A note on $D$-spaces and infinite unions

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Abstract

It is shown that if $X$ is a countably compact space that is the union of a countable family of $D$-spaces, then $X$ is compact. This gives a positive answer to Arhangel’skii’s problem [A.V. Arhangel’skii, $D$-spaces and finite unions, Proc. Amer. Math. Soc. 132 (7) (2004) 2163–2170]. In this note, we also obtain a result that if a regular space $X$ is sequential and has a point-countable $k$-network, then $X$ is a $D$-space.

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1. Introduction

The notion of $D$-space was introduced by van Douwen [5]. A neighborhood assignment for a space $X$ is a function $\phi$ from $X$ to the topology of the space $X$, such that $x \in \phi(x)$ for any $x \in X$. A space $X$ is called a $D$-space, if for any neighborhood assignment $\phi$ for $X$ there exists a closed discrete subset $D$ of $X$, such that $X = \bigcup \{\phi(d): d \in D\}$. By results of [3] and [4], we know that all metrizable spaces, all Moore spaces, and all strong $\Sigma$-spaces are $D$-spaces. From [2], we know that every space with a point-countable base is a $D$-space.

About the unions of $D$-spaces, we do not know whether the union of two $D$-spaces is a $D$-space. But we know a little about the union of special $D$-spaces (such as metrizable spaces or spaces with a point-countable base). In [2], Arhangel’skii and Buzyakova proved that if a space $X$ is the union of a finite family of metrizable subspaces, then $X$ is a $D$-space. In [1], Arhangel’skii proved that if a regular space $X$ is the union of a finite family of subspaces with a point-countable base, then $X$ is a $D$-space.

On the other hand, we know little about the union of countable family of special $D$-spaces. There exists a Hausdorff, locally compact, locally countable, separable, first countable, submetrizable, $\sigma$-discrete, countable extent space that is not a $D$-space (cf. [6]). So Arhangel’skii raised the following problem (cf. [1]) in 2004: Suppose $X$ is a countably compact space that is the union of a countable family of $D$-spaces. Is $X$ compact? In [8], Gruenhage proved the case of finite unions. His result is that if $X$ is a countably compact space that is the union of a finite family of $D$-spaces,
then \( X \) is compact. In this note, we prove that \( X \) is also compact, if \( X \) is countably compact and is the union of an infinite family of \( D \)-spaces. So this conclusion gives a positive answer to the problem raised by Arhangel’skii.

In [2], it was shown that every space with a point-countable base is a \( D \)-space. The author proved that \( X \) is also a \( D \)-space, if \( X \) has a point-countable weak base (cf. [9]). We may know some basic property of weak base from [10] and [11]. By the Proposition 4.7 of [10], we know that if \( X \) has a point-countable weak base, then \( X \) is a sequential space with a point-countable \( k \)-network. In this note, we try to discuss the \( D \)-property of spaces which are sequential and have a point-countable \( k \)-network. We obtain the following result: If \( X \) is a regular sequential space with a point-countable \( k \)-network, then \( X \) is a \( D \)-space. In notation and terminology we will follow [7].

All the spaces in this note are at least \( T_1 \)-spaces. The set of all natural numbers is denoted by \( N \), and \( \omega \) is \( N \cup \{0\} \).

2. Main results

Recall that a space \( X \) is linearly Lindelöf if every increasing open cover of \( X \) has a countable subcover. \( X \) is countably compact if every countable open cover of \( X \) has a finite subcover. So we know that every countably compact linearly Lindelöf space is compact.

The following Lemma 1 appeared in [8].

**Lemma 1.** A space \( X \) is linearly Lindelöf iff whenever \( \mathcal{U} \) is an open cover of \( X \) of cardinality \( \kappa \) and \( \mathcal{U} \) has no subcover of cardinality \( < \kappa \), then \( \text{cf} (\kappa) \leq \omega \).

In [8], it is proved that if \( X \) has countable extent and can be written as the union of finitely many \( D \)-spaces, then \( X \) is linearly Lindelöf. Part of the idea appeared in the following Theorem 2 is inspired by the proof of Theorem 4.2 of [8].

