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SCIENCE DIRECT
Journal of Functional Analysis 213 (2004) 176–205
JOURNAL OF Functional Analysis
http://www.elsevier.com/locate/jfa

The non-microstates free entropy dimension of DT-operators

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Received 4 June 2003; revised 11 November 2003; accepted 30 April 2004

Communicated by Don Voiculescu

Abstract

Dykema and Haagerup introduced the class of DT-operators (Amer. J. Math. 126 (2004) 121–189) and also showed that every DT-operator generate $L(\mathbf{F}_2)$ (J. Funct. Anal. 209 (2004) 332–366), the von Neumann algebra generated by the free group on two generators. In this paper, we prove that Voiculescu's non-microstates free entropy dimension is 2 for all DT-operators.

Keywords: Free probability; DT-operators; Free entropy dimension

1. Introduction

The class of DT-operators was introduced by Dykema and Haagerup [DH1]. For $n \in \mathbb{N}$ let $\mathcal{M}_n(\mathbb{C})$ be the space of $n \times n$-matrices with entries being random variables over a classical probability space having moments of all orders. Let $\tau_n : \mathcal{M}_n(\mathbb{C}) \to \mathbb{C}$ be the expectation of the normalized trace on $\mathcal{M}_n(\mathbb{C})$. Then $(\mathcal{M}_n(\mathbb{C}), \tau_n)$ is a $*$-non-commutative probability space.

Let μ be a compactly supported Borel-measure on $\mathbb{C}$ and let $c > 0$. Let $D_n \in \mathcal{M}_n(\mathbb{C})$ be diagonal matrices with i.i.d diagonal entries all having distribution μ. Let $T_n \in \mathcal{M}_n(\mathbb{C})$ be strictly upper triangular matrices such that the $n(n-1)$ real and imaginary parts of the entries above the diagonal consists of a family of i.i.d.

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0022-1236/$ - see front matter
doi:10.1016/j.jfa.2004.04.012

centered Gaussian random variables with variance $1/2n$. Define $Z_n = D_n + cT_n$. Let Z be an element in a $*$-non-commutative probability space, $(\mathcal{M}, \tau)$. Z is a $\mathrm{DT}(\mu, c)$-element if its $*$-moments, $\tau(Z^{\varepsilon_1} Z^{\varepsilon_2} \cdots Z^{\varepsilon_k})$, is determined by the limit

$$\lim_{n \to \infty} \tau_n(Z_n^{\varepsilon_1} Z_n^{\varepsilon_2} \cdots Z_n^{\varepsilon_k}) \tag{1.1}$$

for all $k \in \mathbb{N}$ and $\varepsilon_1, \ldots, \varepsilon_k \in \{1, *\}$. Limit (1.1) always exists [DH1, Theorem 2.1].

A $\mathrm{DT}(\mu, c)$-operator is a $\mathrm{DT}(\mu, c)$-element constructed in a W^*-probability space, and a DT-operator is a $\mathrm{DT}(\mu, c)$-operator for some μ and c. Haagerup and Dykema has shown [DH2] that actually every DT-operator generate a von Neumann algebra isomorphic to $L(\mathbf{F}_2)$, the von Neumann algebra generated by the free group on two generators.

Voiculescu has introduced two kinds of entropy; a microstates free entropy, χ, and a non-microstates free entropy, χ^*, and he conjectures that $\chi = \chi^*$. Following the definition of the microstates free entropy dimension, δ, from χ we define the non-microstates free entropy dimension, δ^*, similarly from χ^*. So if $\chi = \chi^*$ then of course $\delta = \delta^*$.

Śniady has shown a formula [Śn] for the microstates free entropy of DT-operators. In particular, it follows from his results that a $\mathrm{DT}(\delta_0, 1)$-operator or upper triangular operator which is also just the limit in $*$-moments of the T_n's above, has microstates free entropy $-\infty$. This makes it an interesting problem to compute the free entropy dimension of this operator since $-\infty$-microstates entropy is the only chance that the microstates entropy dimension can be less than 2. We prove the following theorem.

Theorem 1.1. *Let μ be a compactly supported Borel measure on $\mathbb{C}$, let $c > 0$ and let Z be a $\mathrm{DT}(\mu, c)$-operator. Then*

$$\delta^*(Z) = 2. \tag{1.2}$$

We also consider $\delta^*(\cdot, \mathcal{B})$, the non-microstates free entropy dimension with respect to an algebra, $\mathcal{B}$, and we show that $\delta^*(a_1, \ldots, a_n : \mathcal{B})$ of self-adjoint variables $a_1, \ldots, a_n$ can only be different from n if the non-microstates free Fischer information, $\Phi^*(a_1, \ldots, a_n : \mathcal{B})$, is $+\infty$.

2. Review on free Fischer information

The following review on free Fisher information can be read out of Nica et al. [NSS1,NSS2] and Speicher [Sp2]. Free Fischer information was originally introduced by Voiculescu [Voi1,Voi5] and further investigated in [Voi6].

In this section we will, unless otherwise stated, let $(\mathcal{A}, \phi)$ be a W^*-probability space with ϕ a faithful, normal trace, let $1 \in \mathcal{B} \subset A$ be a unital W^*-sub-algebra. Also we let $E_{\mathcal{B}} : \mathcal{A} \to \mathcal{B}$ be a conditional expectation such that $(\mathcal{B} \subset \mathcal{A}, E_{\mathcal{B}})$ is a $\mathcal{B}$-probability space compatible to $(\mathcal{A}, \phi)$ in the sense that $\phi = \phi \circ E_{\mathcal{B}}$. Let

$L^2(\mathscr{A}, \phi)$ be the Hilbert space completion of $\mathscr{A}$ with respect to the norm $||a||_\phi = \sqrt{\phi(a^*a)}$ for $a \in \mathscr{A}$.

If $\mathscr{X}$ is a subset of $\mathscr{A}$ then $L^2(\mathscr{X}, \phi)$ will denote the L^2-completion, $\overline{\text{alg}(\mathscr{X}, \mathscr{X}^*)}^{||\cdot||_\phi}$, of the unital $*$-algebra generated by $\mathscr{X}$.

We define a self-adjoint family, $(a_i)_{i\in I} \subset \mathscr{A}$, to be a family of operators such that for all $i \in I$ there exists $j \in I$ such that $a_i^* = a_j$.

Definition 2.1. Let $(\mathscr{A}, \phi)$ be a W^*-probability space with ϕ a faithful trace. Let $1 \in \mathscr{B} \subset A$ be a unital W^*-sub-algebra, and let $(a_i)_{i\in I}$ be a self-adjoint family of random variables in $\mathscr{A}$. Then a family of vectors $(\xi_i)_{i\in I}$ from $L^2(\mathscr{A}, \phi)$ fulfills the conjugate relations for $(a_i)_{i\in I}$ with respect to $\mathscr{B}$ if

$$\begin{aligned} &\phi(\xi_i b_0 a_{i_1} b_1 a_{i_2} \cdots a_{i_n} b_n) \\ &\quad = \sum_{m=1}^{n} \delta_{i,i_m} \phi(b_0 a_{i_1} \cdots a_{i_{m-1}} b_{m-1}) \phi(b_m a_{i_{m+1}} b_{m+1} \ldots a_{i_n} b_n) \end{aligned} \tag{2.1}$$

for every $n \geqslant 0$, $b_0, b_1, \ldots, b_n \in B$ and $i, i_1, i_2, \ldots i_n \in I$.

A family of vectors $(\xi_i)_{i\in I} \subset L^2(\mathscr{A}, \phi)$ is said to be a conjugate system for a self-adjoint family of operators, $(a_i)_{i\in I} \subset \mathscr{A}$, with respect to $\mathscr{B}$ if it satisfies the conjugate relations (2.1) and if furthermore $(\xi_i)_{i\in I} \subset L^2((a_i)_{i\in I} \cup \mathscr{B}, \phi)$.

Remark 2.2. (a) The above definition is to be understood as $\phi(\xi_i b) = 0$ for all $i \in I$. Since $\mathscr{B}$ is unital we thus have $\phi(\xi_i) = 0$ for all $i \in I$.

(b) If a conjugate system $(\xi_i)_{i\in I}$ exists then it is unique since (2.1) is a prescription for taking inner products with monomials of the form $b_0 a_{i_1} b_1 a_{i_2} \ldots a_{i_n} b_n$ from $L^2((a_i)_{i\in I} \cup \mathscr{B}, \phi)$, so the inner product of an element from $(\xi_i)_{i\in I}$ with an arbitrary element from $L^2((a_i)_{i\in I} \cup \mathscr{B}, \phi)$ is completely determined.

(c) If one can find $(\xi)_{i\in I}$ that fulfills the conjugate relations, (2.1), for a self-adjoint family, $(a_i)_{i\in I} \subset \mathscr{A}$, with respect to a unital sub-algebra $\mathscr{B}$ of $\mathscr{A}$, then if $P: L^2(\mathscr{A}, \phi) \to L^2((a_i)_{i\in I} \cup \mathscr{B}, \phi)$ is the Hilbert space projection then $(P\xi_i)_{i\in I}$ is a conjugate system for $(a_i)_{i\in I}$ with respect to $\mathscr{B}$.

(d) If $(a_i)_{i=1}^n$ are all self-adjoint then Voiculescu originally denoted the conjugate variables of $(a_i)_{i=1}^n$ with respect to $\mathscr{B}$ by

$$\mathscr{J}(a_i\colon \mathscr{B}[a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n])$$

for $i \in \{1, \ldots, n\}$.

Definition 2.3. Let $(\mathscr{A}, \phi)$ be a W^*-probability space with ϕ a faithful trace. Let $\mathscr{B} \subset A$ be a unital W^*-sub-algebra, and let $(a_i)_{i\in I}$ be a self-adjoint family of random variables in $\mathscr{A}$. If $(a_i)_{i\in I}$ has a conjugate system, $(\xi)_{i\in I}$, with respect to $\mathscr{B}$ then we define the free Fischer information of $(a_i)_{i\in I}$ with respect to B as

$$\Phi^*((a_i)_{i\in I} : \mathscr{B}) = \sum_{i\in I} ||\xi_i||_\phi^2. \tag{2.2}$$

If no conjugate system exists for $(a_i)_{i\in I}$ with respect to $\mathscr{B}$ we define $\Phi^*((a_i)_{i\in I} : \mathscr{B}) = +\infty$. If $\mathscr{B} = \mathbb{C}1$ then we define

$$\Phi^*((a_i)_{i\in I}) = \Phi^*((a_i)_{i\in I} : \mathbb{C}1),$$

and we call this the free Fischer information of $(a_i)_{i\in I}$.

Remark 2.4. (a) If $(\xi_i)_{i\in I}$ satisfies the conjugate relations for a self-adjoint family $(a_i)_{i\in I} \subset \mathscr{A}$ with respect to $\mathscr{B}$ and if $P : L^2(\mathscr{A}, \phi) \to L^2((a_i)_{i\in I} \cup \mathscr{B}, \phi)$ is the Hilbert space projection then by Remark 2.2 (c) we know that $(P\xi_i)_{i\in I}$ is a conjugate system for $(a_i)_{i\in I}$ with respect to $\mathscr{B}$ and since projections are norm decreasing we conclude that

$$\Phi^*((a_i)_{i\in I} : \mathscr{B}) \leqslant \sum_{i\in I} ||\xi_i||_\phi^2. \tag{2.3}$$

(b) If $r \in \mathbb{R}$ is a strictly positive scalar then one easily sees that if $(a_i)_{i\in I} \subset \mathscr{A}$ is a self-adjoint family of random variables with conjugate system $(\xi_i)_{i\in I}$ then $(\frac{1}{r}\xi_i)_{i\in I}$ is a conjugate system for $(ra_i)_{i\in I}$ and thus

$$\Phi^*((ra_i)_{i\in I} : \mathscr{B}) = \frac{1}{r^2}\Phi^*((a_i)_{i\in I} : \mathscr{B}). \tag{2.4}$$

(c) The free Fischer information respects inclusion of sub-algebras in the following sense. If $(\mathscr{A}, \phi)$ is a W^*-probability space and $1 \in \mathscr{B}_1 \subset \mathscr{B}_2$ are to unital W^*-sub-algebras then if $(a_i)_{i\in I} \subset \mathscr{A}$ is a self-adjoint system then

$$\Phi^*((a_i)_{i\in I} : \mathscr{B}_1) \leqslant \Phi^*((a_i)_{i\in I} : \mathscr{B}_2) \tag{2.5}$$

because if a conjugate system for $(a_i)_{i\in I}$ exists with respect to $\mathscr{B}_2$ then this conjugate system will also satisfy the conjugate relations for $(a_i)_{i\in I}$ with respect to $\mathscr{B}_1$, and hence (2.5) follows from (2.3) in (a).

