality also holds for all $-\tau \leqslant t \leqslant 0$ and, therefore,

$$\sup_{-\tau \leqslant t \leqslant \rho}\left[e^{\gamma t} E\left|x(t)\right|^p\right] \leqslant \frac{(1 + k^{\frac{1}{p}})^p}{(1-\varepsilon)^{p-1}} \sup_{-\tau \leqslant \theta \leqslant 0} E\left|x(\theta)\right|^p + \frac{ke^{\gamma\tau}}{\varepsilon^{p-1}} \sup_{-\tau \leqslant t \leqslant \rho}\left[e^{\gamma t} E\left|x(t)\right|^p\right].$$

Considering $ke^{\gamma\tau} < \varepsilon^{p-1} < 1$, we see that

$$\sup_{-\tau \leqslant t \leqslant \rho}\left[e^{\gamma t} E|x(t)|^p\right] \leqslant \frac{(1 + k^{\frac{1}{p}})^p}{(1-\varepsilon)^{p-1}}\left(1 - \frac{ke^{\gamma\tau}}{\varepsilon^{p-1}}\right)^{-1} \sup_{-\tau \leqslant \theta \leqslant 0} E\left|x(\theta)\right|^p.$$

Since $ke^{\gamma\tau} < (ke^{\gamma\tau})^{\frac{p-1}{p}}$, it is easy to obtain the desired result letting $\varepsilon = (ke^{\gamma\tau})^{\frac{1}{p}}$. $\square$

Lemma 3. *Let the conditions of Theorem* 1 *be satisfied. Then, for every* $\gamma \in (0, \overline{\gamma})$ *and* $t \geqslant 0$,

$$e^{\gamma t} E\left|x(t) - G(x_t)\right|^p \leqslant \left(1 + k^{\frac{1}{p}}\right)^p \sup_{-\tau \leqslant \theta \leqslant 0} E\left|\xi(\theta)\right|^p. \tag{15}$$

Proof. For every $\xi \in L^p_{\mathcal{F}_0}([-\tau, 0]; R^n)$, we can assume, without loss of generality, that $\sup_{-\tau \leqslant \theta \leqslant 0} E|\xi(\theta)|^p > 0$.

Since $q > (1 - k^{\frac{1}{p}})^{-p}$, then $q/(1 + (kq)^{\frac{1}{p}})^p > 1$, which implies that $\overline{\gamma} > 0$. Then, for an arbitrary $\gamma \in (0, \overline{\gamma})$ we see that

$$0 < \gamma < \min\left\{\lambda, \frac{1}{\tau} \ln \frac{1}{k}\right\}.$$

Since $h(x) = x/(1-(kx)^{\frac{1}{p}})^p$ increases when $kx < 1$, one can choose q such that

$$q > \frac{e^{\gamma\tau}}{(1-(ke^{\gamma\tau})^{\frac{1}{p}})^p} > \frac{1}{(1-(ke^{\gamma\tau})^{\frac{1}{p}})^p} > \frac{1}{(1-k^{\frac{1}{p}})^p}. \tag{16}$$

Let us prove (15) by contradiction. If (15) does not hold for every $t \geqslant 0$, Lemma 1 yields that there exists $\rho \geqslant 0$ so that

$$e^{\gamma t}E\left|x(t)-G(x_t)\right|^p \leqslant e^{\gamma\rho}E\left|x(\rho)-G(x_\rho)\right|^p = \left(1+k^{\frac{1}{p}}\right)^p \sup_{-\tau\leqslant\theta\leqslant 0} E\left|\xi(\theta)\right|^p \tag{17}$$

for all $0 \leqslant t \leqslant \rho$. There is also a sequence $\{t_k\}_{k\geqslant 1}$, $t_k \downarrow \rho$, that satisfies

$$e^{\gamma t_k}E\left|x(t_k)-G(x_{t_k})\right|^p > e^{\gamma\rho}E\left|x(\rho)-G(x_\rho)\right|^p. \tag{18}$$

By applying Lemma 2 to (17) we find for all $-\tau \leqslant t \leqslant \rho$ that

$$e^{\gamma t}E\left|x(t)\right|^p \leqslant \frac{(1+k^{\frac{1}{p}})^p}{(1-(ke^{\gamma\tau})^{\frac{1}{p}})^p} \sup_{-\tau\leqslant\theta\leqslant 0} E\left|x(\theta)\right|^p = \frac{e^{\gamma\rho}}{(1-(ke^{\gamma\tau})^{\frac{1}{p}})^p} E\left|x(\rho)-G(x_\rho)\right|^p.$$