**Theorem 2.** Suppose \( X \) is a countably compact space that is the union of a countable family of \( D \)-spaces. Then \( X \) is a compact space.

**Proof.** Let \( X = \bigcup \{X_i: i \in N\} \), where \( X_i \) is a \( D \)-space for each \( i \in N \). Since \( X \) is countably compact, we only need to prove that \( X \) is linearly Lindelöf. Suppose by way of contradiction that \( X \) is not linearly Lindelöf. Then by Lemma 1, there is an open cover \( \mathcal{U} = \{U_\alpha: \alpha < \kappa\} \) of some cardinality \( \kappa \) with \( \text{cf} (\kappa) > \omega \) and such that \( \mathcal{U} \) has no subcover of cardinality \( < \kappa \). We may assume that \( \text{cf} (\kappa) = \kappa \).

For each \( x \in X \), let \( \alpha_x \) be least such that \( x \in U_{\alpha_x} \) and consider the neighborhood assignment defined by \( \phi(x) = U_{\alpha_x} \). For each \( i \in N \), there is a relative discrete closed subset \( D_i \) of \( X_i \) such that \( \{\phi(d): d \in D_i\} \) covers \( X_i \). Since \( \mathcal{U} \) has no subcover of smaller cardinality and \( \text{cf} (\kappa) > \omega \), then there is some \( i_0 \in N \), such that \( \|\{\alpha_d: d \in D_{i_0}\}\| = \kappa \).

We let \( F_1 = D_{i_0} \setminus D_{i_0} \). Then \( F_1 \) is a closed set of \( X \) and \( F_1 \cap X_{i_0} = \emptyset \) following from that \( D_{i_0} \) is a relative discrete set of \( X_{i_0} \). If \( F_1 = \emptyset \), then \( D_{i_0} \) will be a closed discrete set of \( X \), and \( |D_{i_0}| \geq \kappa \). This contradicts with countably compact property of \( X \). So \( F_1 \) is a non-empty closed set of \( X \).

**Claim.** \( \mathcal{U} \) has no subfamily \( \mathcal{U}_1 \) of cardinality \( < \kappa \), such that \( F_1 \subset \cup \mathcal{U}_1 \).

The following is the proof of the Claim.

Otherwise, we have some \( \beta < \kappa \), such that \( F_1 \subset \bigcup \{U_\alpha: \alpha < \beta\} = V_1 \). Then \( D_{i_0} \setminus V_1 = \overline{D_{i_0}} \setminus V_1 \) is a closed discrete set of \( X \). Thus \( |D_{i_0} \setminus V_1| < \omega \). So \( \|\{\alpha_d: d \in D_{i_0} \cap V_1\}\| = \kappa \). Thus we can get a \( \alpha_{d_0} \), such that \( d_0 \in D_{i_0} \cap V_1 \) and \( \alpha_{d_0} > \beta \). On the other hand, \( d_0 \in V_1 \), there is some \( \alpha_0 < \beta \), such that \( d_0 \in U_{\alpha_0} \). Thus we have \( \alpha_{d_0} \leq \alpha_0 < \beta \). This contradicts with \( \alpha_{d_0} > \beta \). So we have proved the Claim.

To assist the reader, we will give the proof of next step. Let \( A_1 = \{i_0\} \), so \( F_1 = \bigcup \{F_1 \cap X_i: i \in N \setminus A_1\} \). \( F_1 \) is a closed set of \( X \), so \( F_1 \cap X_i \) is a closed subset of subspace \( X_i \) for each \( i \in N \setminus A_1 \). We know that \( D \)-spaces are hereditary with closed sets. So \( F_1 \cap X_i \) is a \( D \)-space for each \( i \in N \setminus A_1 \). Thus \( F_1 \) is also the union of a countable family of \( D \)-spaces. In the following we will repeat the method which is used on \( X \).

