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The asymptotic values of a polynomial function on the real plane

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Abstract

Let a polynomial function f of two real variables be given. We prove the existence of a finite number of unbounded regions of the real plane along which the tangent planes to the graph of f tend to horizontal position, when moving away from the origin. The real limit values of this function on these regions are called asymptotic values. We also define the real critical values at infinity of f and prove the theorem of local trivial fibration at infinity, away from these values.

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0. Introduction

Given a polynomial function $f: \mathbb{R}^2 \to \mathbb{R}$, intuition tells us that the number of unbounded regions of the real plane $\mathbb{R}^2$ along which the tangent planes to the graph of f tend to horizontal position should be finite. We begin the paper by making these concepts precise and proving the theorem. Of course, a necessary condition for one such region to exist is the existence of an unbounded sequence of points in $\mathbb{R}^2$ on which ∇f tends to $\bar{0}$ and f tends to a real value $\lambda \in \mathbb{R}$. Then the curve selection lemma provides a half-branch C at infinity of an algebraic curve such that $\lim_C f = \lambda$ and $\lim_C \nabla f = \bar{0}$. We call these values *asymptotic values for f*. The set of such values λ is related to the smallest set $S \subset \mathbb{R}$ such that $f: \mathbb{R}^2 \setminus f^{-1}[S] \to \mathbb{R} \setminus S$ is a locally trivial fibration; more precisely, every λ in S which is not a critical value of f (in the usual sense) is asymptotic – the converse does not hold, in general. The elements in S which are not critical values are called *real critical values at infinity for f*, in this paper. The

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analogous set, replacing $\mathbb{R}$ by $\mathbb{C}$, has been studied in [5–8]. We know no reference for these questions, in the real case. Back to the fibration problem, the real case does not follow from what is known in the complex case; to begin with, examples show that the definition of critical value at infinity in [5, 8] must be refined to adapt well to the real case (the change of sign of a partial derivative of f has to be taken into account). We conclude the paper with the theorem of local trivial fibration at infinity, away from the set S of real critical values at infinity.

At the time of writing, we were motivated by the following problem: is every open polynomial map $f: \mathbb{R}^2 \to \mathbb{R}^2$ closed? A recent example of Pinchuk [9] answers this question in the negative. However, we trust that our results might be useful in understanding the geometry of such examples.

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0.1. Definitions and notations

Let Φ be the filter of neighborhoods of the point at infinity in the Alexandroff compactification of $\mathbb{R}^2$. Germs of semialgebraic subsets of $\mathbb{R}^2$ with respect to Φ [see 2, ch. I, §§6,7] are called *germs at infinity* or simply *germs*, in this paper. If $\mathscr{A} \subseteq \mathbb{R}^2$ is semialgebraic, then the germ of $\mathscr{A}$ is denoted A and $\mathscr{A}$ is a *representative* of A. However, we will not make the distinction between a germ and one suitably chosen representative of it, when there is no risk of confusion.

Let $k = 1, 2$. A germ is *k-dimensional* if every of its representatives has dimension $\geq k$ and some has dimension k. A germ is *open* (respectively, *closed*) if some of its representatives is open (respectively, closed). A germ A is *connected* if for every $\mathscr{F} \in \Phi$ there exists a connected representative $\mathscr{A}$ of A with $\mathscr{A} \subseteq \mathscr{F}$. All germs considered in this paper are connected, unless otherwise stated. For example, it is easy to realize that the notion of one-dimensional germ coincides with that of half-branch at infinity of an algebraic affine curve (see [1]). Notice that for every continuous function $f: \mathbb{R}^2 \to \mathbb{R}$ and every k-dimensional germ A, if $\lim_A f$ exists in $\mathbb{R} \cup \{\pm\infty\}$ then it is unique, by connectedness.

We fix an embedding of the real affine plane $\mathbb{R}^2$ into the real projective plane $P(\mathbb{R}^2)$, mapping the affine coordinates (x, y) of a point to the projective coordinates $(x : y : 1)$ of it. The *line at infinity* is, as usual, the set $P(\mathbb{R}^2) \backslash \mathbb{R}^2$, denoted L_∞. Let A be a k-dimensional germ ($k = 1, 2$); the set $L_\infty \cap \bar{A}$ is called the *center* of A, where $\bar{A}$ denotes the closure of A with respect to the euclidean topology on $P(\mathbb{R}^2)$. It will be denoted p_A. Since germs are assumed to be connected, p_A is connected too.

By the *degree* of a polynomial function $f: \mathbb{R}^2 \to \mathbb{R}$ we mean the degree of the polynomial expression that defines the function. The degree of the zero polynomial is $-\infty$, by convention.