In particular, if we put $t = \rho+\theta$ for all $-\tau \leqslant \theta \leqslant 0$, and then use (16), we see that

$$E\left|x(\rho+\theta)\right|^p \leqslant \frac{e^{\gamma\tau}}{(1-(ke^{\gamma\tau})^{\frac{1}{p}})^p} E\left|x(\rho)-G(x_\rho)\right|^p < qE\left|x(\rho)-G(x_\rho)\right|^p.$$

Further, if we take $\phi = x_\rho$ in (9), that is $\phi(\theta) = x(\rho+\theta)$, $\phi(0) = x(\rho)$, we find that

$$\begin{aligned} &E\left\{\frac{p}{2}\left|x(\rho)-G(x_\rho)\right|^{p-4}\left(\left|x(\rho)-G(x_\rho)\right|^2\left[2\left(x(\rho)-G(x_\rho)\right)^T f(\rho,x_\rho) + \left|g(\rho,x_\rho)\right|^2\right]\right.\right. \\ &\quad \left.\left. + (p-2)\left|\left(x(\rho)-G(x_\rho)\right)^T g(\rho,x_\rho)\right|^2\right)\right\} \\ &\leqslant -\lambda E\left|x(\rho)-G(x_\rho)\right|^p. \end{aligned}$$

Since $\gamma < \lambda$ and since f, g and G are continuous, we have for all sufficiently small $h > 0$ and $\rho \leqslant t \leqslant \rho+h$,

$$\begin{aligned} &E\left\{\frac{p}{2}\left|x(t)-G(x_t)\right|^{p-4}\left(\left|x(t)-G(x_t)\right|^2\left[2\left(x(t)-G(x_t)\right)^T f(t,x_t) + \left|g(t,x_t)\right|^2\right] + (p-2)\left|\left(x(t)-G(x_t)\right)^T g(t,x_t)\right|^2\right)\right\} \\ &\leqslant -\gamma E\left|x(t)-G(x_t)\right|^p. \end{aligned}$$

On the other hand, the application of the Ito formula to $e^{\gamma t}|x(t)-G(x_t)|^p$ yields

$$\begin{aligned} &e^{\gamma(\rho+h)}E\left|x(\rho+h)-G(x_{\rho+h})\right|^p - e^{\gamma\rho}E\left|x(\rho)-G(x_\rho)\right|^p \\ &= E\int_\rho^{\rho+h} e^{\gamma t}\left[\gamma\left|x(t)-G(x_t)\right|^p + \frac{p}{2}\left|x(t)-G(x_t)\right|^{p-4}\left(\left|x(t)-G(x_t)\right|^2\left[2\left(x(t)-G(x_t)\right)^T f(t,x_t) + \left|g(t,x_t)\right|^2\right]\right.\right. \\ &\quad \left.\left. + (p-2)\left|\left(x(t)-G(x_t)\right)^T g(t,x_t)\right|^2\right)\right] dt + E\int_\rho^{\rho+h} pe^{\gamma t}\left|x(t)-G(x_t)\right|^{p-2}\left(x(t)-G(x_t)\right)^T g(t,x_t)\,dw_t. \end{aligned}$$

Since the last integral is equal to zero, on the basis of (9) we derive that

$$e^{\gamma(\rho+h)}E\left|x(\rho+h)-G(x_{\rho+h})\right|^p - e^{\gamma\rho}E\left|x(\rho)-G(x_\rho)\right|^p \leqslant 0.$$

However, this is a contradiction with respect to (18) and, therefore, the assumption (15) is valid. □

It is now easy to prove Theorem 1.

Proof of Theorem 1. Since (15) holds, we can apply Lemma 2 and (16) to conclude that

$$e^{\gamma t}E\left|x(t)\right|^p \leqslant \frac{(1+k^{\frac{1}{p}})^p}{(1-(ke^{\gamma\tau})^{\frac{1}{p}})^p} \sup_{-\tau\leqslant\theta\leqslant 0} E\left|\xi(\theta)\right|^p \leqslant q\left(1+k^{\frac{1}{p}}\right)^p \sup_{-\tau\leqslant\theta\leqslant 0} E\left|x(\theta)\right|^p$$

for all $t \geqslant 0$. What now remains is to take $\gamma \to \overline{\gamma}$, which completes the proof. □

Let us now introduce some conditions which, combined with the Razumikhin's approach, show that the pth moment exponential stability implies the almost sure exponential stability of the trivial solution to Eq. (1). Before that, we must prove the following lemma.

