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Invariants of solvable Lie algebras with triangular nilradicals and diagonal nilindependent elements

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Abstract

The invariants of solvable Lie algebras with nilradicals isomorphic to the algebra of strongly upper triangular matrices and diagonal nilindependent elements are studied exhaustively. Bases of the invariant sets of all such algebras are constructed by an original purely algebraic algorithm based on Cartan's method of moving frames.

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1. Introduction

The purpose of this paper is to present the advantages of our purely algebraic algorithm for the construction of invariants with examples of solvable Lie algebras with nilradicals isomorphic to the algebra of strongly upper triangular matrices and nilindependent elements represented by diagonal matrices. In contrast to known methods, this approach is powerful enough to construct

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invariants of such Lie algebras in a closed form. First let us present the motivation behind this investigation.

Established work about invariants of Lie algebras can be conditionally divided into two mainstream types that are weakly connected with each other. One of them is more 'physical' and is mainly oriented to applications of invariants. The other one is more 'theoretical' and usually has a stronger mathematical background. We simultaneously survey works on the invariants within the frameworks of both. Note that invariant polynomials in Lie algebra elements are called the *Casimir operators*, while those which are not necessarily polynomials are called *generalized Casimir operators*.

The term 'Casimir operator' arose in the physical literature as a reference to [20]. At that time, only the lowest rank Lie algebras appeared to be of interest. In subsequent years the need to know the invariants of much larger Lie algebras arose more rapidly in physics than in mathematics.

In the mathematics literature it was soon recognized that the universal enveloping algebra $U(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$ contains elements (necessarily polynomial) that commute with any element of $\mathfrak{g}$, that there is a basis for all such invariants, and that the number of basis elements coincides with the rank of $\mathfrak{g}$. The degrees of the basis elements are given by the values of the exponents of the corresponding Weyl group (augmented by 1). The best known are the Casimir operators of degree 2 for semisimple Lie algebras. The explicit form of Casimir operators depends on the choice of the basis of $\mathfrak{g}$. The center of the universal enveloping algebra $U(\mathfrak{g})$ proved to be isomorphic to the space of polynomials on the dual space to $\mathfrak{g}$, which are invariant with respect to the coadjoint action of the corresponding Lie group [24]. This gives a basis for the calculation of Casimir operators by the infinitesimal and algebraic methods.

There are numerous papers on the properties and the specific computation of invariants of Lie algebras, on the estimation of their number and on the application of invariants of various classes of Lie algebras, or even of a particular Lie algebra which appears in physical problems (see the citations of this paper and references therein). Casimir operators are of fundamental importance in physics. They represent such important quantities as angular momentum, elementary particle mass and spin, Hamiltonians of various physical systems and they also provide information on quantum numbers that allow the characterization of the states of a system, etc. Generalized Casimir operators of Lie algebras are of great significance to representation theory as their eigenvalues provide labels to distinguish irreducible representations. For this reason it is of importance to have an effective procedure to determine these invariants explicitly, in order to evaluate them for the different representations of Lie algebras.

Unfortunately, up to the semi-simple case, which was completely solved in the 1960s, there is no general theory that allows the construction of the generalized Casimir invariants of Lie algebras. The standard infinitesimal method became conventional for the calculations of invariants. It is based on integration of overdetermined systems of first-order linear partial differential equations associated with infinitesimal operators of coadjoint action. This is why it is effective only for the algebras of a quite simple structure or of low dimensions.

The interest in finding all independent invariants of Lie algebras was recognized a few decades ago [1,5,34,36,37,41,46]. In particular, functional bases of invariants were calculated for all three-, four-, five-dimensional and nilpotent six-dimensional real Lie algebras in [34]. The same problem was considered in [28] for the six-dimensional real Lie algebras with four-dimensional nilradicals. In [35] the subgroups of the Poincaré group along with their invariants were found. There is a more detailed review of the low-dimensional algebras and their invariants in [6,40]. The cardinality of invariant bases was calculated by different formulas within the framework of the infinitesimal approach [5,14]. Invariants of Lie algebras with various additional structural restrictions were also constructed. Namely, the solvable Lie algebras with the nilradicals isomorphic to the Heisenberg algebras [42], with Abelian nilradicals [29,31], with nilradicals containing Abelian ideals of codimension 1 [43], solvable triangular algebras [45], some solvable rigid Lie algebras [10,11], solvable Lie algebras with graded nilradical of maximal nilindex and a Heisenberg subalgebra [3], different classes of unsolvable algebras [15,16,30]. Empiric techniques were also applied for finding invariants of Lie algebras (e.g. [4]).

The existence of bases consisting entirely of Casimir operators (polynomial invariants) is important for the theory of generalized Casimir operators and for their applications. It was shown that it is the case for the semi-simple, nilpotent, perfect and more general algebraic Lie algebras [1,2]. Properties of Casimir operators of some perfect Lie algebras and estimations for their number were investigated recently in [12,13,30].

In [6–8] an original pure algebraic approach to invariants of Lie algebras was proposed and developed. Within its framework, the technique of Cartan's method of moving frames [18,19] in the Fels–Olver version [22,23] is specialized for the case of coadjoint action of the associated inner automorphism groups on the dual spaces of Lie algebras. (For modern development of the moving frames method and more references see also [33].) Unlike the infinitesimal methods based on solving systems of partial differential equations, such an approach involves only systems of algebraic equations. As a result, it is essentially simpler to extend the field of their application. Note that similar algebraic tools were occasionally applied to construct invariants for the specific case of inhomogeneous algebras [25,26,39]. By the infinitesimal method, such algebras were investigated in [21].

Different versions of the algebraic approach were tested for the Lie algebras of dimensions not greater than 6 [6] and also a wide range of known solvable Lie algebras of arbitrary finite dimensions with fixed structure of nilradicals [7]. A special technique for working with solvable Lie algebras having triangular nilradicals was developed in [8]. Fundamental invariants were constructed with this technique for the algebras $\mathfrak{t}_0(n)$, $\mathfrak{t}(n)$ and $\mathfrak{st}(n)$. Here $\mathfrak{t}_0(n)$ denotes the nilpotent Lie algebra $\mathfrak{t}_0(n)$ of strictly upper triangular $n \times n$ matrices over the field $\mathbb{F}$, where $\mathbb{F}$ is either $\mathbb{C}$ or $\mathbb{R}$. The solvable Lie algebras of non-strictly upper triangle and special upper triangle $n \times n$ matrices are denoted by $\mathfrak{t}(n)$ and $\mathfrak{st}(n)$, respectively.

The invariants of Lie algebras having triangular nilradicals were first studied in [45], by the infinitesimal method. The claim about the Casimir operators of $\mathfrak{t}_0(n)$ and the conjecture on the invariants of $\mathfrak{st}(n)$ from [45] were completely corroborated in [8]. Another conjecture was formulated in [45] on the invariants of solvable Lie algebras having $\mathfrak{t}_0(n)$ as their nilradicals and possessing a minimal (one) number of nilindependent 'diagonal' elements. It was completed and rigourously proved in [9]. Within the framework of the infinitesimal approach, necessary calculations are too cumbersome in these algebras even for small values of n that it demanded the thorough mastery of the method, and probably led to partial computational experiments and to the impossibility of proving the conjectures for arbitrary values of n.

In this paper, bases of the invariant sets of all the solvable Lie algebras with nilradicals isomorphic to $\mathfrak{t}_0(n)$ and s 'diagonal' nilindependent elements are constructed for arbitrary relevant values of n and s (i.e., $n > 1, 0 \leqslant s \leqslant n-1$). We use the algebraic approach first proposed in [6] along with some additional technical tools developed for triangular and closed algebras in [8,9]. The description of the necessary notions and statements, the precise formulation and discussion of technical details of the applied algorithm can be found ibid and are additionally reviewed in Section 2 for convenience. In Section 3 an illustrative example on invariants of a four-dimensional Lie algebra from the above class is given for clear demonstration of features of the developed method.

All the steps of the algorithm are implemented one after another for the Lie algebras under consideration: construction of the coadjoint representation of the corresponding Lie group and its fundamental lifted invariant (Section 4), excluding the group parameters from the lifted invariants by the normalization procedure that results to a basis of the invariants for the coadjoint action (Section 5) and re-writing this basis as a basis of the invariants of the Lie algebra under consideration (Section 6). The calculations for all steps are more complicated than in [8,9], but due to optimization they remain quite useful. The necessary numbers of normalization constraints, their forms and, therefore, the cardinalities of the fundamental invariants depend on the algebra parameters. In Section 7 different particular cases of the solvable Lie algebras with triangular nilradicals and 'diagonal' nilindependent elements, which was investigated earlier, are connected with the obtained results.

2. The algorithm

For convenience of the reader and to introduce some necessary notations, before the description of the algorithm, we briefly repeat the preliminaries given in [6–8] about the statement of the problem of calculating Lie algebra invariants, and on the implementation of the moving frame method [22,23]. The comparative analysis of the standard infinitesimal and the presented algebraic methods, as well as their modifications, is given in [8].

Consider a Lie algebra $\mathfrak{g}$ of dimension $\dim \mathfrak{g} = n < \infty$ over the complex or real field and the corresponding connected Lie group G. Let $\mathfrak{g}^*$ be the dual space of the vector space $\mathfrak{g}$. The map $\mathrm{Ad}^*\colon G \to GL(\mathfrak{g}^*)$, defined for each $g \in G$ by the relation

$$\langle \mathrm{Ad}^*_g x, u\rangle = \langle x, \mathrm{Ad}_{g^{-1}} u\rangle \quad \text{for all } x \in \mathfrak{g}^* \text{ and } u \in \mathfrak{g}$$

is called the *coadjoint representation* of the Lie group G. Here $\mathrm{Ad} : G \to GL(\mathfrak{g})$ is the usual adjoint representation of G in $\mathfrak{g}$, and the image Ad_G of G under Ad is the inner automorphism group of the Lie algebra $\mathfrak{g}$. The image of G under Ad^* is a subgroup of $GL(\mathfrak{g}^*)$ and is denoted by Ad^*_G.

A function $F \in C^\infty(\mathfrak{g}^*)$ is called an *invariant* of Ad^*_G if $F(\mathrm{Ad}^*_g x) = F(x)$ for all $g \in G$ and $x \in \mathfrak{g}^*$. The set of invariants of Ad^*_G is denoted by $\mathrm{Inv}(\mathrm{Ad}^*_G)$. The maximal number $N_\mathfrak{g}$ of functionally independent invariants in $\mathrm{Inv}(\mathrm{Ad}^*_G)$ coincides with the codimension of the regular orbits of Ad^*_G, i.e., it is given by the difference

$$N_\mathfrak{g} = \dim \mathfrak{g} - \mathrm{rank}\,\mathrm{Ad}^*_G.$$

Here $\mathrm{rank}\,\mathrm{Ad}^*_G$ denotes the dimension of the regular orbits of Ad^*_G and will be called the *rank of the coadjoint representation* of G (and of $\mathfrak{g}$). It is a basis independent characteristic of the algebra $\mathfrak{g}$, the same as $\dim \mathfrak{g}$ and $N_\mathfrak{g}$.

