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ScienceDirect
European Journal of Combinatorics 29 (2008) 1219–1234
European Journal of Combinatorics
www.elsevier.com/locate/ejc

Parametrizing complex Hadamard matrices

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Received 9 October 2006; accepted 18 June 2007
Available online 13 August 2007

Abstract

The purpose of this paper is to introduce new parametric families of complex Hadamard matrices in two different ways. First, we prove that every real Hadamard matrix of order $N \geq 4$ admits an affine orbit. This settles a recent open problem of Tadej and Życzkowski [W. Tadej, K. Życzkowski, A concise guide to complex Hadamard matrices, Open Syst. Inf. Dyn. 13 (2006) 133–177], who asked whether a real Hadamard matrix can be isolated among complex ones. In particular, we apply our construction to the only (up to equivalence) real Hadamard matrix of order 12 and show that the arising affine family is different from all previously known examples listed in [W. Tadej, K. Życzkowski, A concise guide to complex Hadamard matrices, Open Syst. Inf. Dyn. 13 (2006) 133–177]. Second, we recall a well-known construction related to real conference matrices, and show how to introduce an affine parameter in the arising complex Hadamard matrices. This leads to new parametric families of orders 10 and 14. An interesting feature of both of our constructions is that the arising families cannot be obtained via Diţă's general method [P. Diţă, Some results on the parametrization of complex Hadamard matrices, J. Phys. A 37 (20) (2004) 5355–5374]. Our results extend the recent catalogue of complex Hadamard matrices [W. Tadej, K. Życzkowski, A concise guide to complex Hadamard matrices, Open Syst. Inf. Dyn. 13 (2006) 133–177], and may lead to direct applications in quantum-information theory.

1. Introduction

In the past few decades complex Hadamard matrices have been extensively studied since it turned out that they are related to many interesting combinatorial and important physical problems. However, despite many years of research only moderate results are known, e.g. the problem of finding all complex Hadamard matrices even of small orders is still open. The first

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0195-6698/$ - see front matter
doi:10.1016/j.ejc.2007.06.009

significant result is due to Haagerup [5], who managed to classify all complex Hadamard matrices up to order 5 in 1997. Only partial results are known about matrices of order 6. Besides some affine families listed in [11], all self-adjoint (Hermitian) complex Hadamard matrices of order 6 were classified by Beauchamp and Nicoara [1], and a symmetric non-affine family was found by Matolcsi and Szöllősi very recently [7].

First, there was an interest in particular examples of (permutation) inequivalent complex Hadamard matrices of low order. However, due to a recent discovery of Diţă [3] the situation has changed dramatically. His powerful method leads to the construction of *parametric* families of Hadamard matrices in composite dimensions. This method was subsequently rediscovered by Matolcsi, Réffy and Szöllősi [8] who used a spectral set construction from [6], and then used another spectral set construction to obtain new families of complex Hadamard matrices. An entirely different approach for parametrization was described in the monument paper of Tadej and Życzkowski [11] who introduced the method of "linear variation of phases", obtaining affine Hadamard families. They successfully obtained all maximal affine Hadamard families stemming from the Fourier matrices F_N for $N \leq 16$. Thus, one is interested in the inequivalent classes of *parametric families* of Hadamard matrices nowadays.

The aim of this paper is to describe two *general* constructions which lead to new parametric families of complex Hadamard matrices in certain dimensions; these matrices arise due to a natural construction from real Hadamard and real conference matrices. We prove that they are non-Diţă-type, which subsequently leads to new results in the sense that they were not included in the recent catalogue. The main point of this paper is to show that these matrices always admit an affine orbit, thus we can introduce new parametric families of complex Hadamard matrices of order 10, 12 and 14. With the aid of our results we can supplement the incomplete catalogue of complex Hadamard matrices of small orders in [11].

2. Preliminaries

First let us introduce some formal definitions and recall previous results from [3,8,11].

Definition 2.1. An Hadamard matrix H is a square complex matrix of order N with $|H_{i,j}| = 1$ for $i, j = 1, 2, \ldots, N$, satisfying $HH^* = NI$, where I is the identity matrix and H^* denotes the Hermitian transpose of H.

Definition 2.2. A complex (real) Hadamard matrix H of order N is dephased (normalized) if $H_{1,i} = H_{i,1} = 1$ for every $i = 1, 2, \ldots, N$. In a given dephased matrix H, the lower right $(N-1) \times (N-1)$ submatrix is called the core of H.

Definition 2.3. Two Hadamard matrices, H_1 and H_2, are equivalent if there exist diagonal unitary matrices D_1 and D_2 and permutation matrices P_1 and P_2 such that $H_1 = D_1 P_1 H_2 P_2 D_2$.

It is clear that every complex Hadamard matrix is equivalent to a dephased one.

Next we recall Diţă's general method of constructing complex Hadamard matrices (his subsequent results on families with some free parameters follow easily from this formula as described very well in his paper [3]).

Construction 2.1. *Let M be a complex Hadamard matrix of order k, and $N_1, N_2, \ldots, N_k$ are complex Hadamard matrices of order n. Then*

$$K := \begin{bmatrix} m_{11}N_1 & \cdot & \cdot & m_{1k}N_k \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ m_{k1}N_1 & \cdot & \cdot & m_{kk}N_k \end{bmatrix} \tag{1}$$

is a complex Hadamard matrix of order nk.

Definition 2.4. A complex Hadamard matrix K is called Diţă-type if it is equivalent to a matrix arising from formula (1).

Definition 2.5. A parametric family of complex Hadamard matrices is called affine if the phases of the entries are sums of a constant and a linear function of the parameters. A family is maximal affine, if it is not properly contained in any other affine family.

Remark 2.2. When we say that H admits an affine orbit, we mean that there exists an affine family stemming from a dephased form of H, consisting purely of dephased complex Hadamard matrices. Since the first row and column entries are fixed at some chosen values, the members of the family cannot be obtained, one from another, by multiplication by unitary diagonal matrices.

Several affine families are listed in [11]. For an example of an affine family in this paper the reader might want to jump ahead to formulas (7)–(9).

In general, deciding whether two Hadamard matrices are equivalent or not is a nontrivial task. However, recently Matolcsi et al. introduced a powerful method, which easily establishes if an Hadamard matrix is a *Diţă-type* one. In fact, it turned out that it is worth investigating the corresponding log-Hadamard matrix (A square matrix L is log-Hadamard if the entrywise exponential matrix, $[\mathrm{e}^{2\pi \mathbf{i} L_{i,j}}]$, $L_{i,j} \in [0, 1)$, is Hadamard.). The following definition and Lemma 2.3 summarize the corresponding results from [8].

Definition 2.6. Let L be an $N \times N$ real matrix. For an index set $I = \{i_1, i_2, \dots, i_n\} \subset \{1, 2, \dots, N\}$ two rows (or columns) $\mathbf{s}$ and $\mathbf{q}$ are called *I-equivalent*, in notation $\mathbf{s} \sim_I \mathbf{q}$, if the positive fractional part of the entrywise differences, $s_i - q_i \mod 1$, are the same for every $i \in I$. Two rows (or columns) $\mathbf{s}$ and $\mathbf{q}$ are called (d)-n-equivalent if there exist n-element disjoint sets of indices $I_1, \dots, I_d$ such that $\mathbf{s} \sim_{I_j} \mathbf{q}$ for all $j = 1, \dots, d$.