Lemma 4. *Let there exist a constant $k \in (0,1)$ such that*

$$\left|G(\varphi)\right|^p \leqslant k \sup_{-\tau \leqslant \theta \leqslant 0} \left|\varphi(\theta)\right|^p \tag{19}$$

for all $\varphi \in C([-\tau,0]; R^n)$. For a continuous function $z : [-\tau,0] \to R^n$ let us denote that $z_t = \{z(t+\theta), -\tau \leqslant \theta \leqslant 0\}$ for $t \geqslant 0$. Let $0 < \gamma < -\frac{1}{\tau}\ln k$ and $H > 0$. If

$$\left|z(t) - G(z_t)\right|^p \leqslant He^{-\gamma t} \quad \text{for all } t \geqslant 0, \tag{20}$$

then

$$\limsup_{t\to\infty} \frac{1}{t} \ln\left|z(t)\right| \leqslant -\frac{\gamma}{p}. \tag{21}$$

Proof. Let $ke^{\gamma\tau} < \varepsilon^{p-1} < 1$. Then, by applying the inequality (4) and the conditions (19) and (20), we find for any $T > 0$ that

$$\begin{aligned}\sup_{0\leqslant t\leqslant T}\left[e^{\gamma t}\left|z(t)\right|^p\right] &\leqslant \frac{1}{(1-\varepsilon)^{p-1}} \sup_{0\leqslant t\leqslant T}\left[e^{\gamma t}\left|z(t)-G(z_t)\right|^p\right] + \frac{1}{\varepsilon^{p-1}} \sup_{0\leqslant t\leqslant T}\left[e^{\gamma t}\left|G(z_t)\right|^p\right] \\ &\leqslant \frac{H}{(1-\varepsilon)^{p-1}} + \frac{ke^{\gamma\tau}}{\varepsilon^{p-1}} \sup_{-\tau\leqslant t\leqslant T}\left[e^{\gamma t}\left|z(t)\right|^p\right].\end{aligned}$$

Since

$$\left(1 - \frac{ke^{\gamma\tau}}{\varepsilon^{p-1}}\right) \sup_{0\leqslant t\leqslant T}\left[e^{\gamma t}\left|z(t)\right|^p\right] \leqslant \frac{H}{(1-\varepsilon)^{p-1}} + \frac{ke^{\gamma\tau}}{\varepsilon^{p-1}} \sup_{-\tau\leqslant t\leqslant 0}\left|z(t)\right|^p,$$

the required relation (21) holds straightforwardly. □

Theorem 2. *Let* (8) *hold for a constant $k \in (0,1)$ and let there exist a constant $K > 0$ such that*

$$E\left(\left|f(t,\phi)\right|^p + \left|g(t,\phi)\right|^p\right) \leqslant K \sup_{-\tau\leqslant\theta\leqslant 0} E\left|\phi(\theta)\right|^p \tag{22}$$

for all $t \geqslant 0$ and $\phi \in L^p_{\mathcal{F}}([-\tau,0]; R^n)$. Let also the trivial solution to Eq. (1) *be the pth moment exponentially stable, that is, there exists a pair of constants γ and M such that* (5) *holds. Then, the trivial solution to Eq.* (1) *is almost surely exponentially stable, that is,*

$$\limsup_{t\to\infty} \frac{1}{t} \ln\left|x(t;\xi)\right| \leqslant -\frac{\overline{\gamma}}{p} \quad \text{a.s.}, \tag{23}$$

where $\overline{\gamma} = \min\{\gamma, -\frac{1}{\tau}\ln k\}$. In particular, if (8), (9) *and* (22) *hold, then the trivial solution to Eq.* (1) *is almost surely exponentially stable.*