To calculate the invariants explicitly, one should fix a basis $\mathcal{E} = \{e_1, \ldots, e_n\}$ of the algebra $\mathfrak{g}$. It leads to fixing the dual basis $\mathcal{E}^* = \{e_1^*, \ldots, e_n^*\}$ in the dual space $\mathfrak{g}^*$ and to the identification of Ad_G and Ad^*_G with the associated matrix groups. The basis elements $e_1, \ldots, e_n$ satisfy the commutation relations $[e_i, e_j] = \sum_{k=1}^n c^k_{ij} e_k$, $i, j = 1, \ldots, n$, where c^k_{ij} are components of the tensor of structure constants of $\mathfrak{g}$ in the basis $\mathcal{E}$.

Let $x \to \check{x} = (x_1, \ldots, x_n)$ be the coordinates in $\mathfrak{g}^*$ associated with $\mathcal{E}^*$. Given any invariant $F(x_1, \ldots, x_n)$ of Ad^*_G, one finds the corresponding invariant of the Lie algebra $\mathfrak{g}$ by symmetrization, $\mathrm{Sym}\,F(e_1, \ldots, e_n)$, of F. It is often called a *generalized Casimir operator* of $\mathfrak{g}$. If F is a polynomial, $\mathrm{Sym}\,F(e_1, \ldots, e_n)$ is a usual *Casimir operator*, i.e., an element of the center of the universal enveloping algebra of $\mathfrak{g}$. More precisely, the symmetrization operator Sym acts only on the monomials of the forms $e_{i_1}\cdots e_{i_r}$, where there are non-commuting elements among $e_{i_1},\dots,e_{i_r}$, and is defined by the formula

$$\mathrm{Sym}(e_{i_1}\cdots e_{i_r})=\frac{1}{r!}\sum_{\sigma\in S_r}e_{i_{\sigma_1}}\cdots e_{i_{\sigma_r}},$$

where $i_1,\dots,i_r$ take values from 1 to n, $r\geqslant 2$. The symbol S_r denotes the permutation group consisting of r elements. The set of invariants of $\mathfrak{g}$ is denoted by $\mathrm{Inv}(\mathfrak{g})$.

A set of functionally independent invariants $F^l(x_1,\dots,x_n)$, $l=1,\dots,N_{\mathfrak{g}}$, forms *a functional basis* (*fundamental invariant*) of $\mathrm{Inv}(\mathrm{Ad}^*_G)$, i.e., each invariant $F(x_1,\dots,x_n)$ can be uniquely represented as a function of $F^l(x_1,\dots,x_n)$, $l=1,\dots,N_{\mathfrak{g}}$. Accordingly the set of $\mathrm{Sym}\,F^l(e_1,\dots,e_n)$, $l=1,\dots,N_{\mathfrak{g}}$, is called a basis of $\mathrm{Inv}(\mathfrak{g})$.

Our task here is to determine the basis of the functionally independent invariants for Ad^*_G, and then to transform these invariants into the invariants of the algebra $\mathfrak{g}$. Any other invariant of $\mathfrak{g}$ is a function of the independent ones.

Let us recall some facts from [22,23] and adapt them to the particular case of the coadjoint action of G on $\mathfrak{g}^*$. Let $\mathscr{G}=\mathrm{Ad}^*_G\times\mathfrak{g}^*$ denote the trivial left principal Ad^*_G-bundle over $\mathfrak{g}^*$. The right regularization $\widehat{R}$ of the coadjoint action of G on $\mathfrak{g}^*$ is the diagonal action of Ad^*_G on $\mathscr{G}=\mathrm{Ad}^*_G\times\mathfrak{g}^*$. It is provided by the map $\widehat{R}_g(\mathrm{Ad}^*_h,x)=(\mathrm{Ad}^*_h\cdot\mathrm{Ad}^*_{g^{-1}},\mathrm{Ad}^*_g x)$, $g,h\in G$, $x\in\mathfrak{g}^*$, where the action on the bundle $\mathscr{G}=\mathrm{Ad}^*_G\times\mathfrak{g}^*$ is regular and free. We call $\widehat{R}_g$ the *lifted coadjoint action* of G. It projects back to the coadjoint action on $\mathfrak{g}^*$ via the Ad^*_G-equivariant projection $\pi_{\mathfrak{g}^*}:\mathscr{G}\to\mathfrak{g}^*$. Any *lifted invariant* of Ad^*_G is a (locally defined) smooth function from $\mathscr{G}$ to a manifold, which is invariant with respect to the lifted coadjoint action of G. The function $\mathscr{I}:\mathscr{G}\to\mathfrak{g}^*$ given by $\mathscr{I}=\mathscr{I}(\mathrm{Ad}^*_g,x)=\mathrm{Ad}^*_g x$ is the *fundamental lifted invariant* of Ad^*_G, i.e., $\mathscr{I}$ is a lifted invariant, and each lifted invariant can be locally written as a function of $\mathscr{I}$. Using an arbitrary function $F(x)$ on $\mathfrak{g}^*$, we can produce the lifted invariant $F\circ\mathscr{I}$ of Ad^*_G by replacing x with $\mathscr{I}=\mathrm{Ad}^*_g x$ in the expression for F. Ordinary invariants are particular cases of lifted invariants, where one identifies any invariant formed as its composition with the standard projection $\pi_{\mathfrak{g}^*}$. Therefore, ordinary invariants are particular functional combinations of lifted ones that happen to be independent of the group parameters of Ad^*_G.

The *algebraic algorithm* for finding invariants of the Lie algebra $\mathfrak{g}$ is briefly formulated in the following four steps.

1. *Construction of the generic matrix $B(\theta)$ of* Ad^*_G. $B(\theta)$ is the matrix of an inner automorphism of the Lie algebra $\mathfrak{g}$ in the given basis $e_1,\dots,e_n$, $\theta=(\theta_1,\dots,\theta_r)$ is a complete tuple of group parameters (coordinates) of Ad_G, and $r=\dim\mathrm{Ad}^*_G=\dim\mathrm{Ad}_G=n-\dim\mathrm{Z}(\mathfrak{g})$, where $\mathrm{Z}(\mathfrak{g})$ is the center of $\mathfrak{g}$.
2. *Representation of the fundamental lifted invariant.* The explicit form of the fundamental lifted invariant $\mathscr{I}=(\mathscr{I}_1,\dots,\mathscr{I}_n)$ of Ad^*_G in the chosen coordinates $(\theta,\check{x})$ in $\mathrm{Ad}^*_G\times\mathfrak{g}^*$ is $\mathscr{I}=\check{x}\cdot B(\theta)$, i.e., $(\mathscr{I}_1,\dots,\mathscr{I}_n)=(x_1,\dots,x_n)\cdot B(\theta_1,\dots,\theta_r)$.
3. *Elimination of parameters by normalization.* We choose the maximum possible number ρ of lifted invariants $\mathscr{I}_{j_1},\dots,\mathscr{I}_{j_\rho}$, constants $c_1,\dots,c_\rho$ and group parameters $\theta_{k_1},\dots,\theta_{k_\rho}$ such that the equations $\mathscr{I}_{j_1}=c_1,\dots,\mathscr{I}_{j_\rho}=c_\rho$ are solvable with respect to $\theta_{k_1},\dots,\theta_{k_\rho}$. After substituting the found values of $\theta_{k_1},\dots,\theta_{k_\rho}$ into the other lifted invariants, we obtain $N_{\mathfrak{g}}=n-\rho$ expressions $F^l(x_1,\dots,x_n)$ without θ's.
4. *Symmetrization.* The functions $F^l(x_1,\dots,x_n)$ necessarily form a basis of $\mathrm{Inv}(\mathrm{Ad}^*_G)$. They are symmetrized to $\mathrm{Sym}\,F^l(e_1,\dots,e_n)$. It is the desired basis of $\mathrm{Inv}(\mathfrak{g})$.

Following the preceding papers [8,9] on invariants of the triangular Lie algebras, here we use, in contrast with the general situation, special coordinates for inner automorphism groups, which naturally harmonize with the canonical matrix representations of the corresponding Lie groups and with special 'matrix' enumeration of a part of the basis elements. The individual approach results in the clarification and a substantial reduction of all calculations. Thus, algebraic systems solved under normalization are reduced to linear ones.

The essence of the normalization procedure by Fels and Olver [22,23] can be presented in the form of on the following statement [8].

Proposition 1. *Let $\mathcal{I}=(\mathcal{I}_1,\dots,\mathcal{I}_n)$ be a fundamental lifted invariant, for the lifted invariants $\mathcal{I}_{j_1},\dots,\mathcal{I}_{j_\rho}$ and some constants $c_1,\dots,c_\rho$ the system $\mathcal{I}_{j_1}=c_1,\dots,\mathcal{I}_{j_\rho}=c_\rho$ be solvable with respect to the parameters $\theta_{k_1},\dots,\theta_{k_\rho}$ and substitution of the found values of $\theta_{k_1},\dots,\theta_{k_\rho}$ into the other lifted invariants result in $m=n-\rho$ expressions $\widehat{\mathcal{I}}_l$, $l=1,\dots,m$, depending only on x's. Then $\rho=\operatorname{rank}\mathrm{Ad}^*_G$, $m=N_\mathfrak{g}$ and $\widehat{\mathcal{I}}_1,\dots,\widehat{\mathcal{I}}_m$ form a basis of $\operatorname{Inv}(\mathrm{Ad}^*_G)$.*

Our experience on the calculation of invariants of a wide range of Lie algebras shows that the version of the algebraic method, which is based on Proposition 1, is most effective. In particular, it provides finding the cardinality of the invariant basis in the process of construction of the invariants. It is the version that is used in this paper.

3. Illustrative example

Before the calculation of invariants for the general case of Lie algebras from the class under consideration, we present an illustrative example on invariants of a low-dimensional Lie algebra from the above class. This demonstrates features of the developed method.

The four-dimensional solvable Lie algebra $\mathfrak{g}^b_{4.8}$ has the following non-zero commutation relations

$$[e_2,e_3]=e_1,\quad [e_1,e_4]=(1+b)e_1,\quad [e_2,e_4]=e_2,\quad [e_3,e_4]=be_3,\quad |b|\leqslant 1.$$

Its nilradical is three-dimensional and isomorphic to the Weil–Heisenberg algebra $\mathfrak{g}_{3.1}$. (Here we use the notations of low-dimensional Lie algebras according to Mubarakzyanov's classification [27].)