If $f(X, Y) = \sum_{i,j=0}^{i+j=d} a_{i,j} X^i Y^j \in \mathbb{R}[X, Y]$ is a polynomial of two real variables of degree $d \geq 0$, let $f^* \in \mathbb{R}[X, Y, Z]$ denote the *standard homogeneization of f*, i.e., $f^*(X, Y, Z) = \sum_{i,j=0}^{i+j=d} a_{i,j} X^i Y^j Z^{d-i-j}$. Let $V(f)$ (respectively $V(f^*)$) denote the set of zeros of f (respectively f^*) in $\mathbb{R}^2$ (respectively $P(\mathbb{R}^2)$). We say that f is *monic in* Y if $a_{0,d} \neq 0$ (this is slightly different from the usual meaning); in this case f may be written $f(X, Y) = b_0 Y^d + b_1(X) Y^{d-1} + \cdots + b_d(X)$, with $\deg(f) = d, b_0 \neq 0$ and $\deg(b_i) \leq i, i = 1, \ldots, d$. In any case, an affine change of coordinates can be found that turns f into a monic polynomial in Y. Write $f = f_d + h$, where $f_d, h \in \mathbb{R}[X, Y]$, f_d homogeneous, $\deg(f) = \deg(f_d) > \deg(h)$ and let $\Sigma(f) := V(f_d) \cap L_\infty$. It is easy to verify that

(a) $V((f - \lambda)^*) \cap L_\infty = \Sigma(f)$, for every $\lambda \in \mathbb{R}$,

(b) $(0: 1: 0) \notin \Sigma(f)$ if and only if f is monic in Y.

1. The behavior of polynomial functions on 1-dimensional germs

Lemma 1.1. *Let* $f: \mathbb{R}^2 \to \mathbb{R}$ *be a polynomial function and let* C *be a* 1*-dimensional germ. If* $f|_C$ *has infinitely many extrema then* $f|_C$ *is constant.*

Proof. If $f|_C$ is not constant, then the set of local extrema is a semialgebraic set of dimension 0, hence finite. □

By the previous lemma, if C is a 1-dimensional germ, it makes sense to say that f is *monotonic along* C. Hence $\lim_C f$ exists in $\mathbb{R} \cup \{\pm\infty\}$.

By a *germ of Nash function* we mean a germ of functions $g: \mathscr{L} \to \mathbb{R}$, where $\mathscr{L} \subset \mathbb{R}$ is an unbounded interval (therefore, $\mathscr{L}$ represents a 1-dimensional germ in $\mathbb{R}$) and g is a Nash (also called analytic-algebraic) function on $\mathscr{L}$. It will be denoted by $g: L \to \mathbb{R}$. We will not make the distinction between a germ of Nash function and one suitably chosen representative of it. Germs of Nash functions and 1-dimensional germs in $\mathbb{R}^2$ are related to each other in the following way. If g is a germ of Nash function then the graph of g is a 1-dimensional germ. Conversely, if C is a 1-dimensional germ, $p_C = (p_1 : p_2 : 0)$, say $p_1 \neq 0$, then there exists $M \in \mathbb{R}^+$ such that $C \subset [(-\infty, -M) \times \mathbb{R}] \cup [(M, +\infty) \times \mathbb{R}]$. Say $C \subset (M, +\infty) \times \mathbb{R}$. From [1, Theorem 2.3.1], it follows that C is represented by the graph of a semialgebraic function defined on the interval $(M, +\infty)$ – in our case, the function can be chosen to be Nash. In this situation we say that *C has a parametrization by a germ of Nash function on the X-axis.* We conclude that a germ of Nash function $g: L \to \mathbb{R}$ is always monotonic and $\lim_L g$ exists in $\mathbb{R} \cup \{\pm\infty\}$ (see also [10, I.6.3]).

Lemma 1.2. *Let* C *be a* 1*-dimensional germ,* $p_C = (p_1 : p_2 : 0), f: \mathbb{R}^2 \to \mathbb{R}$ *a polynomial function such that* $\lim_C f \in \mathbb{R}$ and $\lim_C f_Y = 0$. If $p_1 \neq 0$ *then* $\lim_C f_X = 0$.