Proof. As before, for a fixed ξ let us simply take $x(t)$ instead of the solution $x(t;\xi)$. By applying the elementary inequality $|\sum_{k=1}^s a_k|^p \leqslant s^{p-1}\sum_{k=1}^s |a_k|^p$, $p \geqslant 1$, the Hölder inequality and the well-known Burkholder–Davis–Gundy inequality [4,15], as well as (22), we find for every $n \in N$ that

$$E\sup_{0\leqslant\theta\leqslant\tau}\left|x(n\tau+\theta) - G(x_{n\tau+\theta})\right|^p \leqslant 3^{p-1}E\left|x(n\tau) - G(x_{n\tau})\right|^p + 3^{p-1}K\left(\tau^{p-1} + c_p\tau^{\frac{p}{2}-1}\right)\int_{n\tau}^{(n+1)\tau} \sup_{-\tau\leqslant\theta\leqslant 0} E\left|x(s+\theta)\right|^p ds.$$

In view of (23) and (8) we derive

$$\begin{aligned}E\sup_{0\leqslant\theta\leqslant\tau}\left|x(n\tau+\theta) - G(x_{n\tau+\theta})\right|^p \leqslant 3^{p-1}M\Bigg[&2^{p-1}(1+k)e^{-\overline{\gamma}(n\tau-\tau)} \\ &+ K\left(\tau^{p-1} + c_p\tau^{\frac{p}{2}-1}\right)\int_{n\tau}^{(n+1)\tau} e^{-\gamma(s+\theta)}\,ds\Bigg] \sup_{-\tau\leqslant\theta\leqslant 0} E\left|\xi(\theta)\right|^p.\end{aligned}$$

Since $e^{-\gamma(s+\theta)} \leqslant e^{-\overline{\gamma}(s-\tau)}$ for $n\tau \leqslant s \leqslant (n+1)\tau$ and $-\tau \leqslant \theta \leqslant 0$, we conclude that

$$E \sup_{0\leqslant\theta\leqslant\tau} \left|x(n\tau+\theta) - G(x_{n\tau+\theta})\right|^p \leqslant Ce^{-\overline{\gamma}n\tau},$$

where C is a generic constant independent of n. Then, for an arbitrary $\varepsilon \in (0, \overline{\gamma})$,

$$P\left\{\omega\colon \sup_{0\leqslant\theta\leqslant\tau} \left|x(n\tau+\theta) - G(x_{n\tau+\theta})\right|^p \geqslant e^{-(\overline{\gamma}-\varepsilon)n\tau}\right\} \leqslant Ce^{-\varepsilon n\tau}.$$

The application of the Borel–Cantelly lemma yields that there exists an $n_0(\omega)$ so that, for almost all $\omega \in \Omega$,

$$\sup_{0\leqslant\theta\leqslant\tau} \left|x(n\tau+\theta) - G(x_{n\tau+\theta})\right|^p \leqslant e^{-(\overline{\gamma}-\varepsilon)n\tau}$$

holds whenever $n \geqslant n_0(\omega)$. Moreover,

$$\left|x(t) - G(x_t)\right|^p \leqslant e^{-(\overline{\gamma}-\varepsilon)(t-\tau)} \quad \text{a.s. for } t \geqslant n_0\tau.$$

Since $|x(t) - G(x_t)|^p$ is a.s. finite on $[0, n_0\tau]$, there exists an a.s. finite number $H = H(\omega)$ such that

$$\left|x(t) - G(x_t)\right|^p \leqslant He^{-(\overline{\gamma}-\varepsilon)t} \quad \text{for } t \geqslant 0.$$

Recall that $C([-\tau, 0]; R^n) \subset L^p_{\mathcal{F}}([-\tau, 0]; R^n)$. Since the condition (8) implies (19), the application of Lemma 4 yields

$$\limsup_{t\to\infty} \frac{1}{t} \ln\left|x(t)\right| \leqslant -\frac{\overline{\gamma}-\varepsilon}{p} \quad \text{a.s.}$$

Therefore, what remains is to let $\varepsilon \to 0$. $\square$

3. Some consequences

The pth moment exponential stability of solutions to a more general class of neutral stochastic functional differential equations was recently discussed in paper [5] by S. Janković and M. Jovanović. However, this problem can also be discussed by employing the Razumikhin-type technique. Precisely, we will consider the following equation

$$d\left[x(t) - G(x_t)\right] = \left[f_1\left(t, x(t)\right) + f(t, x_t)\right]dt + g(t, x_t)\,dw(t), \quad t \geqslant 0, \tag{24}$$

with initial data $x_0 = \xi$, where G, f, g and ξ are defined as before, and $f_1 : R_+ \times R^n \to R^n$, $f_1(t, 0) \equiv 0$. Recall that in [13] by X. Mao sufficient conditions on the mean square exponential stability of the solution to this equation are also discussed by using the Razumikhin's approach. In fact, the following theorem represents an extension relative to the results from [13].