We construct a presentation of the inner automorphism matrix $B(\theta)$ of the Lie algebra $\mathfrak{g}$, involving second canonical coordinates on Ad_G as group parameters θ [6,7,8]. The matrices $\widehat{\mathrm{ad}}_{e_i}$, $i=1,\dots,4$, of the adjoint representation of the basis elements e_1, e_2, e_3 and e_4 respectively have the form

$$\begin{pmatrix}0&0&0&1+b\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad \begin{pmatrix}0&-1&0&0\\0&0&0&0\\0&0&0&b\\0&0&0&0\end{pmatrix},\quad \begin{pmatrix}-1-b&0&0&0\\0&-1&0&0\\0&0&-b&0\\0&0&0&0\end{pmatrix}.$$

The inner automorphisms of $\mathfrak{g}^b_{4.8}$ are then described by the triangular matrix

$$B(\theta)=\prod_{i=1}^{3}\exp(\theta_i\widehat{\mathrm{ad}}_{e_i})\cdot\exp(-\theta_4\widehat{\mathrm{ad}}_{e_4})=\begin{pmatrix}\mathrm{e}^{(1+b)\theta_4}&-\theta_3\mathrm{e}^{\theta_4}&\theta_2\mathrm{e}^{b\theta_4}&b\theta_2\theta_3+(1+b)\theta_1\\0&\mathrm{e}^{\theta_4}&0&\theta_2\\0&0&\mathrm{e}^{b\theta_4}&b\theta_3\\0&0&0&1\end{pmatrix}.$$

Therefore, a functional basis of lifted invariants is formed by

$$\begin{aligned}
\mathscr{I}_1 &= \mathrm{e}^{(1+b)\theta_4}x_1,\\
\mathscr{I}_2 &= \mathrm{e}^{\theta_4}(-\theta_3x_1+x_2),\\
\mathscr{I}_3 &= \mathrm{e}^{b\theta_4}(\theta_2x_1+x_3),\\
\mathscr{I}_4 &= (b\theta_2\theta_3+(1+b)\theta_1)x_1+\theta_2x_2+b\theta_3x_3+x_4.
\end{aligned}$$

Further the cases $b=-1$ and $b\neq-1$ should be considered separately.

There are no invariants in case $b\neq-1$ since in view of Proposition 1 the number of functionally independent invariants is equal to zero. Indeed, the system $\mathscr{I}_1=1$, $\mathscr{I}_2=\mathscr{I}_3=\mathscr{I}_4=0$ is solvable with respect to the whole set of the parameters θ.

It is obvious that in the case $b=-1$ the element e_1 generating the center $Z(\mathfrak{g}_{4.8}^{-1})$ is an invariant. (The corresponding lifted invariant $\mathscr{I}_1=x_1$ does not depend on the parameters θ.) Another invariant is easily found via combining the lifted invariants: $\mathscr{I}_1\mathscr{I}_4-\mathscr{I}_2\mathscr{I}_3=x_1x_4-x_2x_3$. After the symmetrization procedure we obtain the following polynomial basis of the invariant set of this algebra

$$e_1,\quad e_1e_4-\frac{e_2e_3+e_3e_2}{2}.$$

The second basis invariant can be also constructed by the normalization technique. We solve the equations $\mathscr{I}_2=\mathscr{I}_3=0$ with respect to the parameters θ_2 and θ_3 and substitute the expressions for them into the lifted invariant $\mathscr{I}_4$. The obtained expression $x_4-x_2x_3/x_1$ does not contain the parameters θ and, therefore, is an invariant of the coadjoint representation. For the basis of invariants to be polynomial, we multiply this invariant by the invariant x_1. It is the technique that is applied below for the general case of the Lie algebras under consideration.

Note that in the above example the symmetrization procedure can be assumed trivial since the symmetrized invariant $e_1e_4-\frac{1}{2}(e_2e_3+e_3e_2)$ differs from the non-symmetrized version $e_1e_4-e_2e_3$ (resp. $e_1e_4-e_3e_2$) on the invariant $\frac{1}{2}e_1$ (resp. $-\frac{1}{2}e_1$). If we take the rational invariant $e_4-e_2e_3/e_1$ (resp. $e_4-e_3e_2/e_1$), the symmetrization is equivalent to the addition of the constant $\frac{1}{2}$ (resp. $-\frac{1}{2}$).

Invariants of $\mathfrak{g}_{4.8}^b$ were first described in [34] within the framework of the infinitesimal approach.

4. Representation of the coadjoint action

Consider the solvable Lie algebra $\mathfrak{t}_\gamma(n)$ with the nilradical $\mathrm{NR}(\mathfrak{t}_\gamma(n))$ isomorphic to $\mathfrak{t}_0(n)$ and s nilindependent element f_p, $p=1,\dots,s$, which act on elements of the nilradical in the way as the diagonal matrices $\Gamma_p=\mathrm{diag}(\gamma_{p1},\dots,\gamma_{pn})$ act on strictly triangular matrices. The matrices Γ_p, $p=1,\dots,s$, and the unity matrix are linear independent since otherwise $\mathrm{NR}(\mathfrak{t}_\gamma(n))\neq\mathfrak{t}_0(n)$. The parameter matrix $\gamma=(\gamma_{pi})$ is defined up to nonsingular $s\times s$ matrix multiplier and homogeneous shift in rows. In other words, the algebras $\mathfrak{t}_\gamma(n)$ and $\mathfrak{t}_{\gamma'}(n)$ are isomorphic iff there exist $\lambda\in\mathrm{M}_{s,s}(\mathbb{F})$, $\det\lambda\neq0$, and $\mu\in\mathbb{F}^s$ such that

$$\gamma'_{pi}=\sum_{p'=1}^{s}\lambda_{pp'}\gamma_{p'i}+\mu_p,\quad p=1,\dots,s,\ i=1,\dots,n.$$

The parameter matrix γ and γ' are assumed equivalent. Up to the equivalence the additional condition $\mathrm{Tr}\Gamma_p = \sum_i \gamma_{pi} = 0$ can be imposed on the algebra parameters. Therefore, the algebra $\mathfrak{t}_\gamma(n)$ is naturally embedded into $\mathfrak{st}(n)$ as a (mega)ideal under identification of $\mathrm{NR}(\mathfrak{t}_\gamma(n))$ with $\mathfrak{t}_0(n)$ and of f_p with Γ_p.

We choose the union of the canonical basis of $\mathrm{NR}(\mathfrak{t}_\gamma(n))$ and the s-element set $\{f_p, p = 1, \dots, s\}$ as the canonical basis of $\mathfrak{t}_\gamma(n)$. In the basis of $\mathrm{NR}(\mathfrak{t}_\gamma(n))$ we use 'matrix' enumeration of basis elements e_{ij}, $i < j$, with the 'increasing' pair of indices similarly to the canonical basis $\{E^n_{ij}, i < j\}$ of the isomorphic matrix algebra $\mathfrak{t}_0(n)$.

Hereafter E^n_{ij} (for the fixed values i and j) denotes the $n \times n$ matrix $(\delta_{ii'}\delta_{jj'})$ with i' and j' running the numbers of rows and column correspondingly, i.e., the $n \times n$ matrix with the unit on the cross of the ith row and the jth column and the zero otherwise. The indices i, j, k and l run at most from 1 to n. Only additional constraints on the indices are indicated. The subscript p runs from 1 to s, the subscript q runs from 1 to s'. The summation convention over repeated indices p and q is used unless otherwise stated. The number s is in the range $0, \dots, n-1$. In the case $s = 0$ we assume $\gamma = 0$, and all terms with the subscript p should be omitted from consideration. The value s' ($s' < s$) is defined in the next section.

Thus, the basis elements $e_{ij} \sim E^n_{ij}$, $i < j$, and $f_p \sim \sum_i \gamma_{pi} E^n_{ii}$ satisfy the commutation relations

$$[e_{ij}, e_{i'j'}] = \delta_{i'j} e_{ij'} - \delta_{ij'} e_{i'j}, \qquad [f_p, e_{ij}] = (\gamma_{pi} - \gamma_{pj}) e_{ij},$$

where δ_{ij} is the Kronecker delta.

The Lie algebra $\mathfrak{t}_\gamma(n)$ can be considered as the Lie algebra of the Lie subgroup

$$T_\gamma(n) = \{B \in T(n) | \exists \varepsilon_p \in \mathbb{F}\colon b_{ii} = \mathrm{e}^{\gamma_{pi}\varepsilon_p}\}$$

of the Lie group $T(n)$ of non-singular upper triangular $n \times n$ matrices.

Let e^*_{ji}, x_{ji} and y_{ij} denote the basis element and the coordinate function in the dual space $\mathfrak{t}^*_\gamma(n)$ and the coordinate function in $\mathfrak{t}_\gamma(n)$, which correspond to the basis element e_{ij}, $i < j$. In particular,

$$\langle e^*_{j'i'}, e_{ij} \rangle = \delta_{ii'}\delta_{jj'}.$$

The reverse order of subscripts of the objects associated with the dual space $\mathfrak{t}^*_\gamma(n)$ is natural (see, e.g., [38, Section 1.4]) and additionally justified by the simplification of a matrix representation of lifted invariants. f^*_p, x_{p0} and y_{p0} denote similar objects corresponding to the basis element f_p. We additionally put $y_{ii} = \gamma_{pi} y_{p0}$ and then complete the sets of x_{ji} and y_{ij} to the matrices X and Y with zeros. Hence X is a strictly lower triangular matrix and Y is a non-strictly upper triangular one. The analogous 'matrix' whose (i, j)th entry is equal to e_{ij} for $i < j$ and 0 otherwise is denoted by $\mathscr{E}$.

Lemma 1. *A complete set of functionally independent lifted invariants of* $\mathrm{Ad}^*_{T_\gamma(n)}$ *is exhausted by the expressions*

$$\mathscr{I}_{ij} = \sum_{i \leqslant i' < j' \leqslant j} b_{ii'} \hat{b}_{j'j} x_{i'j'}, \quad j < i, \qquad \mathscr{I}_{p0} = x_{p0} + \sum_{j<i} \sum_{j \leqslant l \leqslant i} \gamma_{pl} b_{li} \hat{b}_{jl} x_{ij},$$

where $B = (b_{ij})$ *is an arbitrary matrix from* $T_\gamma(n)$, $B^{-1} = (\hat{b}_{ij})$ *is the inverse matrix of* B.