Lemma 2.3. *Permutation of rows and columns, or adding a constant to a row or a column does not change (d)-n-equivalence.*

By formula (1), the structure of an $N \times N$ Diţă-type matrix L (where $N = nk$) implies for the corresponding log-Hadamard matrix $\log L$ that there exists a partition of indices into n-element sets $I_1, \dots, I_k$ and k-tuples of rows $R_j = \{\mathbf{r}_1^j, \dots, \mathbf{r}_k^j\}$ $(j = 1, \dots, n)$ such that any two rows in a fixed k-tuple are equivalent with respect to any of the I_m's. Naturally, the same holds for the transpose of a Diţă-type matrix, with the role of rows and columns interchanged.

The following observation is a trivial consequence of their result:

Lemma 2.4. *Let H be a dephased complex Hadamard matrix of order N, and suppose that $H_{i,j} \neq 1$ for every $1 < i, j \leq N$, i.e. there is no 1 in the core of H. Then H is not of Diţă-type.*

Proof. We argue by contradiction. Assume that H is Diţă-type. Using the notation of the previous paragraph we can arrange (after relabelling the index sets if necessary) that $\{1\} \subseteq I_1$ and (after permuting the columns of H if necessary) that $\{1, 2\} \subseteq I_1$. There must be a row $\mathbf{r}$ of $\log H$ which is I_1-equivalent to the first row. However, as all entries in the first row and first column are 0's, this would imply that **r** contains a 0 in its second coordinate, a contradiction. □

3. Constructing complex Hadamard matrices from real ones

In this section we investigate the structure of real Hadamard matrices. First we prove that they cannot be obtained using Diţă's method in certain dimensions. Next we introduce a somewhat natural construction for obtaining new, parametrized complex Hadamard matrices from real ones. In fact, it was asked in [11] whether all real Hadamard matrices of order $N \geq 4$ can be parametrized and, by Theorem 3.5, we answer this question in the positive. Before doing so we first recall a folklore

Lemma 3.1. *Let $p \geq 3$ be an arbitrary odd number. Suppose that the first four rows of a real $\{-1, 1\}$ matrix of order $4p$ have the following form (note that every real Hadamard matrix is easily seen to be equivalent to one having exactly the same first three rows as the matrix below):*

$$\begin{array}{c} (\mathbf{s}) \\ (\mathbf{t}) \\ (\mathbf{u}) \\ (\mathbf{v}) \end{array} \begin{bmatrix} 1^p & 1^p & 1^p & 1^p \\ 1^p & 1^p & (-1)^p & (-1)^p \\ 1^p & (-1)^p & 1^p & (-1)^p \\ 1^p & (-1)^p & (-1)^p & 1^p \end{bmatrix}, \tag{2}$$

where 1^p means p one's in a row. Then this matrix cannot be extended with a further $\{1, -1\}$ row being orthogonal to all previous ones.

Proof. Suppose, to the contrary, that (2) can be extended by a further row **w**. Let us denote by a, b, c and d the number of 1's in **w** in the first-, second-, third- and fourth quarter, i.e. $\mathbf{w} = \left(1^a, (-1)^{p-a}, 1^b, (-1)^{p-b}, 1^c, (-1)^{p-c}, 1^d, (-1)^{p-d}\right), 0 \leq a, b, c, d \leq p$. Since **w** is orthogonal to all of the rows **s**, **t**, **u** and **v**, we get the following four equations by straightforward computation

$$a - (p - a) + b - (p - b) + c - (p - c) + d - (p - d) = 0 \tag{3}$$

$$a - (p - a) + b - (p - b) - c + (p - c) - d + (p - d) = 0 \tag{4}$$

$$a - (p - a) - b + (p - b) + c - (p - c) - d + (p - d) = 0 \tag{5}$$

$$a - (p - a) - b + (p - b) - c + (p - c) + d - (p - d) = 0. \tag{6}$$

By simple algebra one can check that the solution to equations (3)–(6) is $a = b = c = d = \frac{p}{2}$ and, since p is odd by assumption, this is a contradiction. □

Now we are ready to state our first

Theorem 3.2. *Let p be an odd prime and suppose that H_{4p} is a real Hadamard matrix of order $4p$. Then H_{4p} is not of Diţă-type.*

Proof. We will use the notation of the paragraph following Lemma 2.3 with the exception that instead of taking $\log H$ we apply the notion of I-equivalence to the rows of H itself in a natural way.

Assume, to the contrary, that H_{4p} is of Diţă-type. In this case the only possible values for n are 2, 4, p and $2p$ (with k being $2p$, p, 4 and 2 respectively). Suppose that H_{4p} is dephased, and let us again denote the rows of (2) by **s**, **t**, **u** and **v** respectively. There are four cases to consider according to the choices of n and k:

Case 1. Assume $n = 2p, k = 2$. In this case there should be a partition of indices to $2p$-element sets I_1, I_2 such that in H_{4p} $2p$ pairs of rows are equivalent with respect to I_1 and I_2. After permutation of rows and columns it is trivial to achieve that the first three rows of H_{4p} are **s**, **t** and **u**, respectively, and **s** and **t** form a pair. (First we permute the rows so that the companion of row 1 becomes the second row and then we permute the columns so that the position of 1's and -1's is exactly as in (2).) Then $I_1 = \{1, 2, \dots, 2p\}$ and $I_2 = \{2p+1, 2p+2, \dots, 4p\}$. Now consider **u**. If it formed a pair, then its companion's first $2p$ entries would have to be exactly the same as those in **u**. However, by orthogonality, the last $2p$ entries in **u** and its companion must be opposite. Thus the companion of **u** must be exactly **v**, which is a contradiction since there is no such a row in H_{4p} due to Lemma 3.1 (by our assumptions, of course, H has at least 12 rows).

Case 2. Now assume $n = p, k = 4$. In this case the partitions of indices are p-element sets I_1, I_2, I_3 and I_4, such that in H_{4p} there exist p 4-tuples of rows, such that any two rows in a fixed 4-tuple are equivalent with respect to them. We can suppose that $I_1 = \{1, 2, \dots, p\}$, $I_2 = \{p+1, p+2, \dots, 2p\}$, $I_3 = \{2p+1, 2p+2, \dots, 3p\}$ and $I_4 = \{3p+1, 3p+2, \dots, 4p\}$. Now observe, since **s** contains only 1's, any row equivalent to it with respect to I_1, I_2, I_3 and I_4 must be one of **t**, **u** or **v**. However, we need three rows being equivalent to **s**, thus we need all four rows of (2), which is a contradiction again.