Proof. Since $p_1 \neq 0$, then C has a parametrization by a germ of Nash function $g : L \to \mathbb{R}$ defined on the X-axis. Moreover, $p_1 \neq 0$ implies that $\lim_L g' \notin \{\pm\infty\}$, whence $\lim_L g' \in \mathbb{R}$. Let $h := f \circ (\mathrm{id}, g) : L \to \mathbb{R}$; then h, h' are germs of Nash functions. Since $\lim_L h'$ exists in $\mathbb{R} \cup \{\pm\infty\}$, then the condition $\lim_L h = \lim_C f \in \mathbb{R}$ implies

$\lim_L h' = 0$. Now, from the expression $h'(t) = f_X(t, g(t)) + f_Y(t, g(t))\, g'(t)$, it follows that $\lim_C f_X = 0$. $\square$

Notice that if $\lim_C f = \mu \in \mathbb{R}$ then p_C belongs to $V((f-\mu)^*) \cap L_\infty = \Sigma(f)$.

2. The behavior of polynomial functions on 2-dimensional germs

Let A be an open 2-dimensional germ. Suppose that there exist germs of Nash functions $g_1 < g_2 : L \to \mathbb{R}$ such that A is (represented by) the "band"

$$(g_1, g_2) := \{(x, y) \in \mathbb{R}^2 : x \in \mathscr{L} \text{ and } g_1(x) < y < g_2(x)\}.$$

For instance, this is the case whenever $(0:1:0) \notin p_A$ since then there exists $M \in \mathbb{R}^+$ such that $A \subset [(-\infty, -M) \times \mathbb{R}] \cup [(M, +\infty) \times \mathbb{R}]$ and [1, Theorem 2.3.1] applies (in our situation, the semialgebraic functions found by virtue of this theorem can be chosen to be Nash). If, moreover, $\lim_{|x| \to +\infty} g_2(x) - g_1(x) = 0$, then we say that A is *narrow*.

Let $g_1 \le g_2 : L \to \mathbb{R}$ be germs of Nash functions and let $[g_1, g_2]$ denote the band

$$\{(x, y) \in \mathbb{R}^2 : x \in \mathscr{L} \text{ and } g_1(x) \le y \le g_2(x)\}.$$

Let the bands $[g_1, g_2)$ and $(g_1, g_2]$ have the obvious meaning. If A is a 2-dimensional germ such that A is (represented by) any of these bands, the definition of narrowness above still makes sense.

Lemma 2.1. *Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function, $\deg(f) = d > 0$, A a 2-dimensional germ. If f is bounded over A, then $\# p_A = 1$.*

Proof. The center p_A is connected; if there existed $p \ne q$ in the interior of p_A then A would contain infinitely many half-lines L_i, $i \in I$, all of them centered at p and one more half-line L_0 centered at q so that $L_i \cap L_0 \ne \emptyset$, for all i. Since f is bounded on L_i, then f would be constant on L_i, for all $i \in I \cup \{0\}$. Hence f would be constant, contrary to the assumption. $\square$

Notice that $\# p_A = 1$ does not imply that A is narrow; take, for instance, the 2-dimensional germ (represented by) $\{(x, y) \in \mathbb{R}^2 \colon x > 1 \text{ and } x^2 < y < x^3\}$.

Lemma 2.2. *Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function, $\deg(f) = d > 0$, A a 2-dimensional germ such that f is bounded over A. If $\lim_A f_Y = 0$ and $p_A = (p_1 : p_2 : 0)$ with $p_1 \ne 0$ then $\lim_A f_X = 0$.*

Proof. Let C be any 1-dimensional germ contained in A and apply Lemma 1.2 to C. $\square$

If A is a k-dimensional germ ($k = 1, 2$) and $f: \mathbb{R}^2 \to \mathbb{R}$ is a polynomial function, it is easy to verify that $p_A \cap \Sigma(f) = \emptyset$ implies $\lim_A f \in \{\pm\infty\}$. On the other hand, we have the following result.

Proposition 2.3. *Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function,* $\deg(f) = d > 0$, *A a 2-dimensional germ. If $\lim_A f = \lambda \in \mathbb{R}$ then A is narrow.*