Proof. The adjoint action of $B \in T_\gamma(n)$ on the matrix Y is $\mathrm{Ad}_B Y = BYB^{-1}$, i.e.,

$$\mathrm{Ad}_B\left(y_{p0}f_p + \sum_{i<j} y_{ij}e_{ij}\right) = y_{p0}f_p + y_{p0}\sum_{i<j}\sum_{i\leqslant i'\leqslant j} b_{ii'}\gamma_{pi'}\hat{b}_{i'j}e_{ij} + \sum_{i\leqslant i'<j'\leqslant j} b_{ii'}y_{i'j'}\hat{b}_{j'j}e_{ij}.$$

After changing $e_{ij} \to x_{ji}$, $y_{ij} \to e^*_{ji}$, $f_p \to x_{p0}$, $y_{p0} \to f^*_p$, $b_{ij} \leftrightarrow \hat{b}_{ij}$ in the latter equality, we obtain the representation for the coadjoint action of B

$$\mathrm{Ad}^*_B\left(x_{p0}f^*_p + \sum_{i<j} x_{ji}e^*_{ji}\right) = x_{p0}f^*_p + \sum_{i<j}\sum_{i\leqslant i'\leqslant j} b_{i'j}x_{ji}\hat{b}_{ii'}\gamma_{pi'}f^*_p + \sum_{i\leqslant i'<j'\leqslant j} b_{j'j}x_{ji}\hat{b}_{ii'}e^*_{j'i'}$$
$$= \left(x_{p0} + \sum_{i<j}\sum_{i\leqslant i'\leqslant j} b_{i'j}x_{ji}\hat{b}_{ii'}\gamma_{pi'}\right) f^*_p + \sum_{i'<j'} (BXB^{-1})_{j'i'}e^*_{j'i'}.$$

Therefore, $\mathscr{I}_{p0}$ and the elements $\mathscr{I}_{ij}$, $j < i$, of the matrix $\mathscr{I} = BXB^{-1}$, where $B \in T_\gamma(n)$, form a fundamental lifted invariant of $\mathrm{Ad}^*_{T_\gamma(n)}$. □

Note 1. The complete set of parameters in the above representation of lifted invariants is formed by b_{ij}, $j < i$, and ε_p. The center of the group $T_\gamma(n)$ is nontrivial only if $\gamma_{p1} = \gamma_{pn}$, namely, then $\mathrm{Z}(T_\gamma(n)) = \{E^n + b_{1n}E^n_{1n},\ b_{1n} \in \mathbb{F}\}$. Here $E^n = \mathrm{diag}(1, \ldots, 1)$ is the $n \times n$ unity matrix. In this case the inner automorphism group of $\mathrm{t}_\gamma(n)$ is isomorphic to the factor-group $T_\gamma(n)/\mathrm{Z}(T_\gamma(n))$ and hence its dimension is $\frac{1}{2}n(n-1)$. The parameter b_{1n} in the representation of lifted invariants then is inessential. Otherwise, the inner automorphism group of $\mathrm{t}_\gamma(n)$ is isomorphic to the whole group $T_\gamma(n)$ and all the parameters in the constructed lifted invariants are essential.

5. Invariants of the coadjoint action

Below $A^{i_1,i_2}_{j_1,j_2}$, where $i_1 \leqslant i_2$, $j_1 \leqslant j_2$, denotes the submatrix $(a_{ij})^{i=i_1,\ldots,i_2}_{j=j_1,\ldots,j_2}$ of a matrix $A = (a_{ij})$. The conjugate value of k with respect to n is denoted by $\varkappa$, i.e., $\varkappa = n - k + 1$. The standard notation $|A| = \det A$ is used.

Similarly to [8,9] the following technical lemma on matrices is used in the proof of the below theorem.

Lemma 2. *Suppose* $1 < k < n$. *If* $|X^{\varkappa+1,n}_{1,k-1}| \neq 0$ *then for any* $\beta \in \mathbb{F}$

$$\beta - X^{i,i}_{1,k-1}(X^{\varkappa+1,n}_{1,k-1})^{-1}X^{\varkappa+1,n}_{j,j} = \frac{(-1)^{k+1}}{|X^{\varkappa+1,n}_{1,k-1}|}\begin{vmatrix} X^{i,i}_{1,k-1} & \beta \\ X^{\varkappa+1,n}_{1,k-1} & X^{\varkappa+1,n}_{j,j}\end{vmatrix}.$$

In particular, $x_{\varkappa k} - X^{\varkappa,\varkappa}_{1,k-1}(X^{\varkappa+1,n}_{1,k-1})^{-1}X^{\varkappa+1,n}_{k,k} = (-1)^{k+1}|X^{\varkappa+1,n}_{1,k-1}|^{-1}|X^{\varkappa,n}_{1,k}|$. *Analogously*

$$\left(x_{\varkappa j} - X^{\varkappa,\varkappa}_{1,k-1}(X^{\varkappa+1,n}_{1,k-1})^{-1}X^{\varkappa+1,n}_{j,j}\right)\left(x_{jk} - X^{j,j}_{1,k-1}(X^{\varkappa+1,n}_{1,k-1})^{-1}X^{\varkappa+1,n}_{k,k}\right)$$
$$= \frac{1}{|X^{\varkappa+1,n}_{1,k-1}|}\begin{vmatrix} X^{j,j}_{1,k} & \beta \\ X^{\varkappa,n}_{1,k} & X^{\varkappa,n}_{j,j}\end{vmatrix} + \frac{|X^{\varkappa,n}_{1,k}|}{|X^{\varkappa+1,n}_{1,k-1}|^2}\begin{vmatrix} X^{j,j}_{1,k-1} & \beta \\ X^{\varkappa+1,n}_{1,k-1} & X^{\varkappa+1,n}_{j,j}\end{vmatrix}.$$

Proposition 2. *Up to the equivalence relation on algebra parameters, the following conditions can be assumed satisfied*

$$\exists s' \in \left\{0, \dots, \min\left(s, \left[\frac{n}{2}\right]\right)\right\}, \quad \exists k_q, q = 1, \dots, s', \quad 1 \leqslant k_1 < k_2 < \dots < k_{s'} \leqslant \left[\frac{n}{2}\right]:$$
$$\gamma_{qk} = \gamma_{q\varkappa},\ k < k_q, \quad \gamma_{q\varkappa_q} - \gamma_{qk_q} = 1, \quad \gamma_{pk_q} = \gamma_{p\varkappa_q},\ p \neq q, \quad q = 1, \dots, s',$$
$$\gamma_{pk} = \gamma_{p\varkappa},\ p > s',\ k = 1, \dots, \left[\frac{n}{2}\right].$$

Proof. If $\gamma_{pk} = \gamma_{p\varkappa}$ for all $k \in \{1, \dots, [n/2]\}$ and all $p \in \{1, \dots, s\}$ then put $s' = 0$. Otherwise, we put k_1 equal to the minimal value of k for which there exists p_1 such that $\gamma_{p_1k} \neq \gamma_{p_1\varkappa}$. Permuting, scaling and combining rows of the matrix γ, we make $p_1 = 1$, $\gamma_{1\varkappa_1} - \gamma_{1k_1} = 1$ and $\gamma_{pk_1} = \gamma_{p\varkappa_1}$, $p \neq 1$ that gives the conditions corresponding to $q = 1$.

Then, if $\gamma_{pk} = \gamma_{p\varkappa}$ for all $k \in \{1, \dots, [n/2]\}$ and all $p \in \{2, \dots, s\}$ then we get $s' = 1$. Otherwise, we put k_2 equal to the minimal value of k for which there exists $p_2 > p_1 = 1$ such that $\gamma_{p_2k} \neq \gamma_{p_2\varkappa}$. It follows from the previous step that $k_2 > k_1$. Permuting, scaling and combining rows of the matrix γ, we make $p_2 = 2$, $\gamma_{2\varkappa_2} - \gamma_{2k_2} = 1$ and $\gamma_{pk_2} = \gamma_{p\varkappa_2}$, $p \neq 2$.

By induction, iteration of this procedure leads to the statement. Note that

$$s' = \operatorname{rank}(\gamma_{p\varkappa} - \gamma_{pk})_{k=1,\dots,[n/2]}^{p=1,\dots,s}. \qquad \square$$

We will say that the parameter matrix γ has a *reduced form* if it satisfies the conditions of Proposition 2.

Theorem 1. *Let the parameter matrix γ have a reduced form. A basis of* $\operatorname{Inv}(\operatorname{Ad}^*_{T_\gamma(n)})$ *is formed by the expressions*

$$|X^{\varkappa,n}_{1,k}| \prod_{q=1}^{s'} |X^{\varkappa_q,n}_{1,k_q}|^{\beta_{qk}}, \quad k \in \{1, \dots, [n/2]\} \setminus \{k_1, \dots, k_{s'}\},$$
$$x_{p0} + \sum_{k=1}^{[\frac{n}{2}]} \frac{(-1)^{k+1}}{|X^{\varkappa,n}_{1,k}|} (\gamma_{pk} - \gamma_{p,k+1}) \sum_{k<i<\varkappa} \begin{vmatrix} X^{i,i}_{1,k} & 0 \\ X^{\varkappa,n}_{1,k} & X^{\varkappa,n}_{i,i} \end{vmatrix}, \quad p = s' + 1, \dots, s,$$

where $\beta_{qk} = -\Delta_{qk}/\Delta, = \det(\alpha_{q'k_{q''}})_{q',q''=1,\dots,s'}$, Δ_{qk} *is the determinant obtained from* Δ *with change of the column* $(\alpha_{q'k_q})_{q'=1,\dots,s'}$ *by the column* $(\alpha_{q'k})_{q'=1,\dots,s'}$,

$$\alpha_{qj} = -\sum_{j=1}^{k} (\gamma_{q\varkappa} - \gamma_{qk}).$$

Proof. Under normalization we impose the following restriction on the lifted invariants $\mathscr{I}_{ij}$, $j < i$:

$$\mathscr{I}_{ij} = 0 \quad \text{if } j < i,\ (i, j) \neq (n - j' + 1, j'),\ j' = 1, \dots, \left[\frac{n}{2}\right].$$

It means that we do not fix only values of the elements of the lifted invariant matrix $\mathscr{I}$, which are situated on the secondary diagonal under the main diagonal. The other significant elements of $\mathscr{I}$ are put equal to 0. The choice of just such normalization conditions is a result of a wide preliminary analysis. It can be justified, in particular, by the structure of the entire automorphism group of $\mathfrak{t}_0(n)$, adduced, e.g., in [17].

The decision on what to do with the singular lifted invariant $\mathscr{I}_{p0}$ and the secondary diagonal lifted invariants $\mathscr{I}_{\varkappa k}$, $k = 1, \dots, [n/2]$, is left for the later discussion, since it will turn out that necessity of imposing normalization conditions on them depends on values of γ. As shown below, the final normalization in all the cases provides satisfying the conditions of Proposition 1 and, therefore, is correct.