The following theorem is a special case of Nica et al. [NSS2, Theorem 4.1]. Concerning cumulants we adopt the tensor-product notation of Speicher [Sp1].

Theorem 2.5 (Nica et al. [NSS2, Theorem 4.1]). *Let $(\mathscr{A}, \phi)$ be a W^*-probability space where ϕ is a faithful, normal trace and let $(\mathscr{B} \subset \mathscr{A}, E_{\mathscr{B}})$ be a $\mathscr{B}$-probability space compatible to $(\mathscr{A}, \phi)$. Let $(a_i)_{i\in I}$ be a self-adjoint family of random variables in $\mathscr{A}$. Then $(\xi_i)_{i\in I}$ satisfies the conjugate relations for $(a_i)_{i\in I}$ with respect to $\mathscr{B}$ if* *and only if*

$$\kappa_{n+1}^{\mathscr{B}}\left(\xi_i \bigotimes_{\mathscr{B}} b_0 a_{i_1} \bigotimes_{\mathscr{B}} b_1 a_{i_2} \bigotimes_{\mathscr{B}} \cdots \bigotimes_{\mathscr{B}} b_{n-1} a_{i_n}\right) = \begin{cases} \delta_{ii_1}\phi(b_0)1 & \textit{for } n=1 \\ 0 & \textit{for } n\neq 1 \end{cases} \tag{2.6}$$

for all $b_0, \dots, b_{n-1} \in \mathscr{B}$ *and* $i, i_1, \dots i_n \in I$.

Remark 2.6. Consider a non-self-adjoint random variable $a \in \mathscr{A}$. Then a conjugate system for (a, a^*) must have form (ξ, ξ^*) because of the tracial properties of ϕ and the conjugate relations (2.1). From Theorem 2.5 it is easy to see that $\left(\frac{\xi+\xi^*}{\sqrt{2}}, -\frac{\xi-\xi^*}{\mathrm{i}\sqrt{2}}\right)$ is a conjugate system for $\left(\frac{a+a^*}{\sqrt{2}}, \frac{a-a^*}{\mathrm{i}\sqrt{2}}\right)$ with respect to $\mathscr{B}$ so we conclude that

$$\begin{aligned}\Phi^*(a,a^*) &= ||\xi||_\phi^2 + ||\xi||_\phi^2 = \phi(\xi^*\xi) + \phi(\xi\xi^*) \\ &= \phi\left(\frac{\xi^2 + (\xi^*)^2 + \xi^*\xi + \xi\xi^*}{2}\right) + \phi\left(-\frac{\xi^2 + (\xi^*)^2 - \xi^*\xi - \xi\xi^*}{2}\right) \\ &= \left|\left|\frac{\xi+\xi^*}{\sqrt{2}}\right|\right|_\phi^2 + \left|\left|-\frac{\xi-\xi^*}{\mathrm{i}\sqrt{2}}\right|\right|_\phi^2 = \Phi^*\left(\frac{a+a^*}{\sqrt{2}}, \frac{a-a^*}{\mathrm{i}\sqrt{2}}\right).\end{aligned}$$

Combining with (2.4) of Remark 2.4 we have

$$2\Phi^*(a,a^*) = \Phi^*(\Re a, \Im a).$$

3. Non-microstates free entropy dimension

The non-microstates free entropy for several self-adjoint random variables was originally defined by Voiculescu.

Definition 3.1 (Voiculescu [Voi5, Definition 7.1]). Let $(a_i)_{i=1}^n \subset \mathscr{A}$ be a collection of self-adjoint random variables in a W^*-probability space (A, ϕ) where ϕ is a faithful normal tracial state, and let $1 \in \mathscr{B} \subset \mathscr{A}$ be a unital W^*-sub-algebra of $\mathscr{A}$. The non-microstates free entropy of $(a_i)_{i=1}^n$ with respect to $\mathscr{B}$ is then defined as

$$\begin{aligned}&\chi^*(a_1, \dots, a_n : \mathscr{B}) \\ &\quad = \frac{1}{2}\int_0^\infty \frac{n}{1+t} - \Phi^*(a_1 + \sqrt{t}S_1, \dots a_n + \sqrt{t}S_n)\,\mathrm{d}t + \frac{n}{2}\log(2\pi\mathrm{e}),\end{aligned} \tag{3.1}$$

where $S_1, \dots, S_n$ are standard semicircular elements such that $\{S_1\}, \dots, \{S_n\}$ and $\{\mathrm{alg}((a_i)_{i=1}^n \cup \mathscr{B})\}$ are free sets.

The following property of χ^* is shown by Voiculescu [Voi5].

Proposition 3.2 (Voiculescu [Voi5, Proposition 7.2]). *Let* $(a_i)_{i=1}^n, \mathscr{B}, \mathscr{A}, (S_i)_{i=1}^n$ *and* ϕ *be as in* Definition 3.1. *Let* $C^2 = \phi(a_1^2 + \cdots + a_n^2)$. *Then*

$$\chi^*(a_1, \dots, a_n : \mathscr{B}) \leqslant \frac{n}{2} \log(2\pi e n^{-1} C^2). \tag{3.2}$$

Voiculescu [Voi2] defined the microstates free entropy dimension. We consider the non-microstates free analog.

Definition 3.3. Let $(a_i)_{i=1}^n \subset \mathscr{A}$ be a collection of self-adjoint random variables in a W^*-probability space (A, ϕ) where ϕ is a faithful normal tracial state, and let $1 \in \mathscr{B} \subset \mathscr{A}$ be a unital W^*-sub-algebra of $\mathscr{A}$. The non-microstates free entropy dimension of $(a_i)_{i=1}^n$ with respect to $\mathscr{B}$ is defined by

$$\delta^*(a_1, \dots, a_n : \mathscr{B}) = n + \limsup_{\varepsilon \to 0} \frac{\chi^*(a_1 + \varepsilon S_1, \dots, a_n + \varepsilon S_n : \mathscr{B})}{|\log \varepsilon|}, \tag{3.3}$$

where $S_1, \dots, S_n$ are standard semicircular variables such that $\{S_1\}, \dots, \{S_n\}$ and $\{(a_i)_{i=1}^n \cup \mathscr{B}\}$ are ϕ-free.

An easy upper bound of the non-microstates free entropy dimension follows from Proposition 3.2.

Proposition 3.4. *Let* $(a_i)_{i=1}^n \subset \mathscr{A}$ *be a collection of not all zero self-adjoint random variables in a* W^**-probability space* (A, ϕ) *where* ϕ *is a faithful normal tracial state, and let* $1 \in \mathscr{B} \subset \mathscr{A}$ *be a unital* W^**-sub-algebra of* $\mathscr{A}$. *Let* $(S_i)_{i=1}^n$ *be a standard semicircular family* ϕ*-free from* $(a_i)_{i=1}^n$. *Then*

$$\delta^*(a_1, \dots, a_n : \mathscr{B}) \leqslant n. \tag{3.4}$$

Proof. Let $C^2 = \phi(a_1^2 + \cdots + a_n^2) > 0$. Then

$$\phi((a_1 + \varepsilon S_1)^2 + \cdots + (a_n + \varepsilon S_n)^2) = C^2 + n\varepsilon^2$$

so we infer from (3.2) of Proposition 3.2 that

$$\begin{aligned} \delta^*(a_1, \dots, a_n : \mathscr{B}) &= n + \limsup_{\varepsilon \to 0} \frac{\chi^*(a_1 + \varepsilon S_1, \dots, a_n + \varepsilon S_n : \mathscr{B})}{|\log \varepsilon|} \\ &\leqslant n + \limsup_{\varepsilon \to 0^+} \frac{\frac{n}{2} \log(2\pi e n^{-1}(C^2 + \varepsilon^2 n))}{|\log \varepsilon|} \\ &= n \end{aligned}$$

since C^2 is strictly positive. □

A lower bound of the non-microstates free entropy dimension can be estimated from the non-microstates free Fischer information in the following sense.

Proposition 3.5. *Let $(a_i)_{i=1}^n \subset \mathscr{A}$ be a collection of self-adjoint random variables in a W^*-probability space (A, ϕ) where ϕ is a faithful normal tracial state, and let $1 \in \mathscr{B} \subset \mathscr{A}$ be a unital W^*-sub-algebra of $\mathscr{A}$. The non-microstates free entropy dimension of $(a_i)_{i=1}^n$ with respect to $\mathscr{B}$ is bounded from below by*

$$\delta^*(a_1, \dots, a_n : \mathscr{B}) \geqslant n - \limsup_{t \to 0^+} \left(t\Phi^*(a_1 + \sqrt{t}S_1, \dots, a_n + \sqrt{t}S_n : \mathscr{B})\right), \tag{3.5}$$

where $S_1, \dots, S_n$ is a family of standard semicircular random variables such that $\{S_1\}, \dots, \{S_n\}$ and $\{\mathscr{B} \cup (a_i)_{i=1}^n\}$ are ϕ-free. If the lim sup *on the right-hand side of* (3.5) *is convergent then inequality in* (3.5) *becomes equality.*

Proof. We only show the inequality of (3.5). The last statement about equality follows by trivial modifications of the argument. Let $\mathscr{A}, \mathscr{B}, \phi, (a_i)_{i=1}^n$ and $(S_i)_{i=1}^n$ be as in the proposition. Define

$$a := \limsup_{t \to 0^+} \left(t\Phi^*(a_1 + \sqrt{t}S_1, \dots, a_n + \sqrt{t}S_n : \mathscr{B})\right)$$

and let $\varepsilon > 0$. There exists an interval $\delta > 0$ such that

$$\Phi^*(a_1 + \sqrt{t}S_1, \dots, a_n + \sqrt{t}S_n : \mathscr{B}) \leqslant \frac{a + \varepsilon}{t}$$

for all $0 < t < \delta$.

Using Definition 3.1 carefully implies

$$\begin{aligned}
&\chi^*(a_1 + \sqrt{\delta}S_1, \dots, a_n + \sqrt{\delta}S_n : \mathscr{B}) - \chi^*(a_1 + \sqrt{t}S_1, \dots, a_n + \sqrt{t}S_n : \mathscr{B}) \\
&\quad = \frac{1}{2}\int_t^\delta \frac{n}{1 + s - t} - \Phi^*(a_1 + \sqrt{s}S_1, \dots, a_n + \sqrt{s}S_n : \mathscr{B})\, \mathrm{d}s \\
&\qquad + \frac{1}{2}\int_\delta^\infty \frac{n}{1 + s - \delta} - \frac{n}{1 + s - t}\, \mathrm{d}s \\
&\quad = \frac{1}{2}\int_t^\delta \Phi^*(a_1 + \sqrt{s}S_1, \dots, a_n + \sqrt{s}S_n : \mathscr{B})\, \mathrm{d}s \\
&\quad \leqslant \frac{1}{2}\int_t^\delta \frac{a + \varepsilon}{s}\, \mathrm{d}s = \frac{a + \varepsilon}{2}\log\left(\frac{\delta}{t}\right)
\end{aligned} \tag{3.6}$$

for all $0 < t < \min\{\delta, 1\}$, so

$$\begin{aligned}
&\chi^*(a_1 + \sqrt{t}S_1, \dots, a_n + \sqrt{t}S_n : \mathscr{B}) \\
&\quad \geqslant \chi^*(a_1 + \sqrt{\delta}S_1, \dots, a_n + \sqrt{\delta}S_n : \mathscr{B}) - \frac{a + \varepsilon}{2}\log\delta - \frac{a + \varepsilon}{2}|\log t|.
\end{aligned}$$

From this we deduce that

$$\liminf_{t\to 0^+} \frac{\chi^*(a_1+\sqrt{t}S_1,\dots,a_n+\sqrt{t}S_n:\mathscr{B})}{|\log t|} \geqslant -\frac{a+\varepsilon}{2}$$

and since ε was chosen arbitrarily comparing to (3.3) implies that

$$\begin{aligned}\delta^*(a_1,\dots,a_n:\mathscr{B}) &= n + 2\limsup_{t\to 0^+} \frac{\chi^*(a_1+\sqrt{t}S_1,\dots,a_n+\sqrt{t}S_n:\mathscr{B})}{|\log t|} \\ &\geqslant n + 2\liminf_{t\to 0^+} \frac{\chi^*(a_1+\sqrt{t}S_1,\dots,a_n+\sqrt{t}S_n:\mathscr{B})}{|\log t|} \\ &\geqslant n - a \\ &= n - \limsup_{t\to 0^+} (t\Phi^*(a_1+\sqrt{t}S_1,\dots,a_n+\sqrt{t}S_n:\mathscr{B})). \qquad \square\end{aligned}$$

We remark that it might be tempting to adopt the right-hand side of (3.5) as the definition of the non-microstates free entropy dimension with respect to $\mathscr{B}$. Voiculescu has shown the following Free Stam inequality.