Theorem 3. *Let* (8) *be valid and let there exist positive constants* λ_1 *and* λ_2 *such that*

$$\begin{aligned} &E\left\{\frac{p}{2}\left|\phi(0) - G(\phi)\right|^{p-4}\left(\left|\phi(0) - G(\phi)\right|^2\left[2\left(\phi(0) - G(\phi)\right)^T\left(f_1\left(t, \phi(0)\right) + f(t, \phi)\right) + \left|g(t, \phi)\right|^2\right]\right.\right.\\ &\qquad\left.\left. + (p-2)\left|\left(\phi(0) - G(\phi)\right)^T g(t, \phi)\right|^2\right)\right\}\\ &\quad\leqslant -\lambda_1 E\left|\phi(0)\right|^p + \lambda_2 \sup_{-\tau\leqslant\theta\leqslant0} E\left|\phi(\theta)\right|^p \end{aligned} \tag{25}$$

holds for all $t \geqslant 0$ *and* $\phi \in L^p_{\mathcal{F}}([-\tau, 0]; R^n)$. *If*

$$0 < k < \frac{1}{2^p}, \qquad \lambda_1 > \frac{\lambda_2}{(1 - 2k^{\frac{1}{p}})^p}, \tag{26}$$

then the trivial solution to Eq. (24) *is the* p*th moment exponentially stable.*

Proof. It is easy to check from (26) that there exists $q > 0$ satisfying

$$\frac{1}{k} > q > \frac{1}{(1 - k^{\frac{1}{p}})^p}, \qquad \lambda_1 > \frac{\lambda_2 q}{(1 - (kq)^{\frac{1}{p}})^p}. \tag{27}$$

Since Eq. (24) can be rewritten as Eq. (1) by putting $\widehat{f}(t, \phi) = f_1(t, \phi(0)) + f(t, \phi)$ for all $t \geqslant 0$ and $\phi \in C([-\tau, 0]; R^n)$, it is necessary to verify the condition (9) to prove this assertion.

First, let (10) hold for a chosen q satisfying (27) and for every $t \geqslant 0$ and $\phi \in L^p_{\mathcal{F}}([-\tau, 0]; R^n)$, that is, let

$$E|\phi(\theta)|^p < qE|\phi(0) - G(\phi)|^p, \quad -\tau \leqslant \theta \leqslant 0. \tag{28}$$

On the other hand, by applying the inequality (4) and the conditions (8) and (28), we find for any $\varepsilon \in (0, 1)$ that

$$E|\phi(0) - G(\phi)|^p \leqslant \frac{1}{(1-\varepsilon)^{p-1}} E|\phi(0)|^p + \frac{1}{\varepsilon^{p-1}} E|G(\phi)|^p \leqslant \frac{1}{(1-\varepsilon)^{p-1}} E|\phi(0)|^p + \frac{kq}{\varepsilon^{p-1}} E|\phi(0) - G(\phi)|^p,$$

and hence

$$-E|\phi(0)|^p \leqslant -(1-\varepsilon)^{p-1}\left(1 - \frac{kq}{\varepsilon^{p-1}}\right) E|\phi(0) - G(\phi)|^p. \tag{29}$$

Let $\varepsilon = (kq)^{\frac{1}{p}}$. By using the estimates (28) and (29), we find from (25) that

$$\begin{aligned} E\bigg\{ &\frac{p}{2}|\phi(0) - G(\phi)|^{p-4}\Big(|\phi(0) - G(\phi)|^2\big[2(\phi(0) - G(\phi))^T\big(f_1(t, \phi(0)) + f(t,\phi)\big) + |g(t,\phi)|^2\big] \\ &+ (p-2)\big|(\phi(0) - G(\phi))^T g(t,\phi)\big|^2\Big)\bigg\} \\ &= -\big[\lambda_1\big(1 - (kq)^{\frac{1}{p}}\big)^p - \lambda_2 q\big] E|\phi(0) - G(\phi)|^p. \end{aligned}$$

Since (27) implies $\lambda_1(1 - (kq)^{\frac{1}{p}})^p - \lambda_2 q > 0$, the condition (9) is also satisfied, so that the desired conclusion follows from Theorem 1. □

To compare Theorem 3 with the appropriate results from [5], let us introduce the family $\mathcal{W}([-\tau, 0]; R_+)$ of Borel-measurable bounded non-negative functions $\eta(\theta)$, $-\tau \leqslant \theta \leqslant 0$, such that $\int_{-\tau}^0 \eta(\theta)\,d\theta = 1$ (weighting function) and emphasize the conditions:

(i) There exist a constant $k \in (0, 1)$ and a function $\eta_1 \in \mathcal{W}([-\tau, 0]; R_+)$ such that

$$|G(\varphi)|^p \leqslant k \int_{-\tau}^{0} \eta_1(\theta)|\varphi(\theta)|^p\,d\theta \tag{30}$$

for all $\varphi \in C([-\tau, 0]; R^n)$.