In view of (triangular) structure of the matrices B and X the formula $\mathscr{I} = BXB^{-1}$ determining the matrix part of lifted invariants implies that $BX = \mathscr{I}B$. This matrix equality also is significant for the matrix elements underlying the main diagonals of the left- and right-hand sides, i.e.,

$$e^{\gamma_{pi}\varepsilon_p} x_{ij} + \sum_{i<i'} b_{ii'} x_{i'j} = \mathscr{I}_{ij} e^{\gamma_{pj}\varepsilon_p} + \sum_{j'<j} \mathscr{I}_{ij'} b_{j'j}, \quad j < i.$$

For convenience we divide the latter system under the chosen normalization conditions into four sets of subsystems

$$S_1^k\colon \quad e^{\gamma_{p\varkappa}\varepsilon_p} x_{\varkappa j} + \sum_{i'>\varkappa} b_{\varkappa i'} x_{i'j} = 0, \qquad i = \varkappa, \quad j < k, \quad k = 2, \dots, \left[\frac{n+1}{2}\right],$$

$$S_2^k\colon \quad e^{\gamma_{p\varkappa}\varepsilon_p} x_{\varkappa k} + \sum_{i'>\varkappa} b_{\varkappa i'} x_{i'k} = \mathscr{I}_{\varkappa k} e^{\gamma_{pk}\varepsilon_p}, \quad i = \varkappa, \quad j = k, \quad k = 1, \dots, \left[\frac{n}{2}\right],$$

$$S_3^k\colon \quad e^{\gamma_{p\varkappa}\varepsilon_p} x_{\varkappa j} + \sum_{i'>\varkappa} b_{\varkappa i'} x_{i'j} = \mathscr{I}_{\varkappa k} b_{kj}, \quad i = \varkappa, \ k < j < \varkappa, \quad k = 1, \dots, \left[\frac{n}{2}\right] - 1,$$

$$S_4^k\colon \quad e^{\gamma_{pk}\varepsilon_p} x_{kj} + \sum_{i'>k} b_{ki'} x_{i'j} = 0, \qquad i = k, \quad j < k, \quad k = 2, \dots, \left[\frac{n}{2}\right],$$

and solve them one after another. The subsystem S_2^1 consists of the single equation

$$\mathscr{I}_{n1} = x_{n1} e^{(\gamma_{pn}-\gamma_{p1})\varepsilon_p}.$$

For any fixed $k \in \{2, \dots, [n/2]\}$ the subsystem $S_1^k \cup S_2^k$ is a well-defined system of linear equations with respect to $b_{\varkappa i'}$, $i' > \varkappa$, and $\mathscr{I}_{\varkappa k}$. Analogously, the subsystem S_1^k for $k = \varkappa = [(n+1)/2]$ in the case of odd n is a well-defined system of linear equations with respect to $b_{ki'}$, $i' > k$. The solutions of the above subsystems are expressions of $x_{i'j}$, $i' \geqslant \varkappa$, $j < k$, and ε_p:

$$\mathscr{I}_{\varkappa k} = (-1)^{k+1} \frac{|X_{1,k}^{\varkappa,n}|}{|X_{1,k-1}^{\varkappa+1,n}|} e^{(\gamma_{p\varkappa}-\gamma_{pk})\varepsilon_p}, \quad k = 2, \dots, \left[\frac{n}{2}\right],$$

$$B_{\varkappa+1,n}^{\varkappa,\varkappa} = -e^{\gamma_{p\varkappa}\varepsilon_p} X_{1,k-1}^{\varkappa,\varkappa} (X_{1,k-1}^{\varkappa+1,n})^{-1}, \quad k = 2, \dots, \left[\frac{n+1}{2}\right].$$

Combining of the found values of $\mathscr{I}_{\varkappa k}$ results in the invariants from the statement of the theorem. Functional independence of these invariants is obvious.

After substituting the expressions of $\mathscr{I}_{\varkappa k}$ and $b_{\varkappa i'}$, $i' > \varkappa$, via ε_p and x's into S_3^k, we trivially resolve S_3^k with respect to b_{kj} as uncoupled system of linear equations:

$$b_{1j} = e^{\gamma_{p1}\varepsilon_p} \frac{x_{nj}}{x_{n1}}, \quad 1 < j < n,$$

$$b_{kj}=(-1)^{k+1}\mathrm{e}^{\gamma_{pk}\varepsilon_p}\frac{|X^{\varkappa+1,n}_{1,k-1}|}{|X^{\varkappa,n}_{1,k}|}\left(x_{\varkappa j}-X^{\varkappa,\varkappa}_{1,k-1}(X^{\varkappa+1,n}_{1,k-1})^{-1}X^{\varkappa+1,n}_{j,j}\right)$$
$$=\frac{\mathrm{e}^{\gamma_{pk}\varepsilon_p}}{|X^{\varkappa,n}_{1,k}|}\begin{vmatrix}X^{\varkappa,\varkappa}_{1,k-1} & x_{\varkappa j}\\ X^{\varkappa+1,n}_{1,k-1} & X^{\varkappa+1,n}_{j,j}\end{vmatrix},$$
$$k<j<\varkappa,\quad k=2,\dots,\left[\frac{n}{2}\right]-1.$$

Performing the subsequent substitution of the calculated expressions for b_{kj} to S^k_4, for any fixed appropriate k we obtain a well-defined system of linear equations, e.g., with respect to $b_{ki'}$, $i'>\varkappa$. Its solution is expressed via x's, $b_{k\varkappa}$ and ε_p:

$$B^{k,k}_{\varkappa+1,n}=-\left(\mathrm{e}^{\gamma_{pk}\varepsilon_p}X^{k,k}_{1,k-1}+\sum_{k<j\leqslant\varkappa}b_{kj}X^{j,j}_{1,k-1}\right)(X^{\varkappa+1,n}_{1,k-1})^{-1}$$
$$=-b_{k\varkappa}X^{\varkappa,\varkappa}_{1,k-1}(X^{\varkappa+1,n}_{1,k-1})^{-1}-\frac{\mathrm{e}^{\gamma_{pk}\varepsilon_p}}{|X^{\varkappa,n}_{1,k}|}\sum_{k\leqslant j<\varkappa}\begin{vmatrix}X^{\varkappa,\varkappa}_{1,k-1} & x_{\varkappa j}\\ X^{\varkappa+1,n}_{1,k-1} & X^{\varkappa+1,n}_{j,j}\end{vmatrix}X^{j,j}_{1,k-1}(X^{\varkappa+1,n}_{1,k-1})^{-1},$$
$$k=2,\dots,\left[\frac{n}{2}\right].$$

We rewrite the expression of the lifted invariant $\mathscr{I}_{p0}$, taking into account the already imposed normalization constraints (note that $\varkappa=[(n+1)/2]+1$ if $k=[n/2]$):

$$\mathscr{I}_{p0}=x_{p0}+\sum_l\gamma_{pl}\hat{b}_{ll}\sum_{l<i}b_{li}x_{il}+\sum_{k=2}^{\left[\frac{n+1}{2}\right]}\sum_{j<k}\gamma_{pk}\hat{b}_{jk}\sum_{i\geqslant k}b_{ki}x_{ij}$$
$$+\sum_{k=1}^{\left[\frac{n}{2}\right]}\left(\sum_{j<k}+\sum_{k\leqslant j<\varkappa}\right)\gamma_{p\varkappa}\hat{b}_{j\varkappa}\sum_{i\geqslant\varkappa}b_{\varkappa i}x_{ij}$$
$$=x_{p0}+\sum_l\gamma_{pl}\hat{b}_{ll}\sum_{l<i}b_{li}x_{il}+\sum_{k=1}^{\left[\frac{n}{2}\right]}\gamma_{p\varkappa}\mathscr{I}_{\varkappa k}\sum_{k\leqslant j<\varkappa}b_{kj}\hat{b}_{j\varkappa}$$
$$=x_{p0}+\sum_{k=1}^{\left[\frac{n}{2}\right]}\gamma_{pk}\hat{b}_{kk}\left(\sum_{k<i\leqslant\varkappa}+\sum_{i>\varkappa}\right)b_{ki}x_{ik}$$
$$+\sum_{k=1}^{\left[\frac{n+1}{2}\right]}\gamma_{p\varkappa}\hat{b}_{\varkappa\varkappa}\sum_{i>\varkappa}b_{\varkappa i}x_{i\varkappa}-\sum_{k=1}^{\left[\frac{n}{2}\right]}\gamma_{p\varkappa}\hat{b}_{\varkappa\varkappa}\mathscr{I}_{\varkappa k}b_{k\varkappa}.$$

Then we substitute the found expressions for b's and $I_{\varkappa k}$ to the derived expression of $\mathscr{I}_{p0}$:

$$\mathscr{I}_{p0}=x_{p0}+\gamma_{p1}\mathrm{e}^{-\gamma_{p1}\varepsilon_p}\sum_{1<i\leqslant n}b_{1i}x_{i1}$$
$$+\sum_{k=2}^{\left[\frac{n}{2}\right]}\gamma_{pk}\mathrm{e}^{-\gamma_{pk}\varepsilon_p}\sum_{k<i\leqslant\varkappa}b_{ki}\left(x_{ik}-X^{i,i}_{1,k-1}(X^{\varkappa+1,n}_{1,k-1})^{-1}X^{\varkappa+1,n}_{k,k}\right)$$

$$
\begin{aligned}
&-\sum_{k=2}^{[\frac{n}{2}]}\gamma_{pk}X^{k,k}_{1,k-1}(X^{\varkappa+1,n}_{1,k-1})^{-1}X^{\varkappa+1,n}_{k,k}+\sum_{k=1}^{[\frac{n+1}{2}]}\gamma_{p\varkappa}\hat b_{\varkappa\varkappa}\sum_{i>\varkappa}b_{\varkappa i}x_{i\varkappa}-\sum_{k=1}^{[\frac{n}{2}]}\gamma_{p\varkappa}\hat b_{\varkappa\varkappa}\mathscr{I}_{\varkappa k}b_{k\varkappa}\\
&=x_{p0}+(\gamma_{p1}-\gamma_{pn})\mathrm{e}^{-\gamma_{p1}\varepsilon_p}b_{1n}x_{n1}+\sum_{k=2}^{[\frac{n}{2}]}(\gamma_{pk}-\gamma_{p\varkappa})\mathrm{e}^{-\gamma_{pk}\varepsilon_p}b_{k\varkappa}(-1)^{k+1}\frac{|X^{\varkappa,n}_{1,k}|}{|X^{\varkappa+1,n}_{1,k-1}|}\\
&-\sum_{k=2}^{[\frac{n}{2}]}\gamma_{pk}X^{k,k}_{1,k-1}(X^{\varkappa+1,n}_{1,k-1})^{-1}X^{\varkappa+1,n}_{k,k}-\sum_{k=2}^{[\frac{n+1}{2}]}\gamma_{p\varkappa}X^{\varkappa,\varkappa}_{1,k-1}(X^{\varkappa+1,n}_{1,k-1})^{-1}X^{\varkappa+1,n}_{\varkappa,\varkappa}\\
&+\sum_{k=1}^{[\frac{n}{2}]}\frac{(-1)^{k+1}\gamma_{pk}}{|X^{\varkappa,n}_{1,k}|}\sum_{k<i<\varkappa}\begin{vmatrix}X^{i,i}_{1,k} & 0\\ X^{\varkappa,n}_{1,k} & X^{\varkappa,n}_{i,i}\end{vmatrix}+\sum_{k=2}^{[\frac{n}{2}]}\frac{(-1)^{k+1}\gamma_{pk}}{|X^{\varkappa+1,n}_{1,k-1}|}\sum_{k<i<\varkappa}\begin{vmatrix}X^{i,i}_{1,k-1} & 0\\ X^{\varkappa+1,n}_{1,k-1} & X^{\varkappa+1,n}_{i,i}\end{vmatrix}.
\end{aligned}
$$