Proposition 3.6 (Voiculescu [Voi5, Proposition 6.5]). *Let $(a_i^1)_{i=1}^n, (a_i^2)_{i=1}^n \subset \mathscr{A}$ be a collection of self-adjoint random variables in a W^*-probability space (A,ϕ) where ϕ is a faithful normal tracial state, and let $1 \in \mathscr{B}_1, \mathscr{B}_2 \subset \mathscr{A}$ be a unital W^*-sub-algebras of $\mathscr{A}$. If $\mathrm{alg}((a_i^1)_{i=1}^n, \mathscr{B}_1)$ and $\mathrm{alg}((a_i^2)_{i=1}^n, \mathscr{B}_2)$ are ϕ-free then*

$$\begin{aligned}&\Phi^*(a_1^1+a_1^2,\dots,a_n^1+a_n^2 : W^*(\mathscr{B}_1,\mathscr{B}_2))^{-1} \\ &\quad \geqslant \Phi^*(a_1^1,\dots,a_n^1:\mathscr{B}_1)^{-1} + \Phi^*(a_1^2,\dots,a_n^2:\mathscr{B}_2)^{-1}. \qquad (3.7)\end{aligned}$$

Combining the lower bound of δ^* from Propositions 3.4–3.6 we have the following result which limits the free entropy dimension problem to the case of infinite free Fischer information.

Corollary 3.7. *Let $(a_i)_{i=1}^n \subset \mathscr{A}$ be a collection of self-adjoint random variables in a W^*-probability space (A,ϕ) where ϕ is a faithful normal tracial state, and let $1 \in \mathscr{B} \subset \mathscr{A}$ be a unital W^*-sub-algebra of $\mathscr{A}$. Then*

$$\Phi^*(a_1,\dots,a_n:\mathscr{B}) < \infty \Rightarrow \delta^*(a_1,\dots,a_n:\mathscr{B}) = n. \qquad (3.8)$$

Proof. Apply the Free Stam inequality (3.7) on the algebras $\mathrm{alg}((a_i)_{i=1}^n, \mathscr{B})$ and $\mathrm{alg}((S_i)_{i=1}^n, \mathbb{C})$ and observe that

$$\limsup_{t\to 0^+} t\Phi^*(a_1+\sqrt{t}S_1,\dots,a_n+\sqrt{t}S_n:\mathscr{B}) \leqslant \limsup_{t\to 0^+} \frac{nt}{\frac{n}{\alpha}+t} = 0,$$

when $\alpha = \Phi^*(a_1,\dots,a_n:\mathscr{B}) < \infty$. $\square$

4. χ^* and δ^* for a non-self-adjoint random variable

We translate the results from Section 3 into the context of one non-self-adjoint variable by considering real and imaginary parts. Let $(\mathscr{A}, \phi)$ be W^*-probability space with a normal faithful tracial state, and let $a \in \mathscr{A}$ be non-self-adjoint random variable. For the microstates free entropy we have $\chi(a) = \chi(\Re a, \Im a)$ [HP, Proposition 6.5.5] so it seems natural to define

$$\chi^*(a : \mathscr{B}) := \chi^*(\Re a, \Im a : \mathscr{B}) \tag{4.1}$$

and thus also

$$\delta^*(a : \mathscr{B}) := \delta^*(\Re a, \Im a : \mathscr{B}). \tag{4.2}$$

Using Proposition 3.5, we have the following corollary on non-microstates free entropy dimension of a single non-self-adjoint variable.

Corollary 4.1. *Let $a \in \mathscr{A}$ be a non-self-adjoint family of random variables in a W^*-probability space (A, ϕ) where ϕ is a faithful normal tracial state, and let $1 \in \mathscr{B} \subset \mathscr{A}$ be a unital W^*-sub-algebra of $\mathscr{A}$. Then a lower bound of the non-microstates free entropy dimension of a with respect to $\mathscr{B}$ is*

$$\delta^*(a : \mathscr{B}) \geqslant 2 - \limsup_{t \to 0^+} \, (t\Phi^*(a + \sqrt{t}Y, (a + \sqrt{t}Y)^* : \mathscr{B})), \tag{4.3}$$

where Y is a standard circular random variable $*$-free from a and $\mathscr{B}$. If the lim sup *on the right-hand side of* (4.3) *is convergent then inequality in* (4.3) *becomes equality.*

Proof. Just use (3.5) of Proposition 3.5 on the self-adjoint variables $\Re a$ and $\Im a$ and substitute S_1 and S_2 with a single circular random variable Y $*$-free from a such that $\frac{Y+Y^*}{\sqrt{2}} = S_1$ and $\frac{Y-Y^*}{i\sqrt{2}} = S_2$. Then (3.5) becomes

$$\begin{aligned}\delta^*(a : \mathscr{B}) &= \delta^*(\Re a, \Im a : \mathscr{B}) \\ &\geqslant 2 - \limsup_{t \to 0^+} \, (t\Phi^*(\Re(a + \sqrt{2t}Y), \Im(a + \sqrt{2t}Y) : \mathscr{B})\end{aligned}$$

Now Remark 2.6 implies (4.3). □

5. Properties of DT-operators

In this section, we first review some basic properties of DT-operators which can be read out of Dykema and Haagerup [DH1,DH2]. We then deduce some results relating several DT-operators and standard circular operators.

From now on, unless otherwise stated, we let $\mathcal{M}$ be a von Neumann algebra equipped with a faithful, normal, tracial state, τ, such that $(\mathcal{M}, \tau)$ is a W^*-probability space. We also assume that we are given an injective, unital, normal $*$-homomorphism $\lambda : L^\infty[0,1] \to \mathcal{M}$, such that $\tau \circ \lambda(f) = \int_0^1 f(t)\,\mathrm{d}t$ for $f \in L^\infty[0,1]$. We let $\mathcal{D}$ be the picture of $L^\infty[0,1]$ in $\mathcal{M}$ and let $E_{\mathcal{D}} : \mathcal{M} \to \mathcal{D}$ be the trace-preserving conditional expectation onto $\mathcal{D}$. We will identify $\mathcal{D}$ and $L^\infty[0,1]$ and thus consider elements of $\mathcal{D}$ as L^∞-functions. Assume also that $(X_i)_{i=1}^\infty \subset \mathcal{M}$ is a standard semicircular family of random variables τ-free from $\mathcal{D}$. Then $(W^*(\mathcal{D} \cup \{X_i\}))_{i=1}^\infty$ are $\mathcal{D}$-free with respect to $E_{\mathcal{D}}$. As in [DH1] (we also adopt the notation of Dykema and Haagerup [DH1]) we can now construct upper triangular operators, $T_i = \mathcal{UT}(X_i, \lambda)$ for $i \in \mathbb{N}$. If $D_0 : x \mapsto x \in \mathcal{D}$ then $T_i = \mathcal{UT}(X_i, \lambda)$ is the norm limit of

$$T_n^{(i)} = \sum_{j=1}^{n} 1_{\left[\frac{j-1}{n}, \frac{j}{n}\right]}(D_0) X_i 1_{\left[\frac{j}{n}, 1\right]}(D_0) \tag{5.1}$$

as $n \to \infty$ [DH1, Lemma 4.1]. It follows that $T_i \in W^*(\mathcal{D} \cup \{X_i\})$ for all $i \in \mathbb{N}$ so $(T_i)_{i=1}^\infty$ is a $\mathcal{D}$-free family. (T_i, T_i^*) is a centered $\mathcal{D}$-Gaussian pair for all $i \in \mathbb{N}$ [DH2, Appendix A] and it follows that $(T_i, T_i^*)_{i=1}^\infty$ is a centered $\mathcal{D}$-Gaussian set in the sense of Speicher [Sp1, Definition 4.2.3].

Define for $f \in \mathcal{D}$

$$L^*(f) : x \mapsto \int_0^x f(t)\,\mathrm{d}t \quad \text{and} \quad L(f) : x \mapsto \int_x^1 f(t)\,\mathrm{d}t. \tag{5.2}$$

From the appendix of [DH2] it follows that the covariances of T are given by

Lemma 5.1 (Dykema and Haagerup [DH2, Appendix]). *Let $f \in \mathcal{D}$. Then*

$$E_{\mathcal{D}}(T_i f T_i^*) = L(f) \quad \textit{and} \quad E_{\mathcal{D}}(T_i^* f T_i) = L^*(f), \tag{5.3}$$

$$E_{\mathcal{D}}(T_i f T_i) = 0 \quad \textit{and} \quad E_{\mathcal{D}}(T_i^* f T_i^*) = 0 \tag{5.4}$$

and $(T_i)_{i=1}^\infty$ is a $\mathcal{D}$-$*$-free family with respect to $E_{\mathcal{D}}$.

A $\mathrm{DT}(\mu, c)$-operator, as defined in the introduction, where μ is a compactly supported Borel measure on $\mathbb{C}$ and $c > 0$, can be realized in the W^*-probability space $(\mathcal{M}, \tau)$.

Theorem 5.2 (Dykema and Haagerup [DH1, Theorem 4.4]). *Let $T_1 = \mathcal{UT}(X_1, \lambda)$ where X_1 and λ is as above, and let $c > 0$. Let $f \in L^\infty[0,1]$ and define $D = \lambda(f)$. Then $D + cT_1$ is a $\mathrm{DT}(\mu, c)$-element, where μ is the push-forward measure of the Lebesque-measure by f.*

There is a certain freedom in choosing $D \in \mathcal{D}$.

Lemma 5.3 (Dykema and Haagerup [DH1, Lemma 6.2]). *Let μ be a compactly supported Borel measure on $\mathbb{C}$. Then there is $f \in L^\infty[0,1]$ whose distribution is μ and such that if $D = \lambda(f)$ and if $T_1 = \mathcal{UT}(X_1, \lambda)$ is as above, then for any $c > 0$ D itself lies in the W^*-algebra generated by $D + cT_1$.*

Dykema and Haagerup also showed that surprisingly we have.

Theorem 5.4 (Dykema and Haagerup [DH2, Theorem 2.2]). *Let $T_1 = \mathcal{UT}(X_1, \lambda)$ be as above. If $D_0 : x \mapsto x \in \mathcal{D}$ is the identity then $D_0 \in W^*(T_1)$.*

Summarizing on Theorem 5.2, Lemma 5.3 and Theorem 5.4 then a DT(μ, c)-element can be constructed as $Z := D + cT_1$ where the distribution of $D \in \mathcal{D}$ is μ, $c > 0$ and $T_1 = \mathcal{UT}(X_1, \lambda)$. D can furthermore be chosen in such a way that $D \in W^*(Z) = W^*(T_1) \cong L(\mathbf{F}_2)$.

From [DH1] we know that $T_i + T_i^* = X_i$ for all $i \in \mathbb{N}$. We will need a relation between upper triangular operators and standard circular variables, and for this we will use the following theorem which is a special case of a theorem of Nica et al. [NSS3]. The theorem as it is stated here is from [ŚnSp, Theorem 6].