Case 3. Now assume $n = 4, k = p$. In this case the partitions of indices are 4-element sets $I_1, I_2, \dots, I_p$ such that in H_{4p} there exist 4 disjoint p-tuples of rows such that any two rows in a fixed p-tuple are equivalent with respect to them. Again, we would like to find a companion to **s**. Observe that since every row (different from **s**) contains $2p$ 1's and $2p$ (-1)'s it is impossible to split their entries into odd (p) number of disjoint sets containing exactly the same values. Hence we cannot choose a companion to **s**, equivalent to it with respect to the index sets.

Case 4. Finally assume that $n = 2, k = 2p$. Again (by permuting the columns of H_{4p} if necessary), we can suppose that $I_1 = \{1, 2\}, I_2 = \{3, 4\}$. Since H_{4p} is a real Hadamard matrix, we can suppose that (after permuting some rows if necessary) its first three columns are exactly the same as the transpose of the first three row of (2). Now observe that the first $2p$ and the second $2p$ rows have to belong to a common tuple. To preserve equivalence with respect to I_2, one can see that the fourth column of H_{4p} has to be exactly the transpose of the fourth row of the matrix in (2). And this is a contradiction again. □

Corollary 3.3. *H_{12} is not of Diţă-type.*

Now we turn to the parametrization of real Hadamard matrices. It is well known that H_4 admits a 1-parameter orbit. In [11] a 5-parameter, while in [8] a 4-parameter maximal affine orbit was constructed for H_8 (these orbits are essentially different, but they intersect each other at H_8). In general it is not clear how to introduce affine parameters to an *arbitrary* complex Hadamard matrix. The authors of [11] admit that the "linear variation of phases" method becomes a serious combinatorial problem already for $N = 12$, so it cannot effectively be used for higher order matrices. Now we introduce a general method for parametrization which always works for real matrices and, in some cases, for complex matrices too. The main observation is contained in the following

Lemma 3.4. *Let H be an arbitrary dephased complex Hadamard matrix of order $N \geq 4$. Suppose that H has a pair of columns, say* **u** *and* **v**, *with the following property: $u_i = v_i$ or $u_i + v_i = 0$ holds for every $i = 1, 2, \dots, N$. Then H admits an affine orbit.*

Proof. Consider H satisfying the conditions of Lemma 3.4, and take every pair of coordinates (u_i, v_i) for which $u_i + v_i = 0$ holds. Multiply these elements by $\mathrm{e}^{\mathbf{i}t}$, i.e. modify (u_i, v_i) to

$(u_i e^{\mathbf{i}t}, v_i e^{\mathbf{i}t})$. Now we proceed to show that the arising parametric matrix $H^{(1)}(t)$ is Hadamard. To do this let us consider a pair of rows in $H^{(1)}(t)$. It is easy to see that after taking the inner product of these rows, the parameter (if it existed in at least one of them) vanishes, therefore $H^{(1)}(t)$ is Hadamard independently of the exact value of t. Finally, if $H^{(1)}(t)$ is not dephased (i.e. we have chosen the first column of H to be either $\mathbf{u}$ or $\mathbf{v}$), one should multiply some rows by $e^{-\mathbf{i}t}$ to get a dephased matrix, and it is clear that t will not vanish whenever $N \geq 4$. $\square$

With the aid of Lemma 3.4 we can prove the main theorem of this section. We prove that there is no isolated matrix among real Hadamard matrices except for orders 1 and 2 (the cases $N = 4$ and $N = 8$ were mentioned in the paragraph preceding Lemma 3.4).

Theorem 3.5. *Let H be a real Hadamard matrix of order $N \geq 12$. Then H admits an $\left(\frac{N}{2}+1\right)$-parameter affine orbit.*

Proof. Let $N \geq 12$, and let us take an arbitrary dephased real Hadamard matrix of order N, say H. It is clear that when considering any two columns of H, there will be exactly $\frac{N}{2}$ rows, where the entries of these columns differ, and another $\frac{N}{2}$ rows, where the entries of these columns are the same, so the conditions of Lemma 3.4 hold. Now we apply the construction described in the proof of Lemma 3.4.

Clearly, we can further assume, that H has the following "canonical" form: $H_{2,1} = H_{2,2} = \cdots = H_{2,N/2} = 1$ so $H_{2,N/2+1} = H_{2,N/2+2} = \cdots = H_{2,N} = -1$ and $H_{3,3} = H_{3,4} = 1$ and $H_{3,2} = H_{3,N-1} = H_{3,N} = -1$. Consider the following set containing pairs of indices: $T = \{(2i-1, 2i) : i = 1, 2, \ldots, \frac{N}{2}\}$. Every element of T represents a pair of columns in H. Now the construction is the following: for every $i = 1, 2, \ldots, \frac{N}{2}$ take the respective element of T, and consider the *rows* of the corresponding pair of columns. If the entries in a row are different then multiply them by $e^{\mathbf{i}x_i}$ (again: there are exactly $\frac{N}{2}$ such rows). This yields an $\frac{N}{2}$-parameter family, stemming from H. However, it is not dephased, so one has to multiply some rows by $e^{-\mathbf{i}x_1}$ to get a dephased Hadamard matrix. Since $H_{3,3} = H_{3,4}$ we can see that these entries, after parametrization and dephasing the matrix, depend only on x_1, so $x_1, x_2, \ldots, x_{N/2}$ are independent parameters in the dephased matrix. For convenience, we can substitute x_1 by $-x_1$. Now taking a look at the first two *rows* of H (which are still independent, after parametrization, of any of the x_i's) one can multiply the last (differing) $\frac{N}{2}$ entries of these by $e^{-\mathbf{i}x_{N/2+1}}$, the arising matrix thus being still Hadamard. Again, it is not dephased, but observe that after dephasing the matrix (multiplying the last $\frac{N}{2}$ columns by $e^{\mathbf{i}x_{N/2+1}}$), since $H_{3,N-1} = H_{3,N}$ these entries after parametrization depend only on x_1 and on $x_{N/2+1}$. Note that this last operation left unchanged both the parametrized $H_{3,3}$ and $H_{3,4}$ which still depend only on x_1. This completes the proof. $\square$

Remark 3.6. The same construction also works when we replace "rows" by "columns" and vice versa.

Remark 3.7. It is easy to see (by taking the inner product of $\mathbf{u}$ and $\mathbf{v}$) that Lemma 3.4 can only be applied in even orders. However, the conditions of this lemma hold for many non-real Hadamard matrices, too. For example, the Fourier matrix F_N in even orders *has* two columns in which the entries are either the same or of opposite sign. Other examples are the matrices S_8, S_{12} and S_{16} in [8] which also satisfy the conditions of Lemma 3.4. Thus, this lemma can be used for parametrizing a wide class of complex Hadamard matrices.