Proof. By iteration of the process of taking partial derivatives, it suffices to show that if $\lim_A f = \lambda$ and A is not narrow, then $\lim_A \nabla f = \bar{0}$. Say $p_A = (p_1 : p_2 : 0)$ with $p_1 \neq 0$. By Lemma 2.2, it suffices to prove $\lim_A f_Y = 0$. In order to prove this, we may assume $\lambda = 0$ and A open. Let $g_1 < g_2 : L \to \mathbb{R}$ be germs of Nash functions such that A is the band (g_1, g_2). Let $\alpha = \lim_{|x| \to +\infty} g_2(x) - g_1(x) \in \mathbb{R} \cup \{+\infty\}$. Let $(z_n)_n$ be an unbounded sequence in A and let V_n be a closed vertical segment of length $\alpha/2$, such that $z_n \in V_n \subseteq A$. Let $P_n := f \circ \phi_n$, where ϕ_n is the usual parametrization of V_n on the interval $[0, \alpha/2]$. Then $(P_n)_n$ is a sequence of polynomial functions of degree d, at most, defined on a compact interval. Now, the hypothesis $\lim_A f = 0$ implies that $(P_n)_n$ converges uniformly to zero on $[0, \alpha/2]$. This holds if and only if $(a_{nj})_n$ converges to zero, for all $j = 0, 1, \ldots, d$, where $P_n(t) = \sum_{j=0}^{d} a_{nj} t^j$, with $a_{nj} \in \mathbb{R}$. This implies that $(j a_{nj})_n$ converges to zero, for all $j = 0, 1, \ldots, d$, which means that $(P'_n)_n$ converges uniformly to zero on $[0, \alpha/2]$. Since $P'_n(t) = f_Y(\phi_n(t))$, then $\lim_n f_Y(z_n) = 0$. This shows that $\lim_A f_Y = 0$, as desired. □

3. Asymptotic classes and asymptotic values

Definitions 3.1. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function. The pair (λ, C) is called a *limit pair for f* if $\lambda \in \mathbb{R}$, C is a 1-dimensional germ and $\lim_C f = \lambda$. If, in addition, $\lim_C \nabla f = \bar{0}$, then (λ, C) is called an *asymptotic pair for f*. Accordingly, λ is called a *limit*, (respectively *asymptotic*) *value for f*.

In [3], the problem of relating the degree of a polynomial of two real variables with only nondegenerate critical points to the number of local minima, maxima and saddles is studied. There, emphasis is made on critical points in the usual sense and a notion of critical points at infinity is introduced as an auxiliary tool: the authors show that a polynomial with index greater than one must have critical points at infinity. The notion of critical point at infinity in [3] is weaker than what we call center of a 1-dimensional germ belonging to an asymptotic pair.

Proposition 3.2. *Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function and (λ, C) a limit pair for f. There exists a 2-dimensional germ A containing C such that* $\lim_A f = \lambda$.

Proof. The result is clear if $d = \deg(f) \leq 0$. Let us assume $d > 0$. Let $p_C = (p_1 : p_2 : 0)$, say with $p_1 \neq 0$. Let $g: L \to \mathbb{R}$ be a germ of Nash function parametrizing C on the

X-axis. The functions $g_1(x) = g(x) - x^{-d}$, $g_2(x) = g(x) + x^{-d}$, for $x \in \mathscr{L}$, are Nash too. Let A be the band (g_1, g_2). For every $(x, y) \in A$,

$$\begin{aligned}|f(x,y) - \lambda| &\le |f(x,y) - f(x, g(x))| + |f(x, g(x)) - \lambda| \\ &\le |f_Y(x, \xi)| \cdot |x|^{-d} + |f(x, g(x)) - \lambda|,\end{aligned}$$

where ξ lies in the segment between y and $g(x)$, by the mean value theorem. Now, $|f_Y(x, \xi)|$ is bounded by a polynomial expression in $|x|$ of degree $\le d - 1$. Since the right-hand side tends to 0, as $|x| \to +\infty$, it follows that $\lim_A f = \lambda$. □

Notice that the germ A found in the result above is narrow, if $\deg(f) > 0$, by Proposition 2.3.

Notations 3.3. Let $\{C_j : j = 1, \dots, m\}$ be different 1-dimensional germs. They determine, in an obvious manner, exactly m different closed 2-dimensional germs. These will be called *the 2-dimensional germs determined by* $\{C_j : j = 1, \dots, m\}$ and denoted $\{A_j : j = 1, \dots, m\}$. We may label these two families so that subscripts increase counterclockwise and C_j, C_{j+1} are the boundary of A_j, for all $j = 1, \dots, m$, where $C_{m+1} = C_1$.

Remarks 3.4. (a) Let $\{C_1, C_2\}$ be different 1-dimensional germs and $\{A_1, A_2\}$ be the 2-dimensional germs determined by $\{C_1, C_2\}$. Obviously, either A_1 or A_2 is not narrow. Moreover, if the centers p_i of C_i are different or, being equal, the tangent lines T_i to C_i at p_i are different, then both A_i are not narrow. Otherwise (i.e., $p_1 = p_2$ and $T_1 = T_2$) one of the A_i's is narrow, provided that $T_1 \ne L_\infty$.