Theorem 5.5 (Nica et al. [NSS3]). *Let $(\mathcal{M}, \tau)$ be a probability space and $(\mathcal{D} \subset \mathcal{M}, E_{\mathcal{D}})$ a $\mathcal{D}$-probability space compatible to $(\mathcal{M}, \tau)$ and let $\mathcal{X} \subset \mathcal{M}$. Then $\mathcal{X}$ and $\mathcal{D}$ are free in $(\mathcal{M}, \tau)$ iff for every $n \geq 1$ and $x_1, \ldots, x_n \in \mathcal{X}$ there exists $c_n(x_1, \ldots, x_n) \in \mathbb{C}$ such that for every $d_1, \ldots d_{n-1} \in \mathcal{D}$ we have*

$$\kappa_n^{\mathcal{D}}\left(x_1 d_1 \underset{\mathcal{D}}{\bigotimes} \cdots \underset{\mathcal{D}}{\bigotimes} x_{n-1} d_{n-1} \underset{\mathcal{D}}{\bigotimes} x_n\right) = c_n(x_1, \ldots, x_n)\tau(d_1)\cdots\tau(d_{n-1})1.$$

If the above holds, we have

$$c_n(x_1, \ldots, x_n) = \kappa_n^{\mathbb{C}}(x_1, \ldots, x_n).$$

Lemma 5.6. *Let $T_1 = \mathcal{UT}(X_1, \lambda)$ and $T_2 = \mathcal{UT}(X_2, \lambda)$ be upper triangular operators, and let $Y \in \mathcal{M}$ be a standard circular random variable $*$-free from T_1 with respect to τ. Then for $a, b \geq 0$*

$$\sqrt{a}T_1 + \sqrt{b}Y \overset{*-\text{dist.}}{\sim} \sqrt{a+b}T_1 + \sqrt{b}T_2^*. \tag{5.5}$$

In particular, $T_1 + T_2^*$ is a circular element..

Proof. First, we prove that if $T_3 = \mathcal{UT}(X_3, \lambda)$ is upper triangular then $T_3 + T_2^*$ is a standard circular variable with respect to τ. We know that $T_3 + T_2^*$ is $\mathcal{D}$-Gaussian since T_2 and T_3 are $\mathscr{D}$-free and the $\kappa_2^{\mathscr{D}}$-cumulants are given by

$$\kappa_2^{\mathscr{D}}\left((T_3+T_2^*)\bigotimes_{\mathscr{D}} d(T_3+T_2^*)^*\right) = L(d)+L^*(d) = \tau(d)1,$$

$$\kappa_2^{\mathscr{D}}\left((T_3+T_2^*)^*\bigotimes_{\mathscr{D}} d(T_3+T_2^*)\right) = L^*(d)+L(d) = \tau(d)1,$$

$$\kappa_2^{\mathscr{D}}\left((T_3+T_2^*)\bigotimes_{\mathscr{D}} d(T_3+T_2^*)\right) = 0,$$

$$\kappa_2^{\mathscr{D}}\left((T_3+T_2^*)^*\bigotimes_{\mathscr{D}} d(T_3+T_2^*)^*\right) = 0$$

for all $d\in\mathscr{D}$. We thus see from Theorem 5.5 that actually $T_3+T_2^*$ is $\mathbb{C}$-Gaussian with $\kappa_2^{\mathbb{C}}$-cumulants given by

$$\kappa_2^{\mathbb{C}}(T_3+T_2^*,(T_3+T_2^*)^*) = \kappa_2^{\mathbb{C}}((T_2+T_3^*)^*,T_2+T_3^*) = 1,$$

$$\kappa_2^{\mathbb{C}}(T_3+T_2^*,T_3+T_2^*) = \kappa_2^{\mathbb{C}}((T_3+T_2^*)^*,(T_3+T_2^*)^*) = 0.$$

Hence $T_3+T_2^*$ is a standard circular random variable. Furthermore, Theorem 5.5 implies that $T_3+T_2^*$ is τ-free from $\mathscr{D}$. Since also $T_3+T_2^*$ is τ-free from X_1 we conclude that $T_3+T_2^*$ is τ-free from $W^*(X_1,\mathscr{D})$ and, in particular, $T_3+T_2^*$ is τ-free from T_1. Hence

$$\sqrt{a}T_1+\sqrt{b}(T_3+T_2^*) \overset{*\text{-distr}}{\sim} \sqrt{a}T_1+\sqrt{b}Y,$$

where Y is a standard circular random variable τ-free from T_1.

Now, we just have to observe that $\sqrt{a}T_1+\sqrt{b}T_3$ is also $\mathscr{D}$-Gaussian because of $\mathscr{D}$-freeness of T_1 and T_3. Since

$$\kappa_2^{\mathscr{D}}\left((\sqrt{a}T_1+\sqrt{b}T_3)\bigotimes_{\mathscr{D}} d(\sqrt{a}T_1+\sqrt{b}T_3)^*\right) = (a+b)L(d),$$

$$\kappa_2^{\mathscr{D}}\left((\sqrt{a}T_1+\sqrt{b}T_3)^*\bigotimes_{\mathscr{D}} d(\sqrt{a}T_1+\sqrt{b}T_3)\right) = (a+b)L^*(d),$$

$$\kappa_2^{\mathscr{D}}\left((\sqrt{a}T_1+\sqrt{b}T_3)\bigotimes_{\mathscr{D}} d(\sqrt{a}T_1+\sqrt{b}T_3)\right) = 0,$$

$$\kappa_2^{\mathscr{D}}\left((\sqrt{a}T_1+\sqrt{b}T_3)^*\bigotimes_{\mathscr{D}} d(\sqrt{a}T_1+\sqrt{b}T_3)^*\right) = 0,$$

we conclude that $\sqrt{a}T_1+\sqrt{b}T_3 \overset{*\text{-dist.}}{\sim} \sqrt{a+b}T_1$. Summarizing, we have showed that

$$\sqrt{a}T_1+\sqrt{b}Y \overset{*\text{-dist.}}{\sim} \sqrt{a+b}T_1+\sqrt{b}T_2^*$$

but this is exactly what the lemma states. $\square$

The following two lemmas tells us that in the presence of the algebra $\mathscr{D}$ we can "cut out" the lower (and upper) triangular part of certain products and linear combinations of DT-operators.

Lemma 5.7. *Let $D\in\mathscr{D}$, $c>0$. Let $T_1=\mathscr{UT}(X_1,\lambda)$ and $T_2=\mathscr{UT}(X_2,\lambda)$ be upper triangular operators. Then*

$$[D+cT_1,T_2]\in\overline{W^*(T_1)([D+cT_1,T_2]+[(D+cT_1)^*,T_2^*])W^*(T_1)}^{\|\cdot\|_\tau}.$$

Proof. Let $D_0: x\to x$ be the identity-function in $\mathscr{D}$. By Theorem 5.4, we know that $D_0\in W^*(T_1)$. Define the projections $p_i^{(n)}:=1_{\left[\frac{i-1}{n},\frac{i}{n}\right]}(D_0)$ for $i=1,\dots,n$. If we can show that

$$\lim_{n\to\infty}\left\|\sum_{\substack{i,j=1\\ i<j}}^{n}p_i^{(n)}([D+cT_1,T_2]+[(D+cT_1)^*,T_2^*])p_j^{(n)}-[D+cT_1,T_2]\right\|_\tau=0 \tag{5.6}$$

then the lemma follows immediately from the fact that $D_0\in W^*(T_1)$. From the construction of T_1 in (5.1) we know that

$$T_i=\sum_{1\leqslant i\leqslant j\leqslant n}p_i^{(n)}T_ip_j^{(n)} \tag{5.7}$$

for $i=1,2$, since we can just replace T_i in (5.7) by $T_n^{(i)}$ from (5.1) and take the norm limit. Thus also

$$T_1T_2=\sum_{1\leqslant i\leqslant k\leqslant j\leqslant n}p_i^{(n)}T_1p_k^{(n)}T_2p_j^{(n)}$$

and since $p_i^{(n)}T_1p_k^{(n)}=0$ for $i>k$ and $p_k^{(n)}T_2p_j^{(n)}=0$ for $k>j$ we have

$$\begin{aligned}T_1T_2&=\sum_{\substack{1\leqslant i\leqslant j\leqslant n\\ 1\leqslant k\leqslant n}}p_i^{(n)}T_1p_k^{(n)}T_2p_j^{(n)}.\\ &=\sum_{1\leqslant i\leqslant j\leqslant n}p_i^{(n)}T_1T_2p_j^{(n)}.\end{aligned}$$

For $i\geqslant j$

$$p_i^{(n)}T_1T_2p_j^{(n)}=\sum_{k=1}^{n}p_i^{(n)}T_1p_k^{(n)}T_2p_j^{(n)}=\begin{cases}0 & i>j,\\ p_i^{(n)}T_1p_i^{(n)}T_2p_i^{(n)} & i=j\end{cases}$$

because $p_i^{(n)} T_1 p_k^{(n)} = 0$ for $i > k$ and $p_k^{(n)} T_2 p_j^{(n)} = 0$ for $j < k$. For $i \neq j$ we have $p_i^{(n)} D p_j^{(n)} = 0$ since D commutes with the orthogonal projections $p_i^{(n)}$ and $p_j^{(n)}$, so

$$\begin{aligned}(D + cT_1)T_2 &- \sum_{1 \leqslant i < j \leqslant n} p_i^{(n)}(D + cT_1)T_2 p_j^{(n)} \\ &= \sum_{i=1}^{n} p_i^{(n)}(D + cT_1)p_i^{(n)} T_2 p_i^{(n)} = \sum_{i,j=1}^{n} p_i^{(n)}(D + cT_1)p_i^{(n)} p_j^{(n)} T_2 p_j^{(n)} \\ &= \left(\sum_{i=1}^{n} p_i^{(n)}(D + cT_1)p_i^{(n)}\right)\left(\sum_{j=1}^{n} p_j^{(n)} T_2 p_j^{(n)}\right).\end{aligned}$$

Reversing the role of T_1 and T_2 and subtracting we conclude that

$$[D + cT_1, T_2] - \sum_{1 \leqslant i < j \leqslant n} p_i^{(n)}[D + cT_1, T_2]p_j^{(n)} = [D + cU^{(n)}, V^{(n)}], \tag{5.8}$$

where $U^{(n)} = \sum_{i=1}^{n} p_i^{(n)} T_1 p_i^{(n)}$ and $V^{(n)} = \sum_{j=1}^{n} p_j^{(n)} T_2 p_j^{(n)}$. But $||U^{(n)}|| = ||V^{(n)}|| = \sqrt{\frac{e}{n}}$ because $||T_1|| = ||T_2|| = \sqrt{e}$ [DH1, Corollary 8.11], so

$$|||[U^{(n)}, V^{(n)}]|||_\tau \leqslant |||[U^{(n)}, V^{(n)}]||| \leqslant 2\left(||D|| + \frac{\sqrt{e}}{\sqrt{n}}\right)\frac{\sqrt{e}}{\sqrt{n}}. \tag{5.9}$$

For $i < j$, we have

$$p_i^{(n)}(D + cT_1)^* T_2^* p_j^{(n)} = \sum_{k=1}^{n} p_i^{(n)}(D + cT_1)^* p_k^{(n)} T_2^* p_j^{(n)} = 0$$

because D commute with the involved projections and since $p_i^{(n)} T_1^* p_k^{(n)} = 0$ for $i < k$ and $p_k^{(n)} T_1^* p_j^{(n)} = 0$ for $k < j$. Reversing the role of $(D + cT_1)^*$ and T_2^*, summing and subtracting we have

$$\sum_{1 \leqslant i < j \leqslant n} p_i^{(n)}[(D + cT_1)^*, T_2^*]p_j^{(n)} = 0. \tag{5.10}$$