Now we give an example. The following matrix is the only real Hadamard matrix of order 12 (up to equivalence). We note that it can be constructed from a skew-symmetric conference matrix (see Section 4).

$$H_{12} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \end{bmatrix}. \tag{7}$$

By Theorem 3.5 we can easily construct a 7-parameter family stemming from H_{12}. The notation here is exactly the same as that in [11,8]. We denote by $\circ$ the Hadamard product of two matrices (i.e. $[H_1 \circ H_2]_{i,j} = [H_1]_{i,j} \cdot [H_2]_{i,j}$), while the symbol **EXP** stands for the entrywise exponential operation (i.e. $[\mathbf{EXP} H]_{i,j} = \exp(H_{i,j})$).

$$H_{12}^{(7)}(a,b,c,d,e,f,g) = H_{12} \circ \mathbf{EXP}\left(\mathbf{i} \cdot R_{H_{12}^{(7)}}(a,b,c,d,e,f,g)\right) \tag{8}$$

where

$$R_{H_{12}^{(7)}}(a,b,c,d,e,f,g)$$

$$= \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & a & a & a & a & a+g & a+g & a+e+g & a+e+g & a+g & a+g \\ \bullet & \bullet & a+b & a+b & a+c & a+c & a+d+g & a+d+g & a+e+g & a+e+g & a+f+g & a+f+g \\ \bullet & \bullet & a+b & a+b & a+c & a+c & a+d+g & a+d+g & a+e+g & a+e+g & a+f+g & a+f+g \\ \bullet & \bullet & b & b & \bullet & \bullet & d+g & d+g & e+g & e+g & f+g & f+g \\ \bullet & \bullet & \bullet & \bullet & c & c & d+g & d+g & e+g & e+g & f+g & f+g \\ \bullet & \bullet & b & b & \bullet & \bullet & g & g & g & g & f+g & f+g \\ \bullet & \bullet & a+b & a+b & a+c & a+c & a+g & a+g & a+e+g & a+e+g & a+g & a+g \\ \bullet & \bullet & \bullet & \bullet & c & c & g & g & g & g & f+g & f+g \\ \bullet & \bullet & a & a & a & a & a+d+g & a+d+g & a+g & a+g & a+g & a+g \\ \bullet & \bullet & a+b & a+b & a+c & a+c & a+d+g & a+d+g & a+g & a+g & a+g & a+g \end{bmatrix}. \tag{9}$$

According to Corollary 3.3, H_{12} is not of Diţă-type, so it admits only non-Diţă-type matrices in a small neighbourhood of it, since the set of Diţă-type matrices is closed as shown in the following

Proposition 3.8. *The set of all $N \times N$ Diţă-type matrices is closed in the space of all $N \times N$ matrices.*

Proof. Let $T_l \to T$ be a convergent sequence of Diţă-type matrices. We need to show that T is also Diţă-type.

By definition there exist permutation matrices $P_1^{(l)}, P_2^{(l)}$ and diagonal unitary matrices $D_1^{(l)}, D_2^{(l)}$ such that $P_1^{(l)} D_1^{(l)} T_l D_2^{(l)} P_2^{(l)} = K_l$, where K_l arises in formula (1). Each K_l can be characterized by the values of $k, m_{11}, \dots, m_{kk}$, and the matrices $N_1, \dots, N_k$ in (1) (each depending on l, of course, which we left out to simplify the notation). Since the number of possible permutation matrices and the number of possible choices for k is finite, and all other parameters such as $D_1^{(l)}, D_2^{(l)}, m_{ij}, N_i$ take values in compact spaces, there exists a subsequence l_h along which the permutation matrices and the value of k are constant and all other parameters converge, i.e. $D_1^{(l_h)} \to D_1$, $D_2^{(l_h)} \to D_2$, $m_{ij}^{l_h} \to r_{ij}$, $N_i^{l_h} \to Q_i$. By taking the limit it is clear that T is equivalent to the Diţă-type matrix K characterized by the values $k, r_{11}, \dots, r_{kk}$, and the matrices $Q_1, \dots, Q_k$ in (1). $\square$

As a consequence, we have

Corollary 3.9. *The family $H_{12}^{(7)}(a, b, c, d, e, f, g)$ contains only non-Diţă-type matrices in a small neighbourhood around H_{12}.*

Now we show that H_{12} is inequivalent to any of the order 12 matrices appearing in [11,8]. First we recall a result from Haagerup, who introduced the following set $\Lambda_H = \{h_{ij} h_{kl} \overline{h}_{kj} \overline{h}_{il} : (i, j, k, l) \in \{1, \dots, N\}^{\times 4}\}$ for H of order N. In [5] he claims that this set is invariant under the equivalence preserving operations, see Definition 2.3.

Lemma 3.10. *Two complex Hadamard matrices, say H_1 and H_2, are inequivalent, if they have different Λ_H-sets.*

Now we are ready to prove[1] the following

Lemma 3.11. *H_{12} is inequivalent to any of the* 12×12 *matrices listed in* [11,8].

Proof. The proof relies on the Haagerup condition. First observe that $\Lambda_{H_{12}} = \{1, -1\}$. Now consider the seven families of order 12 in [11] stemming from F_{12}, and notice that $e^{2\pi \mathbf{i}/8} \in \Lambda_{F_{12}}$ for any matrix of any of these families stemming from F_{12}, independently of the values of the parameters. Secondly, observe that $e^{2\pi \mathbf{i}/3} \in \Lambda_{S_{12}}$ for any matrix stemming from S_{12} in [8], again independently of the actual values of the parameters. These observations can be easily verified by taking $h_{11} = \overline{h}_{1j} = \overline{h}_{i1} = 1$, and taking an appropriate element h_{ij} for every matrix in the families stemming from F_{12} and from S_{12}. Since H_{12} and matrices from these families possess different Λ-sets, they cannot be equivalent. There are several other families of order 12 listed in [11], however those families were obtained by Diţă's construction (and thus consist purely of Diţă-type matrices), therefore they cannot contain a non-Diţă-type matrix such as H_{12}. This completes the proof. $\square$

Proposition 3.12. *The family $H_{12}^{(7)}(a, b, c, d, e, f, g)$ is locally inequivalent to the families presented in* [11,8]. $\square$

Proof. This clearly follows from the fact, that the invariant set Λ changes continuously. If we change some entries in H_{12} from ± 1 to $e^{\mathbf{i}t}$ with $0 < |t| < \varepsilon$ or $0 < |t - \pi| < \varepsilon$ (for ε being small) then neither $e^{2\pi \mathbf{i}/8}$ nor $e^{2\pi \mathbf{i}/3}$ will arise in the Λ-set of the modified matrix. Finally, by Proposition 3.8, it is clear that we can choose ε small enough to obtain non-Diţă-type matrices only. $\square$

[1] The author is grateful to M. Matolcsi who suggested the proof of Lemma 3.11.

Finally, we consider dimension 16. The situation here is more complicated since there are 5 inequivalent real Hadamard matrices of that order. Therefore, with the aid of our construction (described in the proof of Theorem 3.5) we can obtain 5, locally inequivalent, parametrized families of complex Hadamard matrices. The fact that parametric families stemming from inequivalent Hadamard matrices are locally inequivalent can be proved by the same argument as in Proposition 3.8.

It is known that the orbit of the Fourier matrix F_{16} passes through one of the 5 inequivalent real Hadamard matrices, namely the matrix $F_2 \otimes F_2 \otimes F_2 \otimes F_2$. Unfortunately we do not know how the other 4 real Hadamard matrices are related to F_{16} or to the recently constructed "spectral set" matrix S_{16} in [8]. However, as we mentioned before, H_8 can be parametrized in at least two essentially different ways, and that is exactly why we conjecture that the parametrized complex Hadamard matrices constructed by Theorem 3.5 are, at least locally, new.