Below it is essential for consideration that γ has a reduced form. For any fixed $q\in\{1,\dots,s'\}$ the lifted invariant $\mathscr{I}_{q0}$ necessarily depends on the parameter $b_{k_q\varkappa_q}$ which are not, under already possessed normalization conditions, in the expressions of the other lifted invariants. Hence in this case we should use additional normalization conditions constraining $\mathscr{I}_{q0}$, e.g., $\mathscr{I}_{q0}=0$. It gives an expression for $b_{k_q\varkappa_q}$, $q=1,\dots,s'$, via x's, other $b_{k\varkappa}$'s and ε_p. The exact form of the expression for $b_{k_q\varkappa_q}$ is inessential. The expressions for $\mathscr{I}_{p0}$, $p>s'$, depend on no group parameters and, therefore, are invariants. Note only that under the supposition on γ the above formula for $\mathscr{I}_{p0}$ with $p>s'$ is transformed into

$$
\begin{aligned}
\mathscr{I}_{p0}=x_{p0}&+\left(\sum_{k=1}^{[\frac{n}{2}]-1}+\sum_{k=[\frac{n}{2}]}^{[\frac{n}{2}]}\right)\frac{(-1)^{k+1}\gamma_{pk}}{|X^{\varkappa,n}_{1,k}|}\sum_{k<i<\varkappa}\begin{vmatrix}X^{i,i}_{1,k} & 0\\ X^{\varkappa,n}_{1,k} & X^{\varkappa,n}_{i,i}\end{vmatrix}\\
&+\sum_{k=2}^{[\frac{n}{2}]}\frac{(-1)^{k+1}\gamma_{pk}}{|X^{\varkappa+1,n}_{1,k-1}|}\sum_{k\leqslant i\leqslant\varkappa}\begin{vmatrix}X^{i,i}_{1,k-1} & 0\\ X^{\varkappa+1,n}_{1,k-1} & X^{\varkappa+1,n}_{i,i}\end{vmatrix}-\sum_{k=[\frac{n}{2}]}^{[\frac{n+1}{2}]}\gamma_{p\varkappa}X^{\varkappa,\varkappa}_{1,k-1}(X^{\varkappa+1,n}_{1,k-1})^{-1}X^{\varkappa+1,n}_{\varkappa,\varkappa}
\end{aligned}
$$

that results to the second subset of invariants from the statement of the theorem after shifting the index k in the third sum by -1 and further permutation and recombination of terms.

Then, we take $\widehat{\mathscr{I}}_1=\mathscr{I}_{n1}$ and the combinations $\widehat{\mathscr{I}}_k=(-1)^{k+1}\mathscr{I}_{\varkappa k}\widehat{\mathscr{I}}_{k-1}$, $k=2,\dots,[n/2]$, i.e.,

$$
\widehat{\mathscr{I}}_k=|X^{\varkappa,n}_{1,k}|\mathrm{e}^{-\alpha_{qk}\varepsilon_q},\qquad \alpha_{qk}=-\sum_{k'=1}^{k}(\gamma_{q\varkappa'}-\gamma_{qk'}),\qquad k=1,\dots,[n/2].
$$

Since $\widehat{\mathscr{I}}_{k_q}$ depends only on $\varepsilon_q,\dots\varepsilon_{s'}$ and $\partial\widehat{\mathscr{I}}_{k_q}/\partial\varepsilon_q\neq 0$ for any fixed q, the Jacobian $|\partial\widehat{\mathscr{I}}_{k_q}/\partial\varepsilon_{q'}|\neq 0$ and we should impose s' more normalization conditions $\widehat{\mathscr{I}}_{k_q}=1$. After solving them with respect to ε_q and substituting to the other $\widehat{\mathscr{I}}_k$'s, we obtain the first subset of invariants from the statement of the theorem.

Under the normalization we express the non-normalized lifted invariants via only x's and compute a part of the parameters b's and ε's of the coadjoint action via x's and the other b's and ε's. The expressions in the obtained tuples of invariants are functionally independent. No equations involving only x's are obtained. In view of Proposition 1, this implies that the choice of normalization constraints, which depends on values of γ, is correct. That is why the number of the found functionally independent invariants is maximal, i.e., they form bases of $\mathrm{Inv}(\mathrm{Ad}^*_{T_\gamma(n)})$. $\square$

Corollary 1. $|X^{\varkappa,n}_{1,k}|$, $k = 1, \dots, [n/2]$, *are functionally independent relative invariants of* $\mathrm{Ad}^*_{T_\gamma(n)}$ *for any admissible value of* γ.

See, e.g., [32] for the definition of relative invariants.

6. Invariants of $\mathfrak{t}_\gamma(n)$

Let us reformulate Theorem 1 in terms of generalized Casimir operators.

Theorem 2. *Let the parameter matrix* γ *have a reduced form. A basis of* $\mathrm{Inv}(\mathfrak{t}_\gamma(n))$ *is formed by the expressions*

$$|\mathscr{E}^{1,k}_{\varkappa,n}| \prod_{q=1}^{s'} |\mathscr{E}^{1,k_q}_{\varkappa_q,n}|^{\beta_{qk}}, \quad k \in \{1, \dots, [n/2]\} \setminus \{k_1, \dots, k_{s'}\},$$

$$f_p + \sum_{k=1}^{[\frac{n}{2}]} \frac{(-1)^{k+1}}{|\mathscr{E}^{1,k}_{\varkappa,n}|}(\gamma_{pk} - \gamma_{p,k+1}) \sum_{k<i<\varkappa} \begin{vmatrix} \mathscr{E}^{1,k}_{i,i} & \mathscr{E}^{1,k}_{\varkappa,n} \\ 0 & \mathscr{E}^{i,i}_{\varkappa,n} \end{vmatrix}, \quad p = s' + 1, \dots, s,$$

where $\varkappa = n - k + 1$; $\mathscr{E}^{i_1,i_2}_{j_1,j_2}$, $i_1 \leqslant i_2$, $j_1 \leqslant j_2$, *denotes the matrix* $(e_{ij})^{i=i_1,\dots,i_2}_{j=j_1,\dots,j_2}$; $\beta_{qk} = -\Delta_{qk}/\Delta$, $\Delta = \det(\alpha_{q'k_{q''}})_{q',q''=1,\dots,s'}$, Δ_{qk} *is the determinant obtained from* Δ *with change of the column* $(\alpha_{q'k_q})_{q'=1,\dots,s'}$ *by the column* $(\alpha_{q'k})_{q'=1,\dots,s'}$,

$$\alpha_{qj} = -\sum_{j=1}^{k}(\gamma_{q\varkappa} - \gamma_{qk}).$$

Proof. Expanding the determinants in each element of the first tuple of invariants from Theorem 1, we obtain an expression of x's containing only such coordinate functions that corresponding basis elements commutate each to other. Therefore, the symmetrization procedure is trivial. Since $x_{ij} \sim e_{ji}$, $j < i$, hereafter it is necessary to transpose the matrices in the obtained expressions of invariants for representation improvement. Finally we construct the first part of the basis of $\mathrm{Inv}(\mathfrak{t}_\gamma(n))$ from the statement.

The symmetrization procedure for the second tuple of invariants presented in Theorem 1 also can be assumed trivial. To show this, we again expand all the determinants. Only the monomials of the determinants

$$\begin{vmatrix} X^{i,i}_{1,k} & 0 \\ X^{\varkappa,n}_{1,k} & X^{\varkappa,n}_{i,i} \end{vmatrix}, \quad k \in \{1, \dots, [n/2]\}, \quad i = k, \dots, \varkappa,$$

contains coordinate functions associated with noncommuting basis elements of the algebra $\mathfrak{t}_\gamma(n)$. More precisely, each of the monomials includes two such coordinate functions, namely, $x_{ii'}$ and $x_{j'i}$ for some values $i' \in \{1, \dots, k\}$ and $j' \in \{\varkappa, \dots, n\}$. It is sufficient to make only the symmetrization of the corresponding pairs of basis elements. As a result, after the symmetrization and the transposition of the matrices we obtain the following expressions for the invariants of $\mathfrak{t}_\gamma(n)$ corresponding to the invariants of the second tuple from Theorem 1:

$$f_p+\sum_{k=1}^{[\frac{n}{2}]}\frac{(-1)^{k+1}}{|\mathscr{E}^{1,k}_{\varkappa,n}|}(\gamma_{pk}-\gamma_{p,k+1})\sum_{k<i<\varkappa}\sum_{i'=1}^{k}\sum_{j'=\varkappa}^{n}\frac{e_{i'i}e_{ij'}+e_{ij'}e_{i'i}}{2}(-1)^{i'j'}|\mathscr{E}^{1,k;\hat{i}'}_{\varkappa,n;\hat{j}'}|,$$

where $p=s'+1,\ldots,s$ and $|\mathscr{E}^{1,k;\hat{i}'}_{\varkappa,n;\hat{j}'}|$ denotes the minor of the matrix $\mathscr{E}^{1,k}_{\varkappa,n}$ complementary to the element $e_{i'j'}$. Since $e_{i'i}e_{ij'}=e_{ij'}e_{i'i}+e_{i'j'}$ then

$$\sum_{i'=1}^{k}\sum_{j'=\varkappa}^{n}\frac{e_{i'i}e_{ij'}+e_{ij'}e_{i'i}}{2}(-1)^{i'j'}|\mathscr{E}^{1,k;\hat{i}'}_{\varkappa,n;\hat{j}'}|=\begin{vmatrix}\mathscr{E}^{1,k}_{i,i} & \mathscr{E}^{1,k}_{\varkappa,n}\\ 0 & \mathscr{E}^{i,i}_{\varkappa,n}\end{vmatrix}\pm\frac{1}{2}|\mathscr{E}^{1,k}_{\varkappa,n}|,$$

where we have to take the sign '+' (resp. '−') if the elements of $\mathscr{E}^{1,k}_{i,i}$ are placed after (resp. before) the elements of $\mathscr{E}^{i,i}_{\varkappa,n}$ in all the relevant monomials. Up to constant summands, this results in the expressions for the elements of the second part of the invariant basis adduced in the statement. These expressions are formally derived from the corresponding expressions from Theorem 1 by the replacement $x_{ij}\to e_{ji}$ and $x_{p0}\to f_p$ and the transposition of all matrices. That is why we assume that the symmetrization procedure is trivial in the sense described. Let us emphasize that a uniform order of elements from $\mathscr{E}^{1,k}_{i,i}$ and $\mathscr{E}^{i,i}_{\varkappa,n}$ has to be fixed in all the monomials under usage of the 'non-symmetrized' form of invariants. □