(ii) There exist constants $\lambda_1, \lambda_2 \geqslant 0$ and a function $\eta_2 \in \mathcal{W}([-\tau, 0]; R_+)$ such that

$$\begin{aligned} &\frac{p}{2}|\varphi(0) - G(\varphi)|^{p-4}\Big(|\varphi(0) - G(\varphi)|^2\big[2(\varphi(0) - G(\varphi))^T\big(f_1(t, \varphi(0)) + f(t,\varphi)\big) + |g(t,\varphi)|^2\big] \\ &\quad + (p-2)\big|(\varphi(0) - G(\varphi))^T g(t,\varphi)\big|^2\Big) \\ &\quad \leqslant -\lambda_1|\varphi(0)|^p + \lambda_2 \int_{-\tau}^{0} \eta_2(\theta)|\varphi(\theta)|^p\,d\theta \end{aligned} \tag{31}$$

for all $t \geqslant 0$ and $\varphi \in C([-\tau, 0]; R^n)$.

Then, the assertion that follows is valid.

Theorem 4. *(See [5].) Let the conditions* (30) *and* (31) *hold for a constant $k \in (0, 1)$ and for some functions $\eta_1, \eta_2 \in \mathcal{W}([-\tau, 0]; R_+)$. If $0 \leqslant \lambda_2 < \lambda_1$, then the trivial solution to Eq.* (24) *is the pth moment exponentially stable.*

Note that the requirement $\lambda_1 > \lambda_2$ is much sharper than (26). Likewise, the conditions (30) and (31) are more restrictive than the conditions of Theorem 3. For instance, Theorem 3 can be applied to the study of the pth moment exponential stability for neutral stochastic differential delay equations, while Theorem 4 cannot. To see this, let

$$d\big[x(t) - \bar{G}(x(t-\tau))\big] = \bar{f}(t, x(t), x(t-\tau))\,dt + \bar{g}(t, x(t), x(t-\tau))\,dw(t) \tag{32}$$

be a neutral stochastic differential delay equation with initial data $x_0 = \xi \in L^p_{\mathcal{F}_0}([-\tau, 0]; R^n)$, where $w(t)$ is an m-dimensional Brownian motion and $\bar{G}: R^n \to R^n$, $\bar{f}: R_+ \times R^n \times R^n \to R^n$, $\bar{g}: R_+ \times R^n \times R^n \to R^{n\times m}$ are Borel-measurable functions satisfying $\bar{G}(0) \equiv 0$, $\bar{f}(t, 0, 0) \equiv 0$, $\bar{g}(t, 0, 0) \equiv 0$. We also assume that they are smooth enough so that there exists a global unique solution $x(t; \xi)$ to Eq. (32). The following assertions hold directly from Theorem 3.

Corollary 1. *Let there exist a constant $k \in (0,1)$ such that*

$$\left|\bar{G}(x)\right| \leqslant k^{\frac{1}{p}}|x|, \quad x \in R^n. \tag{33}$$

In addition, let there exist positive constants λ_1 and λ_2 such that

$$\begin{aligned}&\frac{p}{2}\left|x-\bar{G}(y)\right|^{p-4}\left(\left|x-\bar{G}(y)\right|^2\left[2\left(x-\bar{G}(y)\right)^T\bar{f}(t,x,y)+\left|\bar{g}(t,x,y)\right|^2\right]+(p-2)\left|\left(x-\bar{G}(y)\right)^T\bar{g}(t,x,y)\right|^2\right)\\&\quad\leqslant -\lambda_1|x|^p+\lambda_2|y|^p\end{aligned}$$

for all $t \geqslant 0$ and $x, y \in R^n$. If (26) *holds, then the trivial solution to Eq.* (32) *is the pth moment exponentially stable.*