4. Constructing complex Hadamard matrices from conference matrices

The aim of this section is to describe another general method for constructing parametrized complex Hadamard matrices. First we recall a well-known and widely studied class of matrices:

Definition 4.1. A conference matrix of order N is a square $N \times N$ matrix C, satisfying $CC^{\mathrm{T}} = C^{\mathrm{T}}C = (N-1)I$, $C_{ii} = 0$, $i = 1, 2, \dots, N$ and $C_{ij} \in \{-1, 1\}$ for $i \neq j$.

It is easy to see that for a given conference matrix C either multiplying any row or column by -1, or permuting the rows and columns of C with the same permutation matrix P (i.e. considering PCP^{T} instead of C) we get a conference matrix again. Conference matrices related in these two ways are called equivalent. It is a well-known fact that real conference matrices lead to an obvious construction of Hadamard matrices. Whenever C is a real *symmetric* conference matrix, then '$H = I + \mathbf{i}C$' is a complex Hadamard matrix. (For skew-symmetric conference matrices the formula '$H = I - C$' is used.) In the rest of this paper we will refer to the '$H = I + \mathbf{i}C$' construction as the *conference matrix construction*. It is clear that equivalent conference matrices give rise to equivalent Hadamard matrices. For a survey on conference matrices see e.g. [2] or [4]. There are infinitely many orders for which a symmetric conference matrix exists, however it is still an open problem to give a full characterization of them; it is well known that the order of a conference matrix must be even, moreover the order of a *symmetric* conference matrix must be $N = 4k + 2$ for some nonnegative integer k. However this condition is not sufficient due to a negative result proved by Raghavarao in [9]. In particular, if N is the order of a symmetric conference matrix, then $N - 1$ must be the sum of two squares. For a more or less up-to-date list of the orders of the known conference matrices see the last sections of [10].

Next we prove a general method for introducing an affine parameter to every complex Hadamard matrix arising from the conference matrix construction. We denote this class of complex Hadamard matrices by D, as D_6 in [11] is exactly a matrix arising from a symmetric conference matrix of order 6. The following statements are analogous to Theorems 3.2 and 3.5.

Theorem 4.1. *Complex Hadamard matrices arising from the conference matrix construction are not of Diţă-type.*

Proof. After dephasing $H = I + \mathbf{i}C$, the core of the resulting matrix will contain -1's in the main diagonal and $\pm\mathbf{i}$'s otherwise, therefore the statement follows from Lemma 2.4. □

Theorem 4.2. *Every complex Hadamard matrix D_N arising from the conference matrix construction admits an affine orbit, i.e. there exists an affine family of complex Hadamard matrices of at least one parameter which contains D_N.*

Proof. The proof is completely elementary, but requires many cases to consider. Let D_N be any matrix arising from the conference matrix construction, of order N. Further, we can arrange that it be both symmetric and dephased (of course, after parametrization, D_N can be transformed back to the original form $I + \mathbf{i}C$, and this transformation clearly does not affect the presence of parameters). In [3,11] $D_6^{(1)}(t)$ appeared, as a parametric family of order 6, so we restrict our attention to the next order $N = 4k + 2$, and we suppose that $N \geq 10$. We show that one parameter can be introduced independently of what a conference matrix C was used to construct D_N. Indeed, consider its second ($\mathbf{u}$) and third ($\mathbf{v}$) rows. Because D_N is Hadamard, there are exactly $\frac{N-2}{2}$ places where the entries of $\mathbf{u}$ and $\mathbf{v}$ differ only by a sign. Multiply these entries by $e^{\mathbf{i}t}$. Now consider the second and the third *column* of D_N, and multiply those entries by $e^{-\mathbf{i}t}$ which differ by a sign row-wise. We prove that the obtained 1-parameter matrix $D_N^{(1)}(t)$ will still be Hadamard. We show that the modified rows of D_N are orthogonal to each and every other row of $D_N^{(1)}(t)$ independently of t. There are many trivial cases, but there are two which require some extra considerations:

Case 1: We proceed to show that both $\mathbf{u}$ and $\mathbf{v}$ are orthogonal to any unchanged row. After permuting the rows and the columns of $D_N^{(1)}(t)$, we can suppose that it has the following (symmetric) form as beneath; it is also clear, that (by taking the Hermitian transpose of $D_N^{(1)}(t)$ if it is necessary and, again, permuting) the imaginary elements in the upper left 3×3 submatrix are $\mathbf{i}$'s. Now consider any unchanged row, other than the first row of $D_N^{(1)}(t)$; its first three elements could be either $(1, \mathbf{i}, \mathbf{i})$ or $(1, -\mathbf{i}, -\mathbf{i})$ respectively. We consider the first case, the other could be treated exactly in the same way. Below in the figure one can see a sketch of $D_N^{(1)}(t)$.

$$
\begin{array}{c}
\\ \mathbf{(u)} \\ \mathbf{(v)} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\
\end{array}
\left[\begin{array}{c|cc|c|ccc|ccc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & \dots & 1 & 1 & \dots & 1 & 1 & \dots & 1 & 1 & \dots & 1 \\ \hline
1 & -1 & \mathbf{i} & \mathbf{i} & \mathbf{i} & \dots & \mathbf{i} & \mathbf{i}e^{\mathbf{i}t} & \dots & \mathbf{i}e^{\mathbf{i}t} & -\mathbf{i}e^{\mathbf{i}t} & \dots & -\mathbf{i}e^{\mathbf{i}t} & -\mathbf{i} & \dots & -\mathbf{i} \\
1 & \mathbf{i} & -1 & \mathbf{i} & \mathbf{i} & \dots & \mathbf{i} & -\mathbf{i}e^{\mathbf{i}t} & \dots & -\mathbf{i}e^{\mathbf{i}t} & \mathbf{i}e^{\mathbf{i}t} & \dots & \mathbf{i}e^{\mathbf{i}t} & -\mathbf{i} & \dots & -\mathbf{i} \\ \hline
1 & \mathbf{i} & \mathbf{i} & -1 & a & & b & c & & d & e & & f & g & & h \\ \hline
1 & \mathbf{i} & \mathbf{i} & & -1 & & & & & & & & & & & \\
\vdots & \vdots & \vdots & & & \ddots & & & & & & & & & & \\
1 & \mathbf{i} & \mathbf{i} & & & & -1 & & & & & & & & & \\ \hline
1 & \mathbf{i}e^{-\mathbf{i}t} & -\mathbf{i}e^{-\mathbf{i}t} & & & & & -1 & & & & & & & & \\
\vdots & \vdots & \vdots & & & & & & \ddots & & & & & & & \\
1 & \mathbf{i}e^{-\mathbf{i}t} & -\mathbf{i}e^{-\mathbf{i}t} & & & & & & & -1 & & & & & & \\ \hline
1 & -\mathbf{i}e^{-\mathbf{i}t} & \mathbf{i}e^{-\mathbf{i}t} & & & & & & & & -1 & & & & & \\
\vdots & \vdots & \vdots & & & & & & & & & \ddots & & & & \\
1 & -\mathbf{i}e^{-\mathbf{i}t} & \mathbf{i}e^{-\mathbf{i}t} & & & & & & & & & & -1 & & & \\ \hline
1 & -\mathbf{i} & -\mathbf{i} & & & & & & & & & & & -1 & & \\
\vdots & \vdots & \vdots & & & & & & & & & & & & \ddots & \\
1 & -\mathbf{i} & -\mathbf{i} & & & & & & & & & & & & & -1
\end{array}\right]
$$