Corollary 2. *The algebra $\mathfrak{t}_\gamma(n)$ admits a rational basis of invariants if and only if $\beta_{qk}\in\mathbb{Q}$ for all $k\in\{1,\ldots,[n/2]\}\setminus\{k_1,\ldots,k_{s'}\}$ and all $q\in\{1,\ldots,s'\}$.*

Note 2. It follows from Theorem 2 that the cardinality $N_{\mathfrak{t}_\gamma(n)}$ of fundamental invariants of the algebra $\mathfrak{t}_\gamma(n)$ equals to $[n/2]+s-2s'$, where s is the number of nilindependent elements and

$$s'=\operatorname{rank}(\gamma_{p\varkappa}-\gamma_{pk})^{p=1,\ldots,s}_{k=1,\ldots,[n/2]}.$$

For any fixed s the cardinality $N_{\mathfrak{t}_\gamma(n)}$ is maximal if s' has the minimally possible value. In the case $s\in\{1,\ldots,[n/2]\}$ such value is $s'=0$ and, therefore, $\mathfrak{t}_\gamma(n)=[n/2]+s$. It means that $\gamma_{pk}=\gamma_{p\varkappa}$ for all $k\in\{1,\ldots,[n/2]\}$ and all $p\in\{1,\ldots,s\}$. This condition can be reformulated in terms of commutators in the following way. Any nilindependent element commute with the 'nilpotent' basis elements $e_{k\varkappa}$, $k=1,\ldots,[n/2]$, lying on the significant part of the secondary diagonal of the basis 'matrix' $\mathscr{E}$, i.e., $[f_p,e_{k\varkappa}]=0$, $k=1,\ldots,[n/2]$. If $s\in\{[n/2]+1,\ldots,n\}$ the minimal value of s' is $s'=s-[n/2]$ and, therefore, $N_{\mathfrak{t}_\gamma(n)}=3[n/2]-s$. It is equivalent to the condition that $[n/2]$ nilindependent elements of the algebra commute with the basis elements $e_{k\varkappa}$, $k=1,\ldots,[n/2]$.

Note 3. The elements lying on the secondary diagonal of the matrix of lifted invariants play a singular role under the normalization procedure in all investigated algebras with the nilradicals isomorphic to $\mathfrak{t}_0(n)$: $\mathfrak{t}_0(n)$ itself and $\mathfrak{st}(n)$ [8] as well as $\mathfrak{t}_\gamma(n)$ studied in this paper. (More precisely, in [8] the normalization procedure was realized for $\mathfrak{t}(n)$ and then the results on invariants were extended to $\mathfrak{st}(n)$.) Reasons of such singularity were not evident from the consideration in [8]. Note 2 gives an explanation for it and justifies naturalness of the chosen normalization conditions.

7. Particular cases

Theorem 2 includes, as particular cases, known results on invariants of the nilpotent algebra of strongly upper triangular matrices $\mathfrak{t}_0(n)$ [7,8,45], the solvable algebras $\mathfrak{st}(n)$ and $\mathfrak{t}(n)$ of special upper and non-strictly upper triangular matrices [8,45] and the solvable algebras with the nilradical isomorphic to $\mathfrak{t}_0(n)$ and one nilindependent element [9,45]. We show this below, giving additional comments and rewriting invariants in bases which are more appropriate for the special cases.

Let us remind that $N_\mathfrak{g}$ denotes the maximal number of functionally independent invariants in the set $\mathrm{Inv}(\mathrm{Ad}^*_G)$ of invariants of Ad^*_G, where G is the connected Lie group associated with the Lie algebra $\mathfrak{g}$. We use the short 'non-symmetrized' form for certain basis invariants, where it is uniformly assumed that in all monomials elements of $\mathcal{E}^{1,k}_{i,i}$ is placed before (or after) elements of $\mathcal{E}^{i,i}_{\varkappa,n}$. See the proof of Theorem 2 for details.

The algebra $\mathfrak{t}_0(n)$ has no nilindependent elements, i.e., for it $s=0$ and $|X^{\varkappa,n}_{1,k}|$, $k=1,\dots,[n/2]$, are functionally independent absolute invariants of $\mathrm{Ad}^*_{T_0(n)}$.

Corollary 3. $N_{\mathfrak{t}_0(n)}=[n/2]$. *A basis of* $\mathrm{Inv}(\mathfrak{t}_0(n))$ *is formed by the Casimir operators* (*i.e., polynomial invariants*)

$$\det(e_{ij})^{i=1,\dots,k}_{j=n-k+1,\dots,n},\quad k=1,\dots,\left[\frac{n}{2}\right].$$

In the case of one nilindependent element ($s=1$) we can omit the subscript of f and the first subscript of γ. There are two different cases depending on the value of s' which can be either 0 or 1. The statement on invariant can be easily formulated even for the unreduced form of γ.

Corollary 4. *Let* $s=1$. *If additionally* $s'=0$, *i.e.,* $\gamma_k=\gamma_\varkappa$ *for all* $k\in\{1,\dots,[n/2]\}$ *then* $N_{\mathfrak{t}_0(n)}=[n/2]+1$ *and a basis of* $\mathrm{Inv}(\mathfrak{t}_\gamma(n))$ *is formed by the expressions*

$$|\mathcal{E}^{1,k}_{\varkappa,n}|,\quad k=1,\dots,\left[\frac{n}{2}\right],\qquad f+\sum_{k=1}^{[\frac{n}{2}]}\frac{(-1)^{k+1}}{|\mathcal{E}^{1,k}_{\varkappa,n}|}(\gamma_k-\gamma_{k+1})\sum_{i=k+1}^{n-k}\begin{vmatrix}\mathcal{E}^{1,k}_{i,i} & \mathcal{E}^{1,k}_{\varkappa,n}\\ 0 & \mathcal{E}^{i,i}_{\varkappa,n}\end{vmatrix}.$$

Hereafter $\varkappa=n-k+1$, $\mathcal{E}^{i_1,i_2}_{j_1,j_2}$, $i_1\leqslant i_2$, $j_1\leqslant j_2$, *denotes the matrix* $(e_{ij})^{i=i_1,\dots,i_2}_{j=j_1,\dots,j_2}$.

Otherwise $s'\neq 0$, $N_{\mathfrak{t}_0(n)}=[n/2]-1$ *and a basis of* $\mathrm{Inv}(\mathfrak{t}_\gamma(n))$ *consists of the invariants*

$$|\mathcal{E}^{1,k}_{\varkappa,n}|,\quad k=1,\dots,k_0-1,\qquad |\mathcal{E}^{1,k_0}_{\varkappa_0,n}|^{\alpha_k}|\mathcal{E}^{1,k}_{\varkappa,n}|,\quad k=k_0+1,\dots,\left[\frac{n}{2}\right],$$

where k_0 *the minimal value of* k *for which* $\gamma_k\neq\gamma_\varkappa$ *and*

$$\alpha_k=-\sum_{i=k_0}^{k}\frac{\gamma_{n-i+1}-\gamma_i}{\gamma_{n-k_0+1}-\gamma_{k_0}}.$$

The basis constructed for the first case is formed by $[n/2]$ Casimir operators and a nominally rational invariant. The latter invariant can be replaced by the product of it and the Casimir operators $|\mathcal{E}^{1,k}_{\varkappa,n}|$, $k=1,\dots,[n/2]$. This product is more complicated but polynomial. Therefore, under the conditions $s=1$, $s'=0$ the algebra $\mathfrak{t}_\gamma(n)$ possesses a polynomial fundamental invariant.

In the second case $\mathrm{Inv}(\mathfrak{t}_\gamma(n))$ has a rational basis if and only if $\alpha_k\in\mathbb{Q}$ for all $k\in\{k_0,\dots,[n/2]\}$. Under this condition the obtained basis consists of k_0-1 Casimir operators and $[n/2]-k_0$ rational invariants. If additionally $\alpha_k\geqslant 0$ for all $k\in\{k_0,\dots,[n/2]\}$ then the whole basis is polynomial.

Note that for both the cases of b (i.e., for both $b = -1$ and $b \neq -1$) the results on the algebra $\mathfrak{g}_{4.8}^b$ adduced in Section 3 are easily derived from Corollary 4 via fixing $n = 3$, then identifying $e_1 \sim e_{13}, e_2 \sim e_{12}, e_3 \sim e_{23}$ and $e_4 \sim f$ and putting $\gamma_1 = -1$, $\gamma_2 = 0$ and $\gamma_3 = b$.

In the case of the maximal number $s = n - 1$ of nilindependent elements the algebra $\mathfrak{t}_\gamma(n)$ is isomorphic to the algebra $\mathfrak{st}(n)$ of special upper triangular matrices [8]. For the associated matrix γ of this algebra

$$s' = \operatorname{rank}(\gamma_{p\varkappa} - \gamma_{pk})_{k=1,\dots,[n/2]}^{p=1,\dots,s} = \left[\frac{n}{2}\right].$$

Therefore, $\mathfrak{st}(n)$ has no invariants depending only on elements of the nilradical. The number of zero rows in the matrix $(\gamma_{p\varkappa} - \gamma_{pk})_{k=1,\dots,[n/2]}^{p=1,\dots,s}$ after reduction of γ should equal to $s - s' = n - 1 - [n/2] = [(n-1)/2]$. We choose the basis in $\mathfrak{st}(n)$, which is formed by the elements of the canonical basis of the nilradical and nilindependent elements f_p, $p = 1, \dots, n-1$, corresponding to the matrix γ with

$$\gamma_{pi} = \frac{n-p}{n}, \quad i = 1, \dots, p, \qquad \gamma_{pi} = -\frac{p}{n}, \quad i = p+1, \dots, n.$$

The commutation relations of $\mathfrak{st}(n)$ in the chosen basis are

$$\begin{aligned}
&[e_{ij}, e_{i'j'}] = \delta_{i'j} e_{ij'} - \delta_{ij'} e_{i'j}, \quad i < j, \ i' < j';\\
&[f_k, f_{k'}] = 0, \quad k, k' = 1, \dots, n-1;\\
&[f_k, e_{ij}] = 0, \quad i < j \leqslant k \ \text{ or } \ k \leqslant i < j;\\
&[f_k, e_{ij}] = e_{ij}, \quad i \leqslant k \leqslant j, \ i < j.
\end{aligned}$$

Then we pass to the basis in which the matrix γ has a reduced form. We denote the reduced form by γ'. Only the part of the new basis, which corresponds to the zero rows of $(\gamma'_{p\varkappa} - \gamma'_{pk})_{k=1,\dots,[n/2]}^{p=1,\dots,s}$, is essential for finding a fundamental invariant of $\mathfrak{st}(n)$. As this part, we can take the set consisting of the elements $f'_{s'+p} = f_p - f_{n-p}$, $p = 1, \dots, [(n-1)/2]$. Indeed, they are linearly independent and

$$\gamma'_{s'+p,i} = -2\frac{p}{n}, \quad i = p+1, \dots, n-p, \qquad \gamma_{s'+p,i} = \frac{n-2p}{n} \quad \text{otherwise.}$$

Note also that under $p = 1, \dots, [(n-1)/2]$ and $k = 1, \dots, [n/2]$ the expression $\gamma'_{s'+p,k} - \gamma'_{s'+p,k+1}$ equals to 1 if $k = p$ and vanishes otherwise.