Hence (5.6) follows from combining (5.8)–(5.10). □

Lemma 5.8. *Let $T_1 = \mathscr{UT}(X_1, \lambda)$ and $T_2 = \mathscr{UT}(X_2, \lambda)$ be upper triangular operators as in the beginning of the section. Let $D \in \mathscr{D}$ and let $a, b \in \mathbb{R}_+$ be non-zero positive scalars. Then*

$$T_1, T_2 \in W^*(D + \sqrt{a}T_1 + \sqrt{b}T_2^*, \mathscr{D}).$$

Proof. We proceed as in the proof of Lemma 5.7, and let $D_0 : x \to x$ be the identity-function in $\mathscr{D}$. We only show that $T_1 \in W^*(D + \sqrt{a}T_1 + \sqrt{b}T_2^*, \mathscr{D})$, since the result for T_2 is obtained in a similar way. Define the projections $p_i^{(n)} := 1_{\left[\frac{i-1}{n}, \frac{i}{n}\right]}(D_0)$ for $i = 1, \dots, n$. Remember from the proof of Lemma 5.7 that

$$T_1 = \sum_{1 \leqslant i \leqslant j \leqslant n} p_i^{(n)} T_1 p_j^{(n)}$$

and that

$$U^{(n)} := T_1 - \sum_{1 \leqslant i < j \leqslant n} p_i^{(n)} T_1 p_j^{(n)} = \sum_{i=1}^{n} p_i^{(n)} T_1 p_i^{(n)}.$$

We have $||U^{(n)}|| \leqslant \sqrt{\frac{e}{n}}$ and also

$$\sum_{1 \leqslant i < j \leqslant n} p_i^{(n)} T_2^* p_j^{(n)} = 0. \tag{5.11}$$

We conclude that

$$\lim_{n \to \infty} \left\| \sqrt{a}T_1 - \sum_{\substack{i,j=1 \\ i<j}}^{n} p_i^{(n)} (D + \sqrt{a}T_1 + \sqrt{b}T_2^*) p_j^{(n)} \right\|$$
$$\leqslant \lim_{n \to \infty} \sqrt{a} ||U^{(n)}|| \leqslant \lim_{n \to \infty} \sqrt{\frac{ae}{n}} = 0. \qquad \square$$

6. δ^* for a DT-operator with respect to $\mathscr{D}$

Let $T_1 = \mathscr{UT}(X_1, \lambda)$, $T_2 = \mathscr{UT}(X_2, \lambda)$ be upper triangular operators as in the beginning of Section 5. Define $Z = D + cT_1 \in \mathscr{M}$ where $c > 0$ and $D \in \mathscr{D}$. By Theorem 5.2, Z is a DT(μ, c)-operator where μ is the distribution of D. Let $Y \in \mathscr{M}$ be a standard circular operator $*$-free from D and T_1 and thus also $*$-free from Z. By Corollary 4.1, we are to compute

$$\delta^*(Z : \mathscr{D}) = 2 - \lim_{t \to 0^+} (t\Phi^*(D + cT_1 + \sqrt{t}Y, (D + cT_1 + \sqrt{t}Y)^* : \mathscr{D})), \tag{6.1}$$

where Y is a standard circular element $*$-free from T_1, if the limit exists. By Lemma 5.6, we know that

$$T_1 + \sqrt{t}Y \overset{*\text{-dist.}}{\sim} \sqrt{1+t}T_1 + \sqrt{t}T_2^*$$

so using (2.4) of Remark 2.4 (6.1) becomes

$$\begin{aligned}
&\delta^*(Z:\mathscr{D})\\
&\quad = 2-\lim_{t\to 0^+}(t\Phi^*(D+\sqrt{c^2+t}T_1+\sqrt{t}T_2^*,(D+\sqrt{c^2+t}T_1+\sqrt{t}T_2^*)^*:\mathscr{D}))\\
&\quad = 2-\lim_{t\to 0^+}\Phi^*\left(\frac{1}{\sqrt{t}}D+\sqrt{\frac{c^2+t}{t}}T_1+T_2^*,\left(\frac{1}{\sqrt{t}}D+\sqrt{\frac{c^2+t}{t}}T_1+T_2^*\right)^*:\mathscr{D}\right)
\end{aligned}$$

if the limit on the right-hand side exists. Define

$$(S_t,S_t^*)=\left(\frac{1}{\sqrt{t}}D+\sqrt{\frac{c^2+t}{t}}T_1+T_2^*,\left(\frac{1}{\sqrt{t}}D+\sqrt{\frac{c^2+t}{t}}T_1+T_2^*\right)^*\right).$$

Then it easy to see that

$$(\xi_t,\xi_t^*)=\left(\sqrt{\frac{t}{c^2+t}}T_1^*+T_2,\sqrt{\frac{t}{c^2+t}}T_1+T_2^*\right)$$

satisfies the conjugate relations for (S_t,S_t^*) with respect to $\mathscr{D}$ because $(\xi_t,\xi_t^*,S_t,S_t^*)$ is a (non-centered) $\mathscr{D}$-Gaussian system. For example

$$\begin{aligned}
\kappa_2^{\mathscr{D}}\left(\xi_t\bigotimes_{\mathscr{D}} dS_t\right)&=E_{\mathscr{D}}\left(\left(\sqrt{\frac{t}{c^2+t}}T_1^*+T_2\right)d\left(\sqrt{\frac{c^2+t}{t}}T_1+T_2^*\right)\right)\\
&=L^*(d)+L(d)=\left(\int_0^1 d(y)\,\mathrm{d}y\right)1=\tau(d)1
\end{aligned}$$

for $d\in\mathscr{D}$ because T_1 and T_2 are $\mathscr{D}$-free, and $D\in\mathscr{D}$ is $E_{\mathscr{D}}$-free from everything [Sp1, Lemma 3.2.4]. The other $\kappa_2^{\mathscr{D}}$-identities are checked similarly. Furthermore $\xi_t,\xi_t^*\in L^2(\{S_t,S_t^*\}\cup\mathscr{D},\tau)$ because Lemma 5.8 says that actually $T_1,T_2\in W^*(S_t,\mathscr{D})$.

We thus conclude that

$$\begin{aligned}
\Phi^*(S_t,S_t^*:\mathscr{D})&=||\xi_t||_\tau^2+||\xi_t^*||_\tau^2=2||\xi_t||_\tau^2\\
&=2\frac{t}{c^2+t}\tau(T_1^*T_1)+2\tau(T_2T_2^*)\\
&\quad+2\sqrt{\frac{t}{c^2+t}}(\tau(E_{\mathscr{D}}(T_1^*T_2^*)+E_{\mathscr{D}}(T_2T_1)))\\
&=\frac{t}{c^2+t}+1
\end{aligned}$$

since $E_{\mathscr{D}}$-freeness of T_1 and T_2 implies that

$$E_{\mathscr{D}}(T_1^* T_2^*) = E_{\mathscr{D}}(T_2 T_1) = 0.$$

We have now shown that (ξ_t, ξ_t^*) is a conjugate system for (S_t, S_t^*), so by (6.1)

$$\delta^*(Z : \mathscr{D}) = 2 - \lim_{t\to 0^+}\left(\frac{t}{c^2+t} + 1\right) = 1. \qquad \square$$

7. δ^* for a DT-operator with respect to $\mathbb{C}$

Let $T_1 = \mathscr{U}\mathscr{T}(X_1, \lambda) \in \mathscr{M}$ and $T_2 = \mathscr{U}\mathscr{T}(X_2, \lambda) \in \mathscr{M}$ be two $E_{\mathscr{D}}$-free upper triangular operators and let $D \in \mathscr{D}$ be chosen such that $D \in W^*(D + cT_1)$. As in the previous section we define $S_t = \frac{1}{\sqrt{t}} D + \sqrt{\frac{c^2+t}{t}}\, T_1 + T_2^*$, and $\xi_t = \sqrt{\frac{t}{c^2+t}}\, T_1^* + T_2$. $Z = D + cT_1$ is a DT-operator and by Lemma 5.3 every DT-operator can be realized in this way.

The goal is to compute $\delta^*(Z) \coloneqq \delta^*(Z : \mathbb{C})$. Inspecting the proof of $\delta^*(Z : \mathscr{D}) = 1$ from Section 6, we observe that a crucial point in showing that (ξ_t, ξ_t^*) is a conjugate system for (S_t, S_t^*) is the use of the algebra $\mathscr{D}$ to "cut out" the upper and lower triangular parts of S_t and S_t^*. Since $\mathscr{D}$ cannot be contained in $L^2(\{S_t, S_t^*\}, \tau)$ [Voi6, Proposition p. 123] the above argument does not apply to compute $\delta^*(Z)$. By Remark 2.4 (c) it does however tell us that (ξ_t, ξ_t^*) satisfies the conjugate relations for (S_t, S_t^*) with respect to $\mathbb{C}$. Summarizing we have

$$\delta^*(Z) \geqslant 2 - \limsup_{t\to 0^+} \Phi^*(S_t, S_t^* : \mathbb{C}) \geqslant 2 - \limsup_{t\to 0^+} \Phi^*(S_t, S_t^* : \mathscr{D}) = 1. \tag{7.1}$$

We would like to compute the Hilbert space projection

$$P : L^2(\{S_t, S_t^*\} \cup \mathscr{D}, \tau) \to L^2(\{S_t, S_t^*\}, \tau).$$

This, however, seems to be very difficult. Instead our strategy will be to estimate the distance from ξ_t to a suitable subspace of the orthogonal complement of $L^2(\{S_t, S_t\}, \tau)$ in $L^2(\{S_t, S_t^*\} \cup \mathscr{D}, \tau)$ as $t \to 0^+$. To do this, we first produce an element in the orthogonal complement $(L^2(\{S_t, S_t^*\}, \tau))^\perp$.

Lemma 7.1.

$$\begin{aligned} j_t &\coloneqq [\xi_t, S_t] + [\xi_t^*, S_t^*] \\ &= \left(\sqrt{\frac{c^2+t}{t}} - \sqrt{\frac{t}{c^2+t}}\right)([T_1, T_2] + [T_1^*, T_2^*]) \\ &\quad + \left[\frac{1}{\sqrt{t}} D, \sqrt{\frac{t}{c^2+t}}\, T_1^* + T_2\right] + \left[\frac{1}{\sqrt{t}} D^*, \sqrt{\frac{t}{c^2+t}}\, T_1 + T_2^*\right] \end{aligned} \tag{7.2}$$

belongs to $(L^2(\{S_t, S_t^*\}, \tau))^\perp$ *in* $L^2(\{S_t, S_t^*\} \cup \mathscr{D}, \tau)$.

Remark 7.2. The above element, j_t, is actually just Voiculescu's liberation gradient of $(\mathrm{alg}(\{S_t, S_t^*\}), \mathscr{D})$ [Voi6, Section 5]. Voiculescu shows that the liberation gradient $j(A, B)$, of two unital algebras A and B in a W^*-probability space (M, τ) measures how τ-free A and B are, in the sense that $j(A, B) = 0$ if and only if A and B are τ-free. The liberation gradient $j(A, B)$ satisfies

$$\begin{aligned}\tau(j(A,B)a_1b_1\cdots a_nb_n) = \sum_{k=1}^{n} &(\tau(a_1b_1\dots b_{k-1}a_k)\tau(b_ka_{k+1}b_{k+1}\dots a_nb_n)\\ &- \tau(a_1b_1\dots a_{k-1}b_{k-1})\tau(a_kb_k\dots a_nb_n)) \qquad (7.3)\end{aligned}$$

for all $a_1, \dots, a_n \in A$ and $b_1 \dots, b_n \in B$.

Proof. Since (ξ_t, ξ_t^*) is a conjugate system for (S_t, S_t^*) with respect to $\mathscr{D}$ it satisfies the conjugate relations (2.1). Using (2.1) it is easy to show by direct computation that $j_t := [\xi_t, S_t] + [\xi_t^*, S_t^*]$ satisfies (7.3), that is,

$$\begin{aligned}\tau(j_tS_t^{i_1}d_1\dots S_t^{i_n}d_n) = \sum_{k=1}^{n} &(\tau(S_t^{i_1}d_1\dots d_{k-1}S_t^{i_k})\tau(d_kS_t^{i_{k+1}}d_{k+1}\dots S_t^{i_n}d_n)\\ &- \tau(S_t^{i_1}d_1\dots S_t^{i_{k-1}}d_{k-1})\tau(S_t^{i_k}d_k\dots S_t^{i_n}d_n)) \qquad (7.4)\end{aligned}$$

for all $d_1, \dots, d_n \in \mathscr{D}$ and $i_1, \dots, i_n \in \{1, *\}$. Restricting (7.4) to the case where $d_1 = \dots = d_n = 1 \in \mathscr{D}$, we infer that

$$\tau(j_tS_t^{i_1}\cdots S_t^{i_n}) = 0$$

for all $i_1, \dots, i_n \in \{1, *\}$. The inner product of j_t with monomials of the form $S_t^{i_1}\cdots S_t^{i_n}$ is thus zero, and since these monomials span a dense linear subspace of $L^2(\{S_t, S_t^*\}, \tau)$ we conclude that $j_t \perp L^2(\{S_t, S_t^*\}, \tau)$. The last equality of (7.2) is easily checked by direct computation. □

We now use $j_t \in W^*(S_t)^\perp$ from (7.2) to give a new lower bound of $\delta^*(Z)$.