In the figure above the fourth row is marked as the one considered. In this row, starting with $(1, \mathbf{i}, \mathbf{i})$, let a, c, e and g denote the number of $\mathbf{i}$'s, while b, d, f and h denote the number of $-\mathbf{i}$'s in the corresponding "cells". Note that by taking the inner product of the first three rows of D_N, one can calculate how many vertical pairs $(\mathbf{i}, \mathbf{i})$, $(\mathbf{i}, -\mathbf{i})$, $(-\mathbf{i}, \mathbf{i})$ and $(-\mathbf{i}, -\mathbf{i})$ there can be in rows $(\mathbf{u}, \mathbf{v})$. The following equations are necessary and sufficient conditions for the orthogonality of the first three rows of $D_N^{(1)}(t)$, independently of t.

$$b = \frac{N-2}{4} - 2 - a \tag{10}$$

$$d = \frac{N-2}{4} - c \tag{11}$$

$$f = \frac{N-2}{4} - e \tag{12}$$

$$h = \frac{N-2}{4} - g. \tag{13}$$

The number of $\mathbf{i}$'s is $\frac{N-2}{2}$ in every row, so we have

$$a + c + e + g = \frac{N-2}{2} - 2. \tag{14}$$

Since $D_N^{(1)}(0)$ is Hadamard the fourth row is orthogonal to $\mathbf{u}$, prior to modification, and we get

$$2 + a - b + c - d - e + f - g + h = 0. \tag{15}$$

Now put (10)–(13) into (15), yielding

$$a + c = e + g - 2. \tag{16}$$

Similarly, the fourth row of $D_N^{(1)}(0)$ is orthogonal to $\mathbf{v}$, prior to modification, and we get

$$2 + a - b - c + d + e - f - g + h = 0. \tag{17}$$

Substituting (10)–(13) into (17) implies

$$a + e = c + g - 2. \tag{18}$$

Finally, use (14) in (16) and (18) to obtain

$$a + c = a + e \left(= \frac{N-10}{4} \right). \tag{19}$$

This last equation implies that $c = e$, and from (11) and (12) $d = f$ immediately follows. Now it is only a matter of simple computation, to show that both $\mathbf{u}$ and $\mathbf{v}$ are orthogonal to the chosen row of $D_N^{(1)}(t)$, independently of the value of t.

Case 2: We need to prove that a row with $\mathrm{e}^{-\mathbf{i}t}$-type parameters is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. Consider a row starting with $(1, \mathbf{i}\mathrm{e}^{-\mathbf{i}t}, -\mathbf{i}\mathrm{e}^{-\mathbf{i}t})$ (the case $(1, -\mathbf{i}\mathrm{e}^{-\mathbf{i}t}, \mathbf{i}\mathrm{e}^{-\mathbf{i}t})$ can be treated similarly). The columns of $D_N^{(1)}(t)$ can be permuted so that it takes the form:

$$\begin{array}{c} \\ (\mathbf{u}) \\ (\mathbf{v}) \\ \\ \end{array} \left[\begin{array}{c|cc|c|ccc|ccc|ccc|ccc} 1 & 1 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & -1 & \mathbf{i} & \mathbf{i}\mathrm{e}^{\mathbf{i}t} & \mathbf{i} & \dots & \mathbf{i} & \mathbf{i}\mathrm{e}^{\mathbf{i}t} & \dots & \mathbf{i}\mathrm{e}^{\mathbf{i}t} & -\mathbf{i}\mathrm{e}^{\mathbf{i}t} & \dots & -\mathbf{i}\mathrm{e}^{\mathbf{i}t} & -\mathbf{i} & \dots & -\mathbf{i} \\ 1 & \mathbf{i} & -1 & -\mathbf{i}\mathrm{e}^{\mathbf{i}t} & \mathbf{i} & \dots & \mathbf{i} & -\mathbf{i}\mathrm{e}^{\mathbf{i}t} & \dots & -\mathbf{i}\mathrm{e}^{\mathbf{i}t} & \mathbf{i}\mathrm{e}^{\mathbf{i}t} & \dots & \mathbf{i}\mathrm{e}^{\mathbf{i}t} & -\mathbf{i} & \dots & -\mathbf{i} \\ \hline 1 & \mathbf{i}\mathrm{e}^{-\mathbf{i}t} & -\mathbf{i}\mathrm{e}^{-\mathbf{i}t} & -1 & \mathrm{a} & & \mathrm{b} & \mathrm{c} & & \mathrm{d} & \mathrm{e} & & \mathrm{f} & \mathrm{g} & & \mathrm{h} \end{array} \right]$$

where the fourth row is the one under consideration, and a, b, c, d, e, f, g and h have the same meaning as in Case 1. Again, we express the orthogonality of the first three rows of $D_N^{(1)}(t)$ as:

$$b = \frac{N-2}{4} - 1 - a \tag{20}$$

$$d = \frac{N-2}{4} - 1 - c \tag{21}$$

$$f = \frac{N-2}{4} - e \tag{22}$$

$$h = \frac{N-2}{4} - g. \tag{23}$$

And the allowed number of $\mathbf{i}$'s is

$$a + c + e + g = \frac{N-2}{2} - 1. \tag{24}$$

Again, as $\mathbf{u}$ and $\mathbf{v}$ are orthogonal to the considered parametrized row for $t = 0$, one gets

$$a - b + c - d - e + f - g + h = 0 \tag{25}$$

and

$$2 + a - b - c + d + e - f - g + h = 0. \tag{26}$$

By substituting (20)–(23) into (25) and (26) we get

$$a + c = e + g - 1 \tag{27}$$

and

$$a + e = c + g - 1. \tag{28}$$

Again, use (24) in (27) and (28) to obtain

$$a + c = a + e \left(= \frac{N-6}{4} \right). \tag{29}$$

This last equation implies $c = e$ and from (21) and (22) $d = f - 1$ follows. By applying these identities it is only a matter of simple computation that the considered $\mathrm{e}^{\mathbf{i}t}$-type row is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, independently of t.

Other cases: Considering any other pair of rows in $D_N^{(1)}(t)$ it is trivial to show that they are orthogonal to each other. This completes the proof. □

The last theorem allows introduction of one parameter for every complex Hadamard matrix arising from the conference matrix construction. However the following more complex method seems to be working in general. In some sense this is a natural generalization of Theorem 3.5.