(b) We already saw that given a polynomial f of degree $d > 0$ and a 1-dimensional germ C such that $\lim_C f = \mu \in \mathbb{R}$, then p_C belongs to $V((f - \mu)^*)$. Suppose that $V((f - \mu)^*)$ has $2d$ half-branches at infinity, $C_1, C_2, \dots, C_{2d}$. Then there exists i in $\{1, 2, \dots, 2d\}$ such that C_i is tangent to C at p_C. Indeed, there exists i such that for one of the 2-dimensional germs determined by $\{C, C_i\}$, call it A, we have $\lim_A f = \mu$. Then A is narrow, by Proposition 2.3 and the result follows now from (a).

Proposition 3.5. *Let f: $\mathbb{R}^2 \to \mathbb{R}$ be a polynomial function,* $\deg(f) = d$. *Suppose that (λ, C_j) are different limit pairs for f, $j = 1, 2, \dots, m$.* If $m \ge 2d + 1$, *then there exists $k \in \{1, 2, \dots, m\}$ such that* $\lim_{A_k} f = \lambda$ *(see Notations* 3.3).

Proof. We may assume $\lambda = 0$. By the theorem of Lagrange multipliers, the local extrema of f on the circle $V(X^2 + Y^2 - r)$ lie on the curve $V(Xf_Y - Yf_X)$, for all $r \in \mathbb{R}^+$. Consider the half-branches $\{D_j : j = 1, 2 \dots, 2t\}$ at infinity of $V(Xf_Y - Yf_X)$, with $2 \le 2t \le 2d$, and the 2-dimensional germs $\{B_j : j = 1, 2 \dots, 2t\}$ determined by them. Since $m \ge 2d + 1$, then there exist $k \in \{1, 2, \dots, m\}$ and $s \in \{1, 2, \dots, 2t\}$ such that $C_k, C_{k+1} \subset B_s$. Since $f|_{B_j \cap V(X^2 + Y^2 - r)}$ is monotonic for all $r \in \mathbb{R}^+$ big enough and all $j = 1, 2 \dots, 2t$, then $\lim_{C_k} f = \lim_{C_{k+1}} f = 0$ implies $\lim_{A_k} f = 0$. □

Notice that $f(X, Y) = \prod_{k=0}^{d-1}(Y - kX)$ shows that the bound is sharp.

Definition 3.6. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function. The limit pair (λ, C_1) is f-equivalent to the limit pair (μ, C_2) if $\lambda = \mu$ and either $\lim_{A_1} f = \lambda$ or $\lim_{A_2} f = \lambda$ (*see Notations* 3.3).

This defines an equivalence relation, the equivalence classes of which are called *limit classes for f*. The class of (λ, C) is denoted $[\lambda, C]_f$.

Corollary 3.7. *Let* $f\colon \mathbb{R}^2 \to \mathbb{R}$ *be a polynomial function,* $\lambda \in \mathbb{R}$. *There exists finitely many limit classes* $[\lambda, C]_f$.

Definition 3.8. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function. The asymptotic pairs $(\lambda, C_1), (\mu, C_2)$ are asymptotic f-equivalent if $\lambda = \mu$ and either $[\lim_{A_1} f = \lambda$ and $\lim_{A_1} \nabla f = \bar{0}]$ or $[\lim_{A_2} f = \lambda$ and $\lim_{A_2} \nabla f = \bar{0}]$.

This defines an equivalence relation. The class of (λ, C) will be denoted $\langle \lambda, C\rangle_f$ and called an *asymptotic class for f*.

Remark 3.9. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function and C_1, C_2 be different 1-dimensional germs. Suppose that $\lambda \in \mathbb{R}$ and $[\lambda, C_1]_f = [\lambda, C_2]_f$, $[0, C_1]_{f_X} = [0, C_2]_{f_X}$ and $[0, C_1]_{f_Y} = [0, C_2]_{f_Y}$. Then, there exist $\alpha, \beta, \gamma \in \{1, 2\}$ such that

$$\lim_{A_\alpha} f = \lambda, \quad \lim_{A_\beta} f_X = \lim_{A_\gamma} f_Y = 0.$$

We distinguish three cases:

Case 1: $\deg(f_X) > 0$. Then A_β and A_α are narrow, by Proposition 2.3, whence $A_\alpha = A_\beta$, by Remark 3.4. In addition, either $f_Y \equiv 0$ or $[\deg(f_Y) > 0$ and $A_\alpha = A_\beta = A_\gamma]$.

Case 2: $\deg(f_Y) > 0$. Analogous to case 1.

Case 3: $\deg(f) \le 0$. Obvious.