Corollary 5. $N_{\mathfrak{st}(n)} = [(n-1)/2]$. *A basis of* $\operatorname{Inv}(\mathfrak{st}(n))$ *consists of the rational invariants*

$$\check{\mathscr{I}}_k = f_k - f_{n-k} + \frac{(-1)^{k+1}}{|\mathscr{E}^{1,k}_{\varkappa,n}|} \sum_{j=k+1}^{n-k} \begin{vmatrix} \mathscr{E}^{1,k}_{j,j} & \mathscr{E}^{1,k}_{\varkappa,n} \\ 0 & \mathscr{E}^{j,j}_{\varkappa,n} \end{vmatrix}, \quad k = 1, \dots, \left[\frac{n-1}{2}\right],$$

where $\mathscr{E}^{i_1,i_2}_{j_1,j_2}$, $i_1 \leqslant i_2$, $j_1 \leqslant j_2$, *denotes the matrix* $(e_{ij})_{j=j_1,\dots,j_2}^{i=i_1,\dots,i_2}$, $\varkappa = n - k + 1$.

The algebra $\mathfrak{t}(n)$ of non-strictly upper triangular matrices stands alone from the considered algebras since the nilradical of $\mathfrak{t}(n)$ is wider than $\mathfrak{t}_0(n)$. Similarly to $\mathfrak{t}_0(n)$, the algebra $\mathfrak{t}(n)$ admit the completely matrix interpretations of a basis and lifted invariants. Namely, its basis elements are convenient to enumerate with the 'non-decreasing' pair of indices similarly to the canonical basis $\{E^n_{ij}, i \leqslant j\}$ of the isomorphic matrix algebra. Thus, the basis elements $e_{ij} \sim E^n_{ij}$, $i \leqslant j$, satisfy the commutation relations $[e_{ij}, e_{i'j'}] = \delta_{i'j} e_{ij'} - \delta_{ij'} e_{i'j}$, where δ_{ij} is the Kronecker delta.

The center of $\mathfrak{t}(n)$ is one-dimensional and coincides with the linear span of the sum $e_{11}+\cdots+e_{nn}$ corresponding to the unity matrix E^n. The elements e_{ij}, $i<j$, and $e_{11}+\cdots+e_{nn}$ form a basis of the nilradical of $\mathfrak{t}(n)$, which is isomorphic to $\mathfrak{t}_0(n)\oplus\mathfrak{a}$. Here $\mathfrak{a}$ is the one-dimensional (Abelian) Lie algebra.

Let e^*_{ji}, x_{ji} and y_{ij} denote the basis element and the coordinate function in the dual space $\mathfrak{t}^*(n)$ and the coordinate function in $\mathfrak{t}(n)$, which correspond to the basis element e_{ij}, $i\leqslant j$. We complete the sets of x_{ji} and y_{ij} to the matrices X and Y with zeros. Hence X is a lower triangular matrix and Y is an upper triangular one. In the above notations a fundamental lifted invariant of $\mathrm{Ad}^*_{T(n)}$ is formed by the elements $\mathcal{I}_{ij}$, $j\leqslant i$, of the matrix $\mathcal{I}=BXB^{-1}$, where B is an arbitrary matrix from $T(n)$ (Lemma 2 of [8]). See also Note 3 of [8] for discussion on essential parameters in this fundamental lifted invariant. Due to the matrix representation of lifted invariant, a basis of $\mathrm{Inv}(\mathrm{Ad}^*_{T(n)})$ can be constructed by the normalization procedure in a quite easy way.

At the same time, a basis of $\mathrm{Inv}(\mathrm{Ad}^*_{T(n)})$ is obtained from the basis of $\mathrm{Inv}(\mathrm{Ad}^*_{ST(n)})$ with attaching the central element $e_{11}+\cdots+e_{nn}$. Indeed, the algebra $\mathfrak{t}(n)$ is a central extension of $\mathfrak{st}(n)$, i.e., $\mathfrak{t}(n)=\mathfrak{st}(n)\oplus Z(\mathfrak{t}(n))$, under the natural embedding of $\mathfrak{st}(n)$ into $\mathfrak{t}(n)$. It is well known that if the Lie algebra $\mathfrak{g}$ is decomposable into the direct sum of Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ then the union of bases of $\mathrm{Inv}(\mathfrak{g}_1)$ and $\mathrm{Inv}(\mathfrak{g}_2)$ is a basis of $\mathrm{Inv}(\mathfrak{g})$. A basis of $\mathrm{Inv}(Z(\mathfrak{t}(n)))$ obviously consists of only one element, e.g., $e_{11}+\cdots+e_{nn}$. Therefore, the basis cardinality of equals to $\mathrm{Inv}(\mathfrak{t}(n))$ the basis cardinality of $\mathrm{Inv}(\mathfrak{st}(n))$ plus 1, i.e., $[(n+1)/2]$. We only combine basis elements and rewrite them in terms of the canonical basis of $\mathfrak{t}(n)$. Namely,

$$\widehat{\mathcal{I}}_0:=e_{11}+\cdots+e_{nn},\quad \widehat{\mathcal{I}}_k=(-1)^{k+1}\check{\mathcal{I}}_k+(-1)^k\frac{n-2k}{n}\widehat{\mathcal{I}}_0,\quad k=1,\ldots,\left[\frac{n-1}{2}\right].$$

Corollary 6. $N_{\mathfrak{t}(n)}=[(n+1)/2]$. *A basis of* $\mathrm{Inv}(\mathfrak{t}(n))$ *consists of the rational invariants*

$$\widehat{\mathcal{I}}_k=\frac{1}{|\mathcal{E}^{1,k}_{\varkappa,n}|}\sum_{j=k+1}^{n-k}\begin{vmatrix}\mathcal{E}^{1,k}_{j,j} & \mathcal{E}^{1,k}_{\varkappa,n}\\ e_{jj} & \mathcal{E}^{j,j}_{\varkappa,n}\end{vmatrix},\quad k=0,\ldots,\left[\frac{n-1}{2}\right],$$

where $\mathcal{E}^{i_1,i_2}_{j_1,j_2}$, $i_1\leqslant i_2$, $j_1\leqslant j_2$, *denotes the matrix* $(e_{ij})^{i=i_1,\ldots,i_2}_{j=j_1,\ldots,j_2}$, $|\mathcal{E}^{1,0}_{n+1,n}|:=1$, $\varkappa=n-k+1$.

Note that in [8] the inverse way was preferred due to the simple matrix representation of a fundamental lifted invariant of $\mathrm{Ad}^*_{T(n)}$. Namely, at first a basis of $\mathrm{Inv}(\mathfrak{t}(n))$ was calculated by the normalization procedure and then it was used for construction of a basis of $\mathrm{Inv}(\mathfrak{st}(n))$.

8. Conclusion and discussion

In this paper we investigate invariants of solvable Lie algebras with the nilradicals isomorphic to $\mathfrak{t}_0(n)$ and 'diagonal' nilindependent elements, using our original pure algebraic approach [6,7] and the special technique developed in [8,9] for triangular algebras within the framework of this approach. All such algebras are embedded in $\mathfrak{st}(n)$ as ideals. The number s of nilindependent elements varies from 0 to $n-1$. In the frontier cases $s=0$ and $s=n-1$ the algebras are isomorphic to the universal algebras $\mathfrak{t}_0(n)$ and $\mathfrak{st}(n)$ correspondingly.

The two main steps of the algorithm are the construction of a fundamental lifted invariant of the coadjoint representation of the corresponding connected Lie group and the exclusion of parameters from lifted invariants by the normalization procedure. The realization of both steps for the algebras under consideration are more difficult than for the particular cases investigated earlier. Thus, the constructed fundamental lifted invariant has a more complicated representation. It is divided into two parts which play different roles under the normalization. The part corresponding to the nilradical admits a simple 'matrix' representation which is important for further consideration. The components from the other part involves also nilindependent elements and algebra parameters. That is why the choice of the normalization conditions essentially depends on algebra parameters that leads to the furcation of calculations and final results. The partition of the fundamental lifted invariant induces the partition of normalization conditions and the associated basis of algebra invariants.

The above obstacles are surmounted due to the optimization of the applied technique, taking into account properties of the algebras under consideration, in particular, their standard matrix representations. This technique involves the choice of special parameterizations of the inner automorphism groups, the representation of most of the lifted invariants via matrices and the natural normalization constraints associated with the algebra structure. The cardinality of the invariant bases is determined in process of their construction. Moreover, we only partially constrain lifted invariants in the beginning of the normalization procedure and only with conditions without the algebra parameters. Both the total number of necessary constraints and the additional constraints are specified before completing of normalization depending on values of algebra parameters. As a result of the optimization, excluding the group parameters b's and ε's is in fact reduced to solving linear systems of (algebraic) equations.

We plan to continue investigations of the solvable Lie algebras with the nilradicals isomorphic to $\mathfrak{t}_0(n)$ in the general case where nilindependent elements are not necessarily diagonal. All such algebras were classified in [44], and this classification can be enhanced with adaptation of known results [17] on automorphisms of $\mathfrak{t}_0(n)$. Unfortunately, it is not understandable as of yet whether the partial matrix representation of lifted invariants and other tricks from the developed 'triangular' technique will be applicable in these investigations.

Other possibilities on the usage of the algorithm are outlined in our previous papers [6–9]. We hope that the presented results are of interest in the theory of integrable systems and for labeling of representations of Lie algebras, as well as other applications, since the algorithm provides a powerful purely algebraic alternative to the usual method involving differential equations, and certain ad-hoc methods developed for special classes of Lie algebras.

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