Construction 4.3. *Take an arbitrary dephased, symmetric complex Hadamard matrix D arising from the conference matrix construction, of order N. Use Theorem 4.2 method, involving a pair of rows (and the corresponding columns), to introduce a free parameter in D. Then select another pair of "suitable" rows (and the corresponding columns), if possible, in order to use Theorem 4.2 again to introduce another parameter. A "suitable" pair of rows must satisfy two conditions:*

(i) *all its vertical pairs of entries are formed (taking into account already existing parameters, if any) either by identical entries or entries being negative with respect to each other (except for the inevitable* $(-1, *)$ *and* $(*, -1)$ *pairs);*

(ii) *it has a vertical pair* $(\mathbf{i}, -\mathbf{i})$ *or* $(-\mathbf{i}, \mathbf{i})$*, not yet parametrized.*

If a suitable pair of rows is found, introduce a new parameter in it (and in the corresponding columns) in the manner analogous to that of Theorem 4.2, i.e. multiplying pairs of opposite entries by $\mathrm{e}^{\pm\mathbf{i}t}$*. Repeat this procedure as long as there exist suitable pairs of rows.*

The two conditions above seem to be *necessary* in the following sense. Condition (i) guarantees that the first row of D and the rows of a newly parametrized pair are all orthogonal to each other, while condition (ii) is required to ensure that the newly introduced parameter does not depend on earlier ones. It is not clear, however, that they are indeed sufficient, i.e. we do not have a formal proof that the arising parametric matrices remain Hadamard. Also, if several suitable pairs of rows exist at one stage then it is not clear which pair to favour over the others. The maximal number of parameters that can be introduced in this way is $\frac{N}{2} - 1$ (because the first row definitely does not have a companion to make a pair with). We used this construction to obtain the families stemming from D_{10} and D_{14} below, and the well-known family $D_6^{(1)}(t)$ of [11] also arises in this way. These examples suggest the following

Conjecture 4.4. Construction 4.3 *leads to Hadamard matrices after each step, and for* $N \geq 14$ *the maximum number,* $\frac{N}{2} - 1$*, of parameters can be introduced.*

Remark 4.5. The construction yields only $\frac{N}{2} - 2$ parameters for D_6 and D_{10}, because condition (ii) fails to hold due to the matrices being "too small".

In the recent catalogue [11] only Diţă-type matrices were considered in dimensions $N = 10$ and 14. In view of Theorems 4.1 and 4.2 we can now present new parametric families of complex Hadamard matrices of these orders. Our first example is the matrix D_{10} which is constructed from the only (up to equivalence) conference matrix of order 10.

$$D_{10} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -\mathbf{i} & -\mathbf{i} & -\mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} & \mathbf{i} & \mathbf{i} \\ 1 & -\mathbf{i} & -1 & \mathbf{i} & \mathbf{i} & -\mathbf{i} & -\mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} \\ 1 & -\mathbf{i} & \mathbf{i} & -1 & -\mathbf{i} & \mathbf{i} & -\mathbf{i} & \mathbf{i} & -\mathbf{i} & \mathbf{i} \\ 1 & -\mathbf{i} & \mathbf{i} & -\mathbf{i} & -1 & \mathbf{i} & \mathbf{i} & -\mathbf{i} & \mathbf{i} & -\mathbf{i} \\ 1 & -\mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} & -1 & \mathbf{i} & \mathbf{i} & -\mathbf{i} & -\mathbf{i} \\ 1 & \mathbf{i} & -\mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} & -1 & -\mathbf{i} & -\mathbf{i} & \mathbf{i} \\ 1 & \mathbf{i} & -\mathbf{i} & \mathbf{i} & -\mathbf{i} & \mathbf{i} & -\mathbf{i} & -1 & \mathbf{i} & -\mathbf{i} \\ 1 & \mathbf{i} & \mathbf{i} & -\mathbf{i} & \mathbf{i} & -\mathbf{i} & -\mathbf{i} & \mathbf{i} & -1 & -\mathbf{i} \\ 1 & \mathbf{i} & \mathbf{i} & \mathbf{i} & -\mathbf{i} & -\mathbf{i} & \mathbf{i} & -\mathbf{i} & -\mathbf{i} & -1 \end{bmatrix}. \tag{30}$$

We have already seen that D_{10} is a non-Diţă-type matrix and according to Theorem 4.2 it has an affine orbit stemming from it. Moreover, by Construction 4.3 we could introduce 3 parameters (we chose the "suitable" pairs of rows by an ad hoc method, as follows: (2, 10), (3, 9) and (5, 7)).

$$D_{10}^{(3)}(a, b, c) = D_{10} \circ \mathbf{EXP}\left(\mathbf{i} \cdot R_{D_{10}^{(3)}}(a, b, c)\right) \tag{31}$$

where

$$R_{D_{10}^{(3)}}(a,b,c) = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & a-b & a & -c & \bullet & -c & a & a-b & \bullet \\ \bullet & -a+b & \bullet & b & -c & \bullet & -c & b & \bullet & -a+b \\ \bullet & -a & -b & \bullet & \bullet & \bullet & \bullet & \bullet & -b & -a \\ \bullet & c & c & \bullet & \bullet & \bullet & \bullet & \bullet & c & c \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & c & c & \bullet & \bullet & \bullet & \bullet & \bullet & c & c \\ \bullet & -a & -b & \bullet & \bullet & \bullet & \bullet & \bullet & -b & -a \\ \bullet & -a+b & \bullet & b & -c & \bullet & -c & b & \bullet & -a+b \\ \bullet & \bullet & a-b & a & -c & \bullet & -c & a & a-b & \bullet \end{bmatrix}. \tag{32}$$

We checked with a computer that $D_{10}^{(3)}(a,b,c)$ is indeed Hadamard. The defect (in the sense of [11]) of D_{10} is 16, so we cannot be sure that $D_{10}^{(3)}(a,b,c)$ is maximal affine (the defect is an upper bound for the dimensionality of a family stemming from D_{10}). It is possible that further parameters can be introduced.

Now we turn to $N = 14$. Our starting point Hadamard matrix, constructed from the only (up to equivalence) conference matrix of order 14, is the following D_{14}.