In all cases, it follows that $\langle \lambda, C_1\rangle_f = \langle \lambda, C_2\rangle_f$. Summarizing, $\langle \lambda, C_1\rangle_f \neq \langle \mu, C_2\rangle_f$ if and only if $[\lambda, C_1]_f \neq [\mu, C_2]_f$ or $[0, C_1]_{f_X} \neq [0, C_2]_{f_X}$ or $[0, C_1]_{f_Y} \neq [0, C_2]_{f_Y}$.

Theorem 3.10. *Let* $f\colon \mathbb{R}^2 \to \mathbb{R}$ *be a polynomial function. There exist finitely many asymptotic classes for f.*

Proof. The result is clear if $\deg(f) \le 1$. Let us assume $\deg(f) > 1$. Suppose there exist infinitely many asymptotic classes $\langle \lambda, C_\lambda\rangle_f$. Then, by Remark 3.9, one of the following holds:

(i) there exist infinitely many limit classes $[\lambda, C_\lambda]_f$,
(ii) there exists infinitely many limit classes $[0, C_\lambda]_{f_X}$,
(iii) there exists infinitely many limit classes $[0, C_\lambda]_{f_Y}$.

Now, by Corollary 3.7, (ii) and (iii) are impossible. Then (i) holds. Again by Corollary 3.7, there exist infinitely many limit values λ. Then, we may consider infinitely many asymptotic classes $\langle\lambda, C_\lambda\rangle_f$ so that the values λ are pairwise different. Those yield infinitely many limit pairs $(0, C_\lambda)$ for f_X. By Corollary 3.7, infinitely many of them belong to the same f_X-equivalence class. These are limit pairs for f_Y, as well. Thus we find $(0, C_\lambda), (0, C_\mu)$ limit pairs for both f_X and f_Y such that $[0, C_\lambda]_{f_X} = [0, C_\mu]_{f_X}$ and $[0, C_\lambda]_{f_Y} = [0, C_\mu]_{f_Y}$.

Summarizing, there exist values $\lambda, \mu \in \mathbb{R}$, $\lambda > \mu$, 1-dimensional germs C_1, C_2 and $i \in \{1, 2\}$ such that

$$\lim_{C_1} f = \lambda, \quad \lim_{C_2} f = \mu, \quad \lim_{A} \nabla f = \bar{0}.$$

By our assumption on $\deg(f)$, either $\deg(f_X) > 0$, or $\deg(f_Y) > 0$. Thus A_i is narrow, by Proposition 2.3.

There exist germs of Nash functions $g_1 \leq g_2 : L \to \mathbb{R}$ parametrizing C_1, C_2 respectively and such that A_i is the band $[g_1, g_2]$. Say L is contained in the X-axis. Then for every $x \in L$, the length $l(x) := g_2(x) - g_1(x)$ tends to 0, as $|x| \to +\infty$. Besides, there exists $N \in L$, such that $f(x, g_1(x)) > \lambda - (\lambda - \mu)/4$ and $f(x, g_2(x)) < \mu + (\lambda - \mu)/4$, for every $x \in L$ with $|x| > |N|$. Furthermore, we may assume $\|\nabla f\| \leq (\lambda - \mu)/4$ on A_i. Then, by the mean value theorem,

$$(\lambda - \mu)/4 < f(x, g_1(x)) - f(x, g_2(x)) \leq l(x)\,(\lambda - \mu)/4,$$

which is absurd. □

Corollary 3.11. *Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function. There exist finitely many asymptotic values for f.*

4. Real critical values at infinity

In [8], it is shown that for any polynomial $f \in \mathbb{C}[X, Y]$ and any linear form $l = l_1 X + l_2 Y$ sufficiently general for f, then the following conditions are equivalent:

(a) $\lambda \in \mathbb{C}$ is a critical value corresponding to singularities at infinity – i.e., λ belongs to the smallest set $S \subset \mathbb{C}$ such that $f: \mathbb{C}^2 \backslash f^{-1}[S] \to \mathbb{C} \backslash S$ is a locally trivial fibration and λ is not a critical value for f.

(b) there exists a point $(x(\mu), y(\mu))$ in $f^{-1}(\mu)$ of ramification with respect to l – i.e., $(x(\mu), y(\mu))$ satisfies $f_X(x(\mu), y(\mu)) l_2 - f_Y(x(\mu), y(\mu)) l_1 = 0$ – such that $\lim_{\mu \to \lambda} l_1 x(\mu) + l_2 y(\mu) = \infty$.

In the real case we observe that different phenomena occur.