Lemma 7.3.

$$\begin{aligned}&\limsup_{t\to 0^+} \Phi^*(S_t, S_t^* : \mathbb{C})\\ &\quad\leqslant 2\,\mathrm{dist}_2(T_2, W^*(T_1)([D + cT_1, T_2] + [(D + cT_1)^*, T_2^*])W^*(T_1))^2.\end{aligned}$$

Proof. Define $\tilde{S}_t = \sqrt{t}S_t = D + \sqrt{c^2 + t}T_1 + \sqrt{t}T_2^*$. We know that

$$\Phi^*(S_t, S_t^* : \mathbb{C}) = 2||E_{W^*(\tilde{S}_t)}(\xi_t)||_\tau^2. \qquad (7.5)$$

Let $A \in W^*(D + cT_1)([D + cT_1, T_2] + [(D + cT_1)^*, T_2^*])W^*(D + cT_1)$ and let $\varepsilon > 0$. Choose polynomials p and q in $D + cT_1$ and $(D + cT_1)^*$ such that

$$\begin{aligned}&||A - p(D + cT_1, (D + cT_1)^*)([D + cT_1, T_2]\\&\quad + [(D + cT_1)^*, T_2^*])q(D + cT_1, (D + cT_1)^*)||_\tau < \varepsilon.\end{aligned}$$

Define

$$\begin{aligned}B(t) &= p(\tilde{S}_t, \tilde{S}_t^*)\sqrt{t}j_t q(\tilde{S}_t, \tilde{S}_t^*)\\&= p(\tilde{S}_t, \tilde{S}_t^*)\Bigg(\left(\sqrt{c^2 + t} - \frac{t}{\sqrt{c^2 + t}}\right)([T_1, T_2] + [T_1^*, T_2^*])\\&\quad + \left[D, \sqrt{\frac{t}{c^2 + t}}T_1^* + T_2\right] + \left[D^*, \sqrt{\frac{t}{c^2 + t}}T_1 + T_2^*\right]\Bigg)q(\tilde{S}_t, \tilde{S}_t^*)\end{aligned}$$

for $t \geqslant 0$, and observe that

$$\begin{aligned}B(0) &= p(D + cT_1, (D + cT_1)^*)([D + cT_1, T_2]\\&\quad + [(D + cT_1)^*, T_2^*])q(D + cT_1, (D + cT_1)^*).\end{aligned}$$

Then $||A - B(0)||_\tau < \varepsilon$ and $||B(t) - B(0)|| \to 0$ for $t \to 0^+$. Since by Lemma 7.1 $j_t\sqrt{t} \in W^*(\tilde{S}_t^*)^\perp$ also $B(t) \in W^*(\tilde{S}_t^*)^\perp$ so

$$||E_{W^*(\tilde{S}_t^*)}(A)||_\tau \leqslant ||E_{W^*(\tilde{S}_t)}(B(t))||_\tau + ||A - B(t)||_\tau = 0 + ||A - B(t)||_\tau$$

and thus

$$\limsup_{t \to 0^+} ||E_{W^*(\tilde{S}_t)}(A)||_\tau \leqslant \limsup_{t \to 0^+} ||A - B(t)||_\tau < \varepsilon.$$

Since ε was chosen arbitrarily, we conclude that

$$\lim_{t \to 0^+} ||E_{W^*(\tilde{S}_t)}(A)||_\tau = 0 \tag{7.6}$$

for all $A \in W^*(D + cT_1)([D + cT_1, T_2] + [(D + cT_1)^*, T_2^*])W^*(D + cT_1)$. Now define

$$\alpha = \mathrm{dist}_2(T_2, W^*(T_1)([D + cT_1, T_2] + [(D + cT_1)^*, T_2^*])W^*(T_1)).$$

Since $D \in \mathscr{D}$ is chosen such that $D \in W^*(D + cT_1)$ we have $W^*(T_1) = W^*(D + cT_1)$ so from (7.6) we conclude that

$$\limsup_{t \to 0^+} ||E_{W^*(\tilde{S}_t)}(T_2)||_\tau \leqslant \alpha$$

but since $||E_{W^*(\tilde{S}_t)}||_\tau \leqslant 1$ and since $||\xi_t - T_2||_\tau \to 0$ for $t \to 0^+$ we conclude that

$$\limsup_{t\to 0^+} ||E_{W^*(\tilde{S}_t)}(\xi_t)||_\tau \leqslant \alpha. \tag{7.7}$$

Combining with (7.5) we have

$$\limsup_{t\to 0^+} \Phi^*(S_t, S_t^* : \mathbb{C}) \leqslant 2\alpha^2. \qquad \square$$

Lemma 5.7 can now be used to get rid of the $[(D + cT_1)^*, T_2^*]$-term in Lemma 7.3.

Proposition 7.4.

$$\limsup_{t\to 0^+} \Phi^*(S_t, S_t^* : \mathbb{C}) \leqslant 2\,\mathrm{dist}_2(T_2, W^*(T_1)([D + cT_1, T_2])W^*(T_1))^2. \tag{7.8}$$

Proof. It is immediate that

$$E = \overline{W^*(T_1)([D + cT_1, T_2] + [(D + cT_1)^*, T_2^*])W^*(T_1)}^{||\cdot||_\tau}$$

is invariant under multiplication from the left and right with elements from $W^*(T_1)$. By Lemma 5.7, we know that $[D + cT_1, T_2] \in E$ so

$$W^*(T_1)[D + cT_1, T_2]W^*(T_1) \subseteq E.$$

We thus have

$$\begin{aligned} \mathrm{dist}_2(T_2, W^*(T_1)[D + cT_1, T_2]W^*(T_1)) &\geqslant \mathrm{dist}_2(T_2, E) \\ &= \mathrm{dist}_2(T_2, W^*(T_1)([D + cT_1, T_2] + [(D + cT_1)^*, T_2^*])W^*(T_1)) \end{aligned}$$

so the proposition follows from Lemma 7.3. $\square$

Let $W^*(T_1)^0$ be the opposite algebra, that is, $W^*(T_1)^0$ is just $W^*(T_1)$ with multiplication reversed.

Lemma 7.5. *Let $\hat{\otimes}$ denote the von Neumann algebra tensor product, and equip $W^*(T_1)$ with the usual faithful normal tracial state, τ, and usual conditional expectation, $E_{\mathscr{D}}$, given by restriction to $W^*(T_1)$. There exists a positive functional, $\phi : W^*(T_1) \hat{\otimes} W^*(T_1)^0 \to \mathbb{C}$ such that*

(i) $0 \leqslant \phi \leqslant \tau \otimes \tau^0$,
(ii) $\tau(T_2^* a T_2 b) = \phi(a \otimes b^0)$ *for* $a, b \in W^*(T_1)$,

where $b \mapsto b^0 : W^*(T_1) \to W^*(T_1)^0$ *is the anti-multiplicative isomorphism.*

Proof. For all $a, b \in W^*(T_1)$ we have

$$\tau(T_2^* a T_2 b) = \int_0^1 \left(\int_0^x E_{\mathscr{D}}(a)(t)\,\mathrm{d}t \right) E_{\mathscr{D}}(b)(x)\,\mathrm{d}x$$
$$= \int_0^1 \int_0^1 E_{\mathscr{D}}(a)(t) E_{\mathscr{D}}(b)(x) h(t,x)\,\mathrm{d}t\,\mathrm{d}x,$$

where

$$h(t,x) = \begin{cases} 1, & t \leqslant x, \\ 0, & t > x \end{cases}$$

for $0 \leqslant x \leqslant 1$. $h(t,x)$ corresponds to an element $H \in D \hat{\otimes} D^0$ such that $0 \leqslant H \leqslant 1$, so

$$\tau(T_2^* a T_2 b) = (\tau \otimes \tau^0)(H(E_{\mathscr{D}}(a) \otimes E_{\mathscr{D}^0}(b))).$$

Since $E_{\mathscr{D}} \otimes E_{\mathscr{D}^0}$ is a positive normal operator on $W^*(T_1) \hat{\otimes} W^*(T_1)^0$ we have

$$\phi : z \mapsto (\tau \otimes \tau^0)(H(z)(E_{\mathscr{D}} \otimes E_{\mathscr{D}^0}(z)))$$

is a positive normal functional on $W^*(T_1) \hat{\otimes} W^*(T_1)^0$. For $z \geqslant 0$, we observe that

$$\phi(z) \leqslant (\tau \otimes \tau^0)((E_{\mathscr{D}} \otimes E_{\mathscr{D}^0})(z)) = (\tau \otimes \tau^0)(z)$$

so

$$0 \leqslant \phi \leqslant \tau \otimes \tau^0. \qquad \square$$

We want to estimate the distance $\alpha = \mathrm{dist}_2(T_2, W^*(T_1)[D + cT_1, T_2] W^*(T_1))$ from Proposition 7.4. Let $\phi : W^*(T_1) \hat{\otimes} W^*(T_1)^0 \to \mathbb{C}$ be the state from Lemma 7.5. We observe that for $a_1, \ldots, a_n, b_1, \ldots, b_n \in W^*(T_1)$ Lemma 7.5 implies that

$$\begin{aligned}
&||T_2 - \sum_{i=1}^n a_i T_2 b_i||_\tau^2 \\
&= ||T_2||_\tau^2 - 2\Re\tau\left(\sum_{i=1}^n T_2^* a_i T_2 b_i\right) + \tau\left(\sum_{i,j=1}^n b_i^* T_2^* a_i^* a_j T_2 b_j\right) \\
&= \phi(1 \otimes 1) - 2\Re\phi\left(\sum_{i=1}^n a_i \otimes b_i^0\right) + \phi\left(\sum_{i,j=1}^n a_i^* a_j \otimes (b_j b_i^*)^0\right) \\
&= \phi\left(\left(1 \otimes 1 - \sum_{i=1}^n a_i \otimes b_i^0\right)^* \left(1 \otimes - \sum_{j=1}^n a_j \otimes b_j^0\right)\right) \\
&\leqslant \left\| 1 \otimes 1 - \sum_{i=1}^n a_i \otimes b_i^0 \right\|_{\tau \otimes \tau^0}^2.
\end{aligned}$$

We thus have

$$\begin{aligned}\alpha &= \mathrm{dist}_2(T_2, W^*(T_1)[D+cT_1, T_2]W^*(T_1)) \\ &= \inf_{\substack{n\in\mathbb{N}\\ a_i,b_i\in W^*(T_1)}} \left\| T_2 - \sum_{i=1}^{n} a_i((D+cT_1)T_2 - T_2(D+cT_1))b_i \right\|_\tau \\ &= \inf_{\substack{n\in\mathbb{N}\\ a_i,b_i\in W^*(T_1)}} \left\| T_2 - \sum_{i=1}^{n} (a_i(D+cT_1))T_2 b_i - a_i T_2((D+cT_1)b_i) \right\|_\tau\end{aligned}$$

$$\begin{aligned}&\leqslant \inf_{\substack{n\in\mathbb{N}\\ a_i,b_i\in W^*(T_1)}} \left\| 1\otimes 1 - \sum_{i=1}^{n} (a_i(D+cT_1)\otimes b_i^0 - a_i\otimes b_i^0(D+cT_1)^0) \right\|_{\tau\otimes\tau^0} \\ &= \inf_{\substack{n\in\mathbb{N}\\ a_i,b_i\in W^*(T_1)}} \left\| 1\otimes 1 - \left(\sum_{i=1}^{n} (a_i\otimes b_i^0)\right)((D+cT_1)\otimes 1 - 1\otimes(D+cT_1)^0) \right\|_{\tau\otimes\tau^0}.\end{aligned}$$

If we can show that $\ker(R)=\{0\}$ for $R := (D+cT_1)\otimes 1 - 1\otimes(D+cT_1)^0$ then defining $Q_n = f_n(R^*R)R^*$ where

$$f_n(t) = \begin{cases} 0 & \text{for } 0\leqslant t\leqslant \frac{1}{n}, \\ \frac{1}{t} & \text{for } t\geqslant \frac{1}{n}, \end{cases}$$

we have $Q_nR = 1_{[\frac{1}{n},\infty)}(R^*R)$ so since $\ker(R^*R)=\ker(R)=\{0\}$ we have $\|Q_nR - 1\otimes 1\|_{\tau\otimes\tau^0}\to 0$. Since each $Q_n\in W^*(T_1)\hat{\otimes} W^*(T_1)^0$ can be approximated in the strong operator topology by operators of the form $\sum_{i=1}^m a_i\otimes b_i^0$ where $a_i\in W^*(T_1)$ and $b_i^0\in W^*(T_1)^0$ it follows that $\alpha = 0$.