$$D_{14} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} & -\mathbf{i} & -\mathbf{i} & -\mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} & -\mathbf{i} & \mathbf{i} \\ 1 & \mathbf{i} & -1 & \mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} & -\mathbf{i} & -\mathbf{i} & -\mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} & -\mathbf{i} \\ 1 & -\mathbf{i} & \mathbf{i} & -1 & \mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} & -\mathbf{i} & -\mathbf{i} & -\mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} \\ 1 & \mathbf{i} & -\mathbf{i} & \mathbf{i} & -1 & \mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} & -\mathbf{i} & -\mathbf{i} & -\mathbf{i} & -\mathbf{i} & \mathbf{i} \\ 1 & \mathbf{i} & \mathbf{i} & -\mathbf{i} & \mathbf{i} & -1 & \mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} & -\mathbf{i} & -\mathbf{i} & -\mathbf{i} & -\mathbf{i} \\ 1 & -\mathbf{i} & \mathbf{i} & \mathbf{i} & -\mathbf{i} & \mathbf{i} & -1 & \mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} & -\mathbf{i} & -\mathbf{i} & -\mathbf{i} \\ 1 & -\mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} & -\mathbf{i} & \mathbf{i} & -1 & \mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} & -\mathbf{i} & -\mathbf{i} \\ 1 & -\mathbf{i} & -\mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} & -\mathbf{i} & \mathbf{i} & -1 & \mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} & -\mathbf{i} \\ 1 & -\mathbf{i} & -\mathbf{i} & -\mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} & -\mathbf{i} & \mathbf{i} & -1 & \mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} \\ 1 & \mathbf{i} & -\mathbf{i} & -\mathbf{i} & -\mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} & -\mathbf{i} & \mathbf{i} & -1 & \mathbf{i} & -\mathbf{i} & \mathbf{i} \\ 1 & \mathbf{i} & \mathbf{i} & -\mathbf{i} & -\mathbf{i} & -\mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} & -\mathbf{i} & \mathbf{i} & -1 & \mathbf{i} & -\mathbf{i} \\ 1 & -\mathbf{i} & \mathbf{i} & \mathbf{i} & -\mathbf{i} & -\mathbf{i} & -\mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} & -\mathbf{i} & \mathbf{i} & -1 & \mathbf{i} \\ 1 & \mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} & -\mathbf{i} & -\mathbf{i} & -\mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} & -\mathbf{i} & \mathbf{i} & -1 \end{bmatrix}. \tag{33}$$

Again, this is a non-Diţă-type matrix, and a 6-parameter affine family stems from it (which we constructed with the aid of Construction 4.3; the considered "suitable" pairs of rows were (2, 3), (4, 5), (6, 9), (7, 13), (8, 12) and (11, 14)). The defect of the matrix is 36 so it might be possible to introduce further parameters. We do not claim that *all* the matrices contained in the family stemming from D_{14} are non-Diţă-type, but it is obviously true in a small neighborhood of it.

$$D_{14}^{(6)}(a,b,c,d,e,f) = D_{14} \circ \mathbf{EXP}\left(\mathbf{i} \cdot R_{D_{14}^{(6)}}(a,b,c,d,e,f)\right) \tag{34}$$

where

$$R_{D_{14}^{(6)}}(a,b,c,d,e,f)$$

$$= \begin{bmatrix}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & a-b & a-b & -c & a & -e & -c & \bullet & a & -e & a & a \\
\bullet & \bullet & \bullet & a-b & a-b & -c & a & -e & -c & \bullet & a & -e & a & a \\
\bullet & b-a & b-a & \bullet & \bullet & b & b & -e & b & \bullet & -f & -e & b & -f \\
\bullet & b-a & b-a & \bullet & \bullet & b & b & -e & b & \bullet & -f & -e & b & -f \\
\bullet & c & c & -b & -b & \bullet & c-d & c & \bullet & \bullet & \bullet & c & c-d & \bullet \\
\bullet & -a & -a & -b & -b & d-c & \bullet & d-e & d-c & \bullet & d-f & d-e & \bullet & d-f \\
\bullet & e & e & e & e & -c & e-d & \bullet & -c & \bullet & -f & \bullet & e-d & -f \\
\bullet & c & c & -b & -b & \bullet & c-d & c & \bullet & \bullet & \bullet & c & c-d & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & -a & -a & f & f & \bullet & f-d & f & \bullet & \bullet & \bullet & f & f-d & \bullet \\
\bullet & e & e & e & e & -c & e-d & \bullet & -c & \bullet & -f & \bullet & e-d & -f \\
\bullet & -a & -a & -b & -b & d-c & \bullet & d-e & d-c & \bullet & d-f & d-e & \bullet & d-f \\
\bullet & -a & -a & f & f & \bullet & f-d & f & \bullet & \bullet & \bullet & f & f-d & \bullet
\end{bmatrix}. \tag{35}$$

To summarize the cases $N = 10, 14$ we conclude that

Corollary 4.6. *The families $D_{10}^{(3)}(a,b,c)$ and $D_{14}^{(6)}(a,b,c,d,e,f)$ are locally inequivalent to the families contained in* [11].

Remark 4.7. Note that D_{10} and D_{14} are unique in the sense that according to [4] the number of inequivalent symmetric conference matrices is 1 for orders $N = 2, 6, 10, 14$ and 18, while already for order $N = 26$ there exist 4 inequivalent symmetric conference matrices. This implies that in higher dimensions it may be possible to construct locally inequivalent families stemming from inequivalent starting point matrices. Recall that there is no conference matrix of order 22 and 34 due to Raghavarao's theorem [9].

Let us summarize our results. In this paper we have described two general constructions of parametric families of complex Hadamard matrices. We have presented new matrices of order 10, 12 and 14, thus we have supplemented the recent catalogue of complex Hadamard matrices of small orders [11]. We pointed out that certain real Hadamard matrices cannot be constructed using Diţă's formula, so in order to find all inequivalent complex Hadamard matrices of a given order one should look for and resort to other construction methods.

It would be interesting to see whether the hereby presented families can be extended with further parameters. It also remains to be checked whether Construction 4.3 leads indeed to parametric families of complex Hadamard matrices in general.

Acknowledgements

The author thanks Máté Matolcsi for many insightful comments. He is also greatly indebted to the referee whose suggestions have substantially improved the presentation of the paper.

References

[1] K. Beauchamp, R. Nicoara, Orthogonal maximal Abelian $*$-subalgebras of the 6×6 matrices. math.OA/0609076, 2006, Preprint.
[2] R. Craigen, Regular conference matrices and complex Hadamard matrices, Util. Math. 45 (1994) 65–69.
[3] P. Diţă, Some results on the parametrization of complex Hadamard matrices, J. Phys. A 37 (20) (2004) 5355–5374.
[4] C. Elster, A. Neumaier, Screening by conference designs, Biometrika 82 (1995) 589–602.
[5] U. Haagerup, Orthogonal maximal Abelian $*$-subalgebras of the $n \times n$ matrices and cyclic n-roots, in: Operator Algebras and Quantum Field Theory (Rome), MA International Press, Cambridge, 1997, pp. 296–322.
[6] M. Kolountzakis, M. Matolcsi, Tiles with no spectra, Forum Math. 18 (2006) 519–528.
[7] M. Matolcsi, F. Szöllősi, Towards a classification of 6×6 complex Hadamard matrices. math.CA/0702043, 2007, Preprint.
[8] M. Matolcsi, J. Réffy, F. Szöllősi, Constructions of complex Hadamard matrices via tiling Abelian groups, Open Syst. Inf. Dyn. (2007) (in press).
[9] D. Raghavarao, Constructions and Combinatorial Problems in Designs of Experiments, Dover, New York, 1988.
[10] J. Seberry, X. Zhang, Some orthogonal matrices constructed by strong Kronecker multiplication, Australas. J. Combin. 7 (1993) 213–224.
[11] W. Tadej, K. Życzkowski, A concise guide to complex Hadamard matrices, Open Syst. Inf. Dyn. 13 (2006) 133–177.