Example A: Let $f(X, Y) = Y^3, f_X \equiv 0, f_Y = 3Y^2$. Then $\lambda = 0$ satisfies (b) above. Yet f induces a trivial fibration on $\mathbb{R}^2$.

Moreover, let $f \in \mathbb{R}[X, Y]$ be monic in Y. Suppose that (b) above holds for some $\lambda, \mu \in \mathbb{R}$ and $(x(\mu), y(\mu)) \in \mathbb{R}^2$. We have that $l = X$ is a linear form sufficiently general for f. Then the points $(x(\mu), y(\mu))$ lie on a half-branch at infinity C of $V(f_Y)$ and $\lim_C f = \lambda$. By Lemma 1.2, (λ, C) is in fact an asymptotic pair. Yet, the existence of asymptotic pairs does not imply the existence of values (other than critical values) in a neighborhood of which no local triviality holds.

Example B: Let $f(X, Y) = Y^5 + X^2Y^3 - 3XY^2 + 3Y$, $f_Y = 5Y^4 + 3(XY - 1)^2$, $f_X = Y^2(2XY - 3)$. Obviously $V(f_Y) = \emptyset$. However, $(0, C)$ is an asymptotic pair for f, where C is the germ (represented by) $\{(x, y) \in \mathbb{R}^2\colon xy - 1 = 0 \text{ and } x > 0\}$. The fibration that f induces on $\mathbb{R}^2$ is trivial.

Definitions 4.1. (a) Given a 1-dimensional germ C and $g \in \mathbb{R}[X, Y]$, we say that *g changes sign along C* if $C = C_k$ and either [g is positive on A_{k-1} and negative on A_{k+1}] or [g is negative on A_{k-1} and positive on A_{k+1}], where $\{C_j : j = 1, 2, \dots, 2t\}$ are the half-branches of $V(g)$ at infinity (see Notations 3.3).

(b) Let $f \in \mathbb{R}[X, Y]$, $\deg(f) = d > 0$, f monic in Y. A *real critical value at infinity for f* is a value $\lambda \in \mathbb{R}$ such that there exists a 1-dimensional germ C with $\lim_C f = \lambda$ and f_Y changes sign along C.

Notice that (λ, C) is then an asymptotic pair, by Lemma 1.2.

Theorem 4.2. *Let $f \in \mathbb{R}[X, Y]$, $\deg(f) = d > 0$, f monic in Y. If $(\alpha, \beta) \subset \mathbb{R}$ is an open bounded interval not containing any real critical value at infinity for f, then there exists $K \subset \mathbb{R}^2$ compact such that $f\colon f^{-1}[(\alpha, \beta)] \cap [\mathbb{R}^2 \backslash K] \to (\alpha, \beta)$ is a trivial fibration. Moreover, if $(\alpha, \beta) \subset \mathbb{R}$ is an open bounded interval containing a real critical value at infinity for f, then there exists no compact set $K \subset \mathbb{R}^2$ such that $f\colon f^{-1}[(\alpha, \beta)] \cap [\mathbb{R}^2 \backslash K] \to (\alpha, \beta)$ is a trivial fibration.*

Proof. By [1, Theorem 2.3.1], there exist $M \in \mathbb{R}^+$, $n, m \in \mathbb{N}$ and semialgebraic continuous functions $\xi_1 < \cdots < \xi_n\colon (-\infty, -M) \to \mathbb{R}$ and $\chi_1 < \cdots < \chi_m\colon (M, +\infty) \to \mathbb{R}$ such that $f^{-1}[(\alpha, \beta)] \cap [(-\infty, -M) \times \mathbb{R}]$ and $f^{-1}[(\alpha, \beta)] \cap [(M, +\infty) \times \mathbb{R}]$ are unions of bands, i.e.,

$$f^{-1}[(\alpha, \beta)] \cap [(-\infty, -M) \times \mathbb{R}] = \bigcup_{i \in I} (\xi_i, \xi_{i+1}),$$

for a certain subset $I \subseteq \{1, \dots, n\}$ and

$$f^{-1}[(\alpha, \beta)] \cap [(M, +\infty) \times \mathbb{R}] = \bigcup_{j \in J} (\chi_j, \chi_{j+1}),$$

for a certain $J \subseteq \{1, \dots, m\}$.

Suppose first that f_Y does not change sign on these bands. Thus, for every $x > M$, $i \in I$ and $j \in J$, the functions $f|_{V(X+x) \cap (\xi_i, \xi_{i+1})}$ and $f|_{V(X-x) \cap (\chi_j, \chi_{j+1})}$ are monotonous from α to β. We may further assume that $f(x, y) \notin (\alpha, \beta)$, when $|x| \le M$ and $|y| > M$ (this can be done considering the points in $\Sigma(f)$ closest to $(0:1:0)$ and rescaling). If we choose $K := [-M, M] \times [-M, M]$, then the first assertion follows.