Combining this with Theorem 7.4 we have

$$\ker((D+cT_1)\otimes 1 - 1\otimes(D+cT_1)^0) = \{0\} \Rightarrow \limsup_{t\to 0^+} \Phi^*(S_t, S_t^* : \mathbb{C}) = 0. \quad (7.9)$$

By (7.1) this will imply that $\delta^*(T)\geqslant 2$. So now the only remaining problem is to show that $\ker((D+cT_1)\otimes 1 - 1\otimes(D+cT_1)^0)=\{0\}$, and this will follow from the following "eigenspace"-results.

Lemma 7.6. *Let $A,B\in\mathbb{B}(H)$ be bounded operators on a Hilbert space, H, such that* $\ker(B)=\{0\}$. *Define $E_\lambda = \{x\in H \mid Ax=\lambda Bx\}$. If $\lambda_1,\dots,\lambda_n\in\mathbb{C}$ are mutually different, then the corresponding subspaces $E_{\lambda_1},\dots,E_{\lambda_n}$ are all linearly independent.*

Proof. Let $\lambda_1,\dots,\lambda_n\in\mathbb{C}$ be mutually different complex numbers. We must show that $\sum_{i=1}^n x_i = 0$ implies $x_1=\cdots=x_n=0$ when $x_i\in E_{\lambda_i}$ for all $i\in\{1,\dots,n\}$. But

$\sum_{i=1}^{n} x_i = 0$ implies that

$$\sum_{i=1}^{n} A^k x_i = 0$$

for all $k \in \{0, \dots, n-1\}$. Since $x_i \in E_{\lambda_i}$ we have

$$0 = \sum_{i=1}^{n} \lambda_i^k B^k x_i.$$

Multiplying by B an appropriate number of times we obtain

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} B^{n-1}x_1 \\ B^{n-1}x_2 \\ \vdots \\ B^{n-1}x_n \end{bmatrix} = 0.$$

Since the determinant on the left-hand side is a van der Monde determinant, which is exactly invertible for $\lambda_1, \dots, \lambda_n$ all mutually different, we infer that $B^{n-1}x_1 = \cdots B^{n-1}x_n = 0$. Since $\ker(B) = \{0\}$ also $\ker(B^{n-1}) = \{0\}$ so $x_1 = \cdots = x_n = 0$. Thus the "eigenspaces" $E_{\lambda_1}, \dots, E_{\lambda_n}$ are linearly independent. $\square$

Lemma 7.7. *Let $\mathcal{N} \subseteq \mathbb{B}(H)$ be a finite W^*-algebra represented on a Hilbert space H. Let $A, B \in \mathcal{N}$, and define $E_\lambda = \{x \in H | Ax = \lambda Bx\}$. If* $\ker B = \{0\}$ *then*

$$E_{\lambda_n} \cap \overline{E_{\lambda_1} + \cdots + E_{\lambda_{n-1}}} = \{0\}, \tag{7.10}$$

when $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ are mutually different.

Proof. By Lemma 7.6, we know that $E_{\lambda_1}, \dots, E_{\lambda_n}$ are linearly independent for $\lambda_1, \dots, \lambda_n$ mutually different, so we have

$$E_{\lambda_n} \cap (E_{\lambda_1} + \cdots + E_{\lambda_{n-1}}) = \{0\}. \tag{7.11}$$

Obviously $E_{\lambda_1} + \cdots + E_{\lambda_{n-1}}$ and E_{λ_n} are subspaces of H affiliated with $\mathcal{N}$, so Theorem A.2 implies that

$$\begin{aligned} E_{\lambda_n} \cap \overline{E_{\lambda_1} + \cdots + E_{\lambda_{n-1}}} &\subseteq \overline{E_{\lambda_n}} \cap \overline{E_{\lambda_1} + \cdots + E_{\lambda_{n-1}}} \\ &= \overline{E_{\lambda_n} \cap (E_{\lambda_1} + \cdots + E_{\lambda_{n-1}})} = \{0\}. \qquad \square \end{aligned}$$

Proposition 7.8. *Let $A, B \in \mathbb{B}(H)$ be bounded operators on a Hilbert space, H, such that* $\ker B = \{0\}$ *and define $E_\lambda = \{x \in H | Ax = \lambda Bx\}$. Assume that $A, B \in \mathcal{N}$ where $\mathcal{N}$ is a II_1-factor with a faithful tracial state, τ. Let p_λ be the projection onto E_λ. If*

$\lambda_1, \ldots, \lambda_n$ are mutually different, then

$$\tau\left(\bigvee_{j=1}^{n} p_{\lambda_j}\right) = \sum_{j=1}^{n} \tau(p_{\lambda_j}). \tag{7.12}$$

Proof. If p, q are two projections in $\mathcal{N}$ then it follows from Kaplansky's formula that

$$\tau(p \vee q) = \tau(p) + \tau(q) - \tau(p \wedge q). \tag{7.13}$$

Define $q_k = \bigvee_{i=1}^{k} p_j$ for $k = 1, \ldots, n$, and put $q_0 = 0$. Then $q_k = p_k \vee q_{k-1}$ and by Lemma 7.7 $p_k \wedge q_{k-1} = 0$ for $k \geqslant 1$ since $E_{\lambda_k} \cap \overline{E_{\lambda_1} + \cdots + E_{\lambda_{k-1}}} = \{0\}$. We thus conclude from (7.13) that

$$\tau(q_k) = \tau(p_k) + \tau(q_{k-1})$$

and thus also

$$\tau(q_n) = \sum_{j=1}^{n} \tau(p_j). \qquad \square$$

Corollary 7.9. *Let $A, B \in \mathbb{B}(H)$ be bounded operators on a Hilbert space, H, such that* $\ker B = \{0\}$ *and define $E_\lambda = \{x \in H | Ax = \lambda Bx\}$. Assume that $A, B \in \mathcal{N}$ where $\mathcal{N}$ is a II_1-factor with a faithful tracial state, τ. Let p_λ be the projection onto E_λ. Then $E_\lambda = \{0\}$ except for countably many $\lambda \in \mathbb{C}$.*

Proof. For any finite subset $F \subset \mathbb{C}$ Proposition 7.8 implies that

$$\sum_{\lambda \in F} \tau(p_\lambda) = \tau\left(\bigvee_{\lambda \in F} p_\lambda\right) \leqslant 1,$$

so

$$\sum_{\lambda \in \mathbb{C}} \tau(p_\lambda) \leqslant 1$$

and thus $\tau(p_\lambda) = 0$ for all but countably many $\lambda \in \mathbb{C}$. $\square$

Corollary 7.10.

$$\ker((D + cT_1) \otimes 1 - 1 \otimes (D + cT_1)^0) = \{0\}.$$

Proof. Define $A_1 = T_1 \otimes 1$ and $B_1 = 1 \otimes T_1^0$. It follows from [DH1, Theorem 8.9] that $\ker(B_1) = \ker(1 \otimes T_1^0) = \{0\}$ so Corollary 7.9 applies. $\ker(T_1 \otimes 1 - 1 \otimes T_1^0) \neq \{0\}$ would imply that $\ker(T_1 \otimes 1 - 1 \otimes \lambda T_1^0) \neq \{0\}$ for any $\lambda \in \mathbb{T}$ in the unit circle of $\mathbb{C}$, since T_1 and λT_1 have the same $*$-distribution [DH1, Proposition 2.12]. But then $E_\lambda \neq \{0\}$ for all $\lambda \in \mathbb{T}$ which is not countable. This contradicts Corollary 7.9 so we must have $\ker(T_1 \otimes 1 - 1 \otimes T_1^0) = \{0\}$. Next define $A = D \otimes 1 - 1 \otimes D^0$ and $B = T_1 \otimes 1 - 1 \otimes T_1^0$. Then we have just seen that $\ker(B) = \{0\}$, so since (A, B) and $(A, \lambda B)$ are equally $*$-distributed for any $\lambda \in \mathbb{T}$ we can use Corollary 7.9 once more to conclude that $\ker((D + cT) \otimes 1 - 1 \otimes (D + cT)^0) = \{0\}$. $\square$

Corollary 7.10 finishes the proof of $\delta^*(T) \geqslant 2$. It follows from Proposition 3.4 used on the real and imaginary part of T that $\delta^*(T) \leqslant 2$ so we conclude that $\delta^*(T) = 2$.

Proceeding as in the proof of Corollary 7.10, we have following corollary on the point spectrum of DT-operators.

Corollary 7.11. *Every* DT-*operator has empty point spectrum.*

Proof. Let $D + cT$ be a DT-operator for $D \in \mathscr{D}$ and $T \in \mathscr{UT}(X, c)$ for some compactly supported complex probability measure, μ, and some $c > 0$. Let $\gamma \in \mathbb{C}$ be fixed. Then $(-\gamma 1 + D) + cT$ is again a DT-operator and the $*$-distribution of $(-\gamma 1 + D) + cT$ is completely determined by the distribution of $(-\gamma 1 + D)$ and c. Since λT and T are equally $*$-distributed for all $\lambda \in \mathbb{T}$ we infer that $(-\gamma 1 + D) + cT$ and $(-\gamma 1 + D) + \lambda cT$ are equally $*$-distributed for all $\lambda \in \mathbb{T}$. By an argument similar to the one given in Corollary 7.10, we infer that

$$\ker(\gamma 1 - (D + cT)) = \ker((-\gamma 1 + D) + cT) = \{0\}$$

for all $\gamma \in \mathbb{C}$, so $\sigma_p(D + cT) = \emptyset$. $\square$

Appendix A. Affiliated subspaces

This appendix is part of an unpublished master thesis [AZ] done in 1985 by Ainsworth under supervision of Haagerup. The translated title of the thesis is *Unbounded operators affiliated with a finite von Neumann algebra*.

Definition A.1. Let $\mathscr{A}$ be a finite von Neumann algebra represented on a Hilbert space, H. Let $\mathscr{E}$ be a subspace of H. We say that $\mathscr{E}$ is *affiliated* to $\mathscr{A}$ if for all A' in the commutant, $\mathscr{A}'$, of $\mathscr{A}$, and for all $\xi \in \mathscr{E}$ we have $A'\xi \in \mathscr{E}$.

We say that an operator, T, is affiliated with the von Neumann algebra, $\mathscr{A}$ if T for every $A' \in \mathscr{A}'$ we have $A'\mathscr{D}(T) \subseteq \mathscr{D}(T)$ and for all $\xi \in \mathscr{D}(T)$ we have $A'T\xi = TA'\xi$.

It is an easy exercise to see that closures and intersections of affiliated subspaces are again affiliated subspaces.

It is the purpose of this appendix is give a proof of the following theorem.

Theorem A.2. *Let $\mathscr{A}$ be a finite von Neumann algebra represented on a Hilbert space H. Let $\mathscr{E}$ and $\mathscr{F}$ be subspaces of H affiliated with $\mathscr{A}$. Then*

$$\overline{\mathscr{E}} \cap \overline{\mathscr{F}} = \overline{\mathscr{E} \cap \mathscr{F}}.$$

The proof relies on the T-theorem.