The proof follows straightforwardly from Theorem 3 since Eq. (32) can be rewritten as Eq. (24), by putting

$$\begin{aligned}&G(\varphi)=\bar{G}\big(\varphi(-\tau)\big), \qquad f_1(t,x)=\bar{f}(t,x,0),\\&f(t,\varphi)=-\bar{f}\big(t,\varphi(0),0\big)+\bar{f}\big(t,\varphi(0),\varphi(-\tau)\big),\\&g(t,\varphi)=\bar{g}\big(t,\varphi(0),\varphi(-\tau)\big)\end{aligned}$$

for all $t \geqslant 0$, $x \in R^n$ and $\varphi \in C([-\tau,0];R^n)$.

As usual, let $L^p_{\mathcal{F}}(\Omega;R^n)$ be a family of R^n-valued $\mathcal{F}$-measurable random variables X such that $E|X|^p < \infty$.

Theorem 5. *Let the condition* (8) *be satisfied for a constant $k \in (0,1)$ and let $q > (1-k^{\frac{1}{p}})^{-p}$. If there exists a constant $\lambda > 0$ such that*

$$\begin{aligned}&E\left\{\frac{p}{2}\left|X-\bar{G}(Y)\right|^{p-4}\left(\left|X-\bar{G}(Y)\right|^2\left[2\left(X-\bar{G}(Y)\right)^T\bar{f}(t,X,Y)+\left|\bar{g}(t,X,Y)\right|^2\right]+(p-2)\left|\left(X-\bar{G}(Y)\right)^T\bar{g}(t,X,Y)\right|^2\right)\right\}\\&\quad\leqslant -\lambda E\left|X-\bar{G}(Y)\right|^p\end{aligned} \tag{34}$$

for all $t \geqslant 0$ and those $X, Y \in L^p_{\mathcal{F}}(\Omega;R^n)$ satisfying $E|Y|^p < qE|X-\bar{G}(Y)|^p$, then the trivial solution to Eq. (32) *is the pth moment exponentially stable. Moreover, if there exists a constant $K > 0$ such that*

$$\left|\bar{f}(t,x,y)\right|^p+\left|\bar{g}(t,x,y)\right|^p \leqslant K\left(|x|^p+|y|^p\right) \tag{35}$$

for all $t \geqslant 0$ and $x, y \in R^n$, then the trivial solution to Eq. (32) *is almost surely exponentially stable.*

Since Eq. (32) can be transformed into Eq. (1) by putting

$$G(\varphi)=\bar{G}\big(\varphi(-\tau)\big), \qquad f(t,\varphi)=\bar{f}\big(t,\varphi(0),\varphi(-\tau)\big), \qquad g(t,\varphi)=\bar{g}\big(t,\varphi(0),\varphi(-\tau)\big)$$

for all $t \geqslant 0$ and $\varphi \in C([-\tau,0];R^n)$, the proof follows directly from Theorems 1 and 2.

Example. By applying Theorem 5, let us determine stability conditions under which the trivial solution to the following neutral stochastic differential delay equation

$$d\big[x(t)-\bar{G}\big(x(t-\tau)\big)\big]=-ax(t)+\bar{g}\big(t,x(t),x(t-\tau)\big)\,dw(t), \quad t \geqslant 0,\ x_0=\xi,$$

is the pth moment and almost surely exponentially stable. We assume that $w(t)$ is an m-dimensional Brownian motion, the functions $\bar{G}: R^n \to R^n$ and $\bar{g}: R_+ \times R^n \times R^n \to R^{n\times m}$ are Borel-measurable, $\bar{G}(0) \equiv 0$, $\bar{g}(t,0,0) \equiv 0$ and

$$\left|\bar{G}(y)\right| \leqslant \alpha|y| \quad \text{for all } y, \qquad \left|\bar{g}(t,x,y)\right| \leqslant b|y| \quad \text{for all } t,x,y,$$

where $0 < \alpha < 1$ and a, b are positive constants. Since $E|\bar{G}(Y)|^p \leqslant \alpha^p E|Y|^p$ for all $Y \in L^p_{\mathcal{F}}(\Omega;R^n)$, the condition (8) holds with $k = \alpha^p < 1$.