Now, suppose that f_Y changes sign along C, $\lim_C f = 0$ and $0 \in (\alpha, \beta)$. Reducing (α, β), we may assume 0 is the only real critical value at infinity for f in the interval (α, β). Say $C \subset (\chi_1, \chi_2)$. Let $g: (M, +\infty) \to \mathbb{R}$ be a germ of Nash function parametrizing C. Say f_Y is positive on the band (χ_1, g) and negative on the band (g, χ_2). Then $(x, g(x))$ is a global maximum for $f|_{V(X-x)\cap(\chi_1,\chi_2)}$, for all $x \in (M, +\infty)$. Consider the germ of Nash function $h := f \circ (\mathrm{id}, g)$. We find three cases:

Case 1: $h \equiv 0$. Then $V(f+\varepsilon)$ meets (χ_1, χ_2) in two connected components and $V(f-\varepsilon)$ does not meet (χ_1, χ_2), for ε positive and small enough. Moreover, $V(f)$ meets (χ_1, χ_2) in C. Thus, there exists no compact $K \subset \mathbb{R}^2$ such that the fibration $f: f^{-1}[(\alpha, \beta)] \cap [\mathbb{R}^2 \backslash K] \to (\alpha, \beta)$ is trivial.

Case 2: *h is strictly decreasing*. We construct inductively x_n in $(M, +\infty)$, (x_n, y_n'), (x_n, y_n'') and $\varepsilon_n > 0$, for all n in $\mathbb{N}$, as follows. Given $x_n \in (M, +\infty)$, let $\varepsilon_n = h(x_n)$. Since $(x_n, g(x_n))$ is a global maximum for $f|_{V(X-x_n)\cap(\chi_1,\chi_2)}$, then there exist points (x_n, y_n') in (χ_1, g) and (x_n, y_n'') in (g, χ_2) and $0 < \varepsilon_{n+1} < \varepsilon_n$ such that $f(x_n, y_n') = f(x_n, y_n'') = \varepsilon_{n+1}$. Moreover, there exists $x_{n+1} \in (M, +\infty)$ with $x_n < x_{n+1}$ such that $h(x_{n+1}) = \varepsilon_{n+1}$. Thus, there exists no compact $K \subset \mathbb{R}^2$ such that the fibration $f: f^{-1}[(\alpha, \beta)] \cap [\mathbb{R}^2 \backslash K] \to (\alpha, \beta)$ is trivial.

Case 3: *h is strictly increasing*. It is similar to case 2. □

Let f be a polynomial in $\mathbb{R}[X, Y]$, monic in Y. The question arises whether given a critical value at infinity $\lambda \in \mathbb{R}$ (in the sense of [8]) such that there exist points $(x(\mu), y(\mu))$ satisfying (b) which tend to a real point on the line at infinity of $P(\mathbb{C}^2)$, then there exists a 1-dimensional germ D such that (λ, D) is an asymptotic pair for f. Conversely, is it true that if (λ, D) is an asymptotic pair for f, then λ is a critical value at infinity for f (in the sense of [8])? We presume that a result in the spirit of proposition 3.2 could provide affirmative answers to these two questions. One further question is whether the hypothesis (λ, C) asymptotic pair for f with λ real critical value at infinity implies that p_C is a critical point in $V((f-\lambda)^*)$.

In [4], the irregular values at infinity for a polynomial of two complex variables are defined as those $\lambda \in \mathbb{C}$ such that $\bar{F}|_D$ is not a local diffeomorphism onto $P(\mathbb{C})$ in a neighborhood of $\bar{F}^{-1}(\lambda)$ – here $\bar{F}$ is obtained as an extension of the original function to some algebraic variety $\bar{X}$ containing $\mathbb{C}^2$, by successive blowing-ups and D is the exceptional divisor with normal crossings. This technique can be applied to our setting: the polynomial function is considered as a rational morphism from $P(\mathbb{R}^2)$ to $P(\mathbb{R})$ having some indeterminacy points on L_∞. This would provide an alternative approach to the problems studied in this paper and it would be interesting to compare the results.

We leave also the question of studying these problems in more general settings, for instance, for $\mathbb{R}^n$ instead of $\mathbb{R}^2$ and for an arbitrary real closed field R instead of $\mathbb{R}$.

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