Theorem A.3 (T-theorem, Skau [Sk, Corollary 2]). *Let $\mathscr{A}$ be a finite von Neumann algebra on a Hilbert space H and let $\xi \in H$. If $\eta \in \overline{\mathscr{A}\xi}$ there exists a closed densely defined operator, T, affiliated with $\mathscr{A}$ such that $\xi \in \mathscr{D}(T)$ and $T\xi = \eta$.*

Actually we need the following version of the T-theorem for the commutant of a finite von Neumann algebra.

Corollary A.4. *Let $\mathscr{A}$ be a finite von Neumann algebra on a Hilbert space H. Then the T-theorem is valid for the commutant of $\mathscr{A}$, that is for all $\eta \in \overline{\mathscr{A}'\xi}$ there exists a closed densely defined operator, T, affiliated with $\mathscr{A}'$ such that $\xi \in \mathscr{D}(T)$ and $T\xi = \eta$.*

Proof. Let $\xi, \eta \in H$ such that $\eta \in \overline{\mathscr{A}'\xi}$ and let p_ξ and p_η be the projections onto $\overline{\mathscr{A}\xi}$ and $\overline{\mathscr{A}\eta}$, respectively. By Kadison and Ringrose [KR, Proposition 9.1.2.] p_ξ and p_η are finite projections in $\mathscr{A}'$. By Kadison and Ringrose [KR, 6.3.8.] the finiteness of p_ξ and p_η implies that $p := p_\xi \vee p_\eta$ is a finite projection in $\mathscr{A}'$. Now $\xi, \eta \in p(H)$, and since the commutant of $\mathscr{A}\restriction_{p(H)}$ is the finite von Neumann algebra $p\mathscr{A}'p$ the T-theorem (for the finite W^*-algebra $p\mathscr{A}'p$) says that there exists a closed, densely defined (in $p(H)$) operator, T, affiliated with $p\mathscr{A}'p$ such $T\xi = \eta$. Now extend T to a closed operator which is densely defined in H by defining $T = 0$ on the orthogonal complement of $p(H)$. Then T is a closed, densely defined operator affiliated with $\mathscr{A}'$ such that $\xi \in \mathscr{D}(T)$ and $T\xi = \eta$. $\square$

Lemma A.5. *Let $\mathscr{A}$ be a finite von Neumann algebra on a Hilbert space H. If $(p_i)_{i\in I}$ is an increasing net of projections in $\mathscr{A}$ and if p and q are projections in $\mathscr{A}$ such that $p_i \nearrow p$ and $q \leqslant p$ then $p_i \wedge q \nearrow p \wedge q = q$.*

Proof. By Kaplansky's formula, we have

$$q - q \wedge p_i \sim q \vee p_i - p_i.$$

Since $p_i \leqslant p$ and $q \leqslant p$ we have $p_i \vee q \leqslant p$, so

$$q - q \wedge p_i \sim q \vee p_i - p_i \leqslant p - p_i.$$

$(q \wedge p_i)_{i\in I}$ is an increasing net of projections bounded above by p, so $(q \wedge p_i)_{i\in I}$ converges to some projection $r \leqslant p$. For every normal tracial state, τ, we now have

$$\tau(q - r) = \lim_i \tau(q - q \wedge p_i) \leqslant \lim_i \tau(p - p_i) = 0.$$

Since a finite von Neumann algebra has a faithful family of normal tracial states [Ta, Theorem 2.4.] we infer that $q = r$, so $q \wedge p_i \nearrow q$. $\square$

As a final step towards proving Theorem A.2, we need the following key result.

Theorem A.6. *Let $\mathscr{A}$ be a finite von Neumann algebra represented on a Hilbert space H. Let $\mathscr{E}$ be a subspace of H affiliated with $\mathscr{A}$. Then for all $\xi \in \overline{\mathscr{E}}$ and for all $\delta > 0$ there exists $A' \in \mathscr{A}'$ such that $A'\xi \in \mathscr{E}$ and $\|A'\xi - \xi\| < \delta$.*

Proof. The proof has three parts. First, assume that $\mathscr{E} = \mathscr{A}'\eta$ for some $\eta \in H$. Let $\xi \in \overline{\mathscr{E}}$. By Corollary A.4, the T-theorem is true for $\mathscr{A}'$ so there exists a closed densely defined operator, T, affiliated with $\mathscr{A}'$ such that $\eta \in \mathscr{D}(T)$ and $\xi = T\eta$. Let $T = |T^*|U$ be the right polar decomposition of T and define $p_n = 1_{[0,n]}(|T^*|)$. Then $p_n\xi \to \xi$ in norm as $n \to \infty$. Now

$$p_n\xi = 1_{[0,n]}(|T^*|)|T^*|U\eta = \left(\int_0^n \lambda \, \mathrm{d}e_\lambda\right) U\eta,$$

where e_λ is the spectral measure of $|T^*|$. Since $\int_0^n \lambda \, \mathrm{d}e_\lambda \in \mathscr{A}'$ and $U \in \mathscr{A}'$ we have $p_n\xi \in \mathscr{A}'\eta = \mathscr{E}$. Now choose n large enough to ensure that $\|p_n\xi - \xi\| < \delta$. This proves the theorem in the first case.

Secondly, assume that $\mathscr{E} = \sum_{k=1}^n \mathscr{A}'\eta_k$ for some $\eta_1, \dots, \eta_n \in H$. Define $\tilde{\eta} = (\eta_1, \dots, \eta_n) \in H^{\oplus n}$ and observe that for $A'_1, \dots, A'_n \in \mathscr{A}'$ we have

$$\begin{pmatrix} A'_1 & \cdots & A'_n \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^n A'_k\eta_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

so $\tilde{\xi} := (\xi, 0, \dots, 0) \in \overline{M_n(\mathscr{A}')\tilde{\eta}}$. Identify $\mathscr{A}$ with its unital embedding as diagonal operators in $M_n(\mathscr{A})$. With this identification $\mathscr{A}$ is the commutant of $M_n(\mathscr{A}')$, so the idea is now to use the first part of the proof on the subspace $M_n(\mathscr{A}')\tilde{\eta}$ of $H^{\oplus n}$ and the von Neumann algebra $\mathscr{A}$ represented on $H^{\oplus n}$ with commutant $M_n(\mathscr{A}')$. The first part of the proof thus gives us an $A' \in M_n(\mathscr{A}')$ such that $\|A'\tilde{\xi} - \tilde{\xi}\| < \delta$, and $A'\tilde{\xi} \in M_n(\mathscr{A}')\tilde{\eta}$. If $A' = (A'_{ij})_{i,i=1}^n$ then of course

$$A'\tilde{\xi} = \begin{pmatrix} A'_{11} & \cdots & A'_{1n} \\ \vdots & \ddots & \vdots \\ A'_{n1} & \cdots & A'_{nn} \end{pmatrix} \begin{pmatrix} \xi \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} A'_{11}\xi \\ \vdots \\ A'_{n1}\xi \end{pmatrix}$$

so

$$\delta^2 > ||A'\tilde{\xi} - \tilde{\xi}||^2 > ||A'_{11}\xi - \xi||^2$$

and furthermore since $A'\tilde{\xi} \in M_n(\mathscr{A}')\tilde{\eta}$ there exists $B' = (B'_{ij})_{i,j=1}^n \in M_n(\mathscr{A}')$ such that

$$A'_{11}\xi = (A'\tilde{\xi})_1 = (B'\tilde{\eta})_1 = \sum_{k=1}^n B'_{1k}\eta_k \in \mathscr{E}$$

so $A'_{11} \in \mathscr{A}'$ satisfies the theorem in the second case.

As the third and final case assume that $\mathscr{E}$ is an arbitrary subspace of H affiliated with $\mathscr{A}$. Define $\mathscr{E}_J = \sum_{\eta \in J} \mathscr{A}'\eta$ for all finite subsets, J, of $\mathscr{E}$. Then $(\mathscr{E}_J)$ is a upward filtering net ordered by inclusion with upper bound $\mathscr{E}$, so if p_J denotes the projection onto the closure of the subspace $\mathscr{E}_J$ and p denotes the projection onto the closure of the subspace $\mathscr{E}$ then (p_J) is an increasing net of projections that converges strongly to p.

Let $\xi \in \overline{\mathscr{E}}$ and let q be the projection onto the closure of the subspace $\mathscr{A}'\xi$. Then $q \in \mathscr{A}$ and since $\overline{\mathscr{A}'\xi} \subseteq \overline{\mathscr{E}}$ we have $q \leqslant p$ so $p_J \wedge q \nearrow q$ by Lemma A.5. Choose J a finite subset of $\mathscr{E}$ such that

$$||(p_J \wedge q)\xi - \xi|| < \frac{\delta}{3}. \tag{A.1}$$

We know that $(p_J \wedge q)\xi \in \overline{\mathscr{A}'\xi}$ so the first part of the proof gives us an $A' \in \mathscr{A}'$ such that $A'(p_J \wedge q)\xi \in \mathscr{A}'\xi$ and

$$||A'(p_J \wedge q)\xi - (p_J \wedge q)\xi|| < \frac{\delta}{3}. \tag{A.2}$$

Choose $B' \in \mathscr{A}'$ such that $B'\xi = A'(p_J \wedge q)\xi$. Since

$$B'\xi = A'(p_J \wedge q)\xi = (p_J \wedge q)A'\xi \subseteq p_J(H) = \overline{\sum_{\eta \in J} \mathscr{A}'\eta}$$

the second part of the proof gives us a $C' \in \mathscr{A}'$ such that

$$||C'B'\xi - B'\xi|| < \frac{\delta}{3} \tag{A.3}$$

and $C'B'\xi \in \sum_{\eta \in J} \mathscr{A}'\eta \subseteq \mathscr{E}$. Combining (A.1)–(A.3), we have

$$\begin{aligned} ||C'B'\xi - \xi|| &\leqslant ||C'B'\xi - B'\xi|| \\ &\quad + ||A'(p_J \wedge q)\xi - (p_J \wedge q)\xi|| + ||(p_J \wedge q)\xi - q\xi|| \\ &< \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta \end{aligned}$$

so $C'B' \in \mathscr{A}'$ proves the theorem in the third case. $\square$

Proof of Theorem A.2. Let $\mathscr{E}$ and $\mathscr{F}$ be subspaces of H affiliated with the finite von Neumann algebra $\mathscr{A}$ as stated in the theorem. Let $\xi \in \overline{\mathscr{E}} \cap \overline{\mathscr{F}} \subset \overline{\mathscr{E}}$. By Theorem A.6 there exists an $A' \in \mathscr{A}'$ such that $||A'\xi - \xi|| < \frac{\delta}{2}$ for arbitrary $\delta > 0$ and such that $A'\xi \in \mathscr{E}$ and $A'\xi \in \overline{\mathscr{E}} \cap \overline{\mathscr{F}} \subseteq \overline{\mathscr{F}}$. Applying Theorem A.6 again we obtain a $B' \in \mathscr{A}'$ such that $||B'(A'\xi) - A'\xi|| < \frac{\delta}{2}$ and such that $B'(A'\xi) \in \mathscr{F}$. Since $\mathscr{E}$ is affiliated with $\mathscr{A}$ we have $B'(A'\xi) \in \mathscr{E}$ and thus $B'A'\xi \in \mathscr{E} \cap \mathscr{F}$. The inequalities imply that

$$||B'A'\xi - \xi|| \leqslant ||B'A'\xi - A'\xi|| + ||A'\xi - \xi|| < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Since $\delta > 0$ is arbitrary we conclude that $\overline{\mathscr{E}} \cap \overline{\mathscr{F}} \subseteq \overline{\mathscr{E} \cap \mathscr{F}}$. Conversely $\mathscr{E} \cap \mathscr{F} \subseteq \overline{\mathscr{E}} \cap \overline{\mathscr{F}}$ which is closed in H so $\overline{\mathscr{E} \cap \mathscr{F}} \subseteq \overline{\mathscr{E}} \cap \overline{\mathscr{F}}$. □

Acknowledgments

It is a great pleasure for me to thank my advisor Professor Uffe Haagerup for suggesting the problem to me and for stimulating and fruitful discussions on the subject. I thank Peter Ainsworth for letting me publish the results in the appendix in this paper. Finally, I thank Nica for pointing out Ref. [NSS2].

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