The condition (34) yields, for any $\varepsilon > 0$,

$$\begin{aligned}L&=E\left[\frac{p}{2}\left|X-\bar{G}(Y)\right|^{p-4}\left(\left|X-\bar{G}(Y)\right|^2\left[2\left(X-\bar{G}(Y)\right)^T(-aX)+\left|\bar{g}(t,X,Y)\right|^2\right]+(p-2)\left|\left(X-\bar{G}(Y)\right)^T\bar{g}(t,X,Y)\right|^2\right)\right]\\&\leqslant E\left[\frac{p}{2}\left|X-\bar{G}(Y)\right|^{p-2}\left(-2a\left|X-\bar{G}(Y)\right|^2-2a\left(X-\bar{G}(Y)\right)^TG(Y)+(p-1)b^2|Y|^2\right)\right]\\&\leqslant \frac{p}{2}\left[(-2a+\varepsilon a)E\left|X-\bar{G}(Y)\right|^p+\left(\frac{a\alpha^2}{\varepsilon}+(p-1)b^2\right)E\left(\left|X-\bar{G}(Y)\right|^{p-2}|Y|^2\right)\right].\end{aligned}$$

For $p > 2$, the application of the Hölder inequality to the last term yields, letting $\mu = p/(p-2)$, $\nu = p/2$, $1/\mu + 1/\nu = 1$,

$$L \leqslant \frac{p}{2}\left[(-2a + \varepsilon a)E\left|X - \bar{G}(Y)\right|^p + \left(\frac{a\alpha^2}{\varepsilon} + (p-1)b^2\right)\left(E\left|X - \bar{G}(Y)\right|^p\right)^{\frac{p-2}{p}}\left(E|Y|^p\right)^{\frac{2}{p}}\right].$$

Then, for those $X, Y \in L^p_{\mathcal{F}}(\Omega; R^n)$ satisfying $E|Y|^p < qE|X - \bar{G}(Y)|^p$, where $q > (1-\alpha)^{-p}$, we have

$$L < \frac{p}{2}\left[-2a + \varepsilon a + \left(\frac{a\alpha^2}{\varepsilon} + (p-1)b^2\right)q^{\frac{2}{p}}\right]E\left|X - \bar{G}(Y)\right|^p.$$

Having in mind (34), we will require that

$$2a > \varepsilon a + \left(\frac{a\alpha^2}{\varepsilon} + (p-1)b^2\right)q^{\frac{2}{p}} \equiv f(\varepsilon),$$

that is, $2a > \min_{\varepsilon>0} f(\varepsilon)$. Since $\min_{\varepsilon>0} f(\varepsilon) = f(\alpha q^{\frac{1}{p}}) = 2a\alpha q^{\frac{1}{p}} + (p-1)b^2 q^{\frac{2}{p}}$ and since $q > (1-\alpha)^{-p}$, the pth moment exponential stability condition has the form

$$2a(1-\alpha)(1-2\alpha) > (p-1)b^2, \tag{36}$$

where $0 < \alpha < 1/2$. Then,

$$L < \frac{p}{2}\left[-2a + f\left(\alpha q^{\frac{1}{p}}\right)\right]E\left|X - \bar{G}(Y)\right|^p = \frac{p}{2}\left[-2a + 2a\alpha q^{\frac{1}{p}} + (p-1)b^2 q^{\frac{2}{p}}\right]E\left|X - \bar{G}(Y)\right|^p \tag{37}$$

and, therefore, $\limsup_{t\to\infty} \frac{1}{t} \ln E|x(t)|^p \leqslant -\overline{\gamma}$, where, from (12) and (37),

$$\overline{\gamma} = \min\left\{-\frac{p}{2}\left[-2a + 2a\alpha q^{\frac{1}{p}} + (p-1)b^2 q^{\frac{2}{p}}\right], -\frac{1}{\tau}\ln\frac{q}{(1+\alpha q^{\frac{1}{p}})^p}\right\}.$$

Moreover, from Theorem 2 it follows that the trivial solution is also almost surely exponentially stable, i.e.,

$$\limsup_{t\to\infty} \frac{1}{t}\ln\left|x(t)\right| \leqslant -\frac{\tilde{\gamma}}{p}, \quad \text{where } \tilde{\gamma} = \min\left\{\overline{\gamma}, -\frac{p}{\tau}\ln\alpha\right\}.$$

Consequently, it is easy to see that the condition (36) is also valid for $p = 2$.

Note that if $\bar{G}(y) = \alpha y$ and $\bar{g}(t, x, y) = by$, then (36) is the stability condition of the autonomous linear neutral stochastic differential delay equation.

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