For each \( x \in F_1 \cap X_i \), where \( i \in N \setminus A_1 \). Let \( \alpha_x \) be least such that \( x \in U_{\alpha_x} \) and consider the neighborhood assignment defined by \( \phi(x) = U_{\alpha_x} \cap (F_1 \cap X_i) \). For each \( i \in N \setminus A_1 \), there is a relative closed discrete subset \( D_i \) of \( X_i \cap F_1 \) such that \( F_1 \cap X_i \subset \bigcup \{\phi(d): d \in D_i\} \). Since \( \mathcal{U} \) has no subfamily \( \mathcal{U}_1 \) of smaller cardinality such that \( F_1 \subset \cup \mathcal{U}_1 \),
and \( cf(\kappa) > \omega \). Then there is some \( i_1 \in N \setminus A_1 \), such that \(|\{\alpha_\xi: d \in D_{i_1}\}| = \kappa \). \( D_{i_1} \subset F_1 \), so \( \overline{D_{i_1}} \subset F_1 \). We let \( F_2 = \overline{D_{i_1}} \setminus D_{i_1} \). Thus \( F_2 \subset F_1 \). By the same proof as the Claim, we may know that \( F_2 \) is a non-empty closed set of \( X \) and \( F_2 \cap X_{i_1} = \emptyset \). And also \( \mathcal{U} \) has no subfamily \( \mathcal{U}_2 \) of cardinality \( < \kappa \), such that \( F_2 \subset \cup \mathcal{U}_2 \). Let \( A_2 = A_1 \cup \{i_1\} \). So we have \( F_1 \cap X_{i_0} = \emptyset \), \( F_2 \cap X_{i_1} = \emptyset \), and \( i_0 \neq i_1 \).

Let \( \alpha < \omega_1 \), and suppose for every \( \beta < \alpha \), we have a non-empty closed set \( F_\beta \), and a set \( A_\beta \subset N \), and a relative closed discrete set \( D_\beta \) of \( X_\beta, i_\beta \subset N \) for each \( r < \beta \). Satisfying the following conditions:

1. \( \mathcal{U} \) has no subfamily \( \mathcal{U}_\beta \) of cardinality \( < \kappa \), such that \( F_\beta \subset \cup \mathcal{U}_\beta \), and \( X_\beta \cap F_{\beta+1} = \emptyset \) (\( \beta + 1 < \alpha \)), \( F_\beta = \cup \{X_i \cap F_\beta: i \in N \setminus A_\beta\} \);
2. \( i_\beta \neq i_\beta \), if \( \beta_1 \neq \beta_2 \) and \( \beta_1, \beta_2 < \alpha \);
3. \( D_\beta \) is a relative closed discrete set of \( F_\beta \cap X_{i_\beta} \), where \( i_\beta \in N \setminus A_{r'} \) for each \( r' < r \);
4. If \( \beta = r + 1 \), then \( F_\beta = \overline{D_{i_\beta}} \setminus D_{i_\beta} \), and \( A_\beta = A_r \cup \{i_\beta\} \);
5. If \( \beta \) is a limit ordinal, then \( F_\beta = \cap \{F_r: r < \beta\} \) and \( A_\beta = \cup \{A_i: r < \beta\} \).

For each \( \beta < \alpha \) we will define sets \( F_\alpha, A_\alpha \) and \( D_\beta \) that satisfy the above conditions.

Firstly, \( \alpha = \beta + 1 \) for some ordinal \( \beta \).

We know that \( F_\beta = \cup \{X_i \cap F_\beta: i \in N \setminus A_\beta\} \). For each \( i \in N \setminus A_\beta \), \( X_i \cap F_\beta \) is a closed subset of subspace \( X_i \). For each \( x \in F_\beta \cap X_i \), let \( \alpha_x \) be least such that \( x \in U_{\alpha_x} \) and consider the neighborhood assignment defined by \( \phi(x) = U_{\alpha_x} \cap (X_i \cap F_\beta) \). For each \( i \in N \setminus A_\beta \), \( X_i \cap F_\beta \) is a \( D \)-space. Then there is a relative closed discrete subset \( D_i \) of \( X_i \cap F_\beta \) such that \( F_\beta \cap X_i \subset \cup \{\phi(d) : d \in D_i\} \). Since \( \mathcal{U} \) has no subfamily \( \mathcal{U}_\beta \) of smaller cardinality such that \( F_\beta \subset \cup \mathcal{U}_\beta \), and \( cf(\kappa) > \omega \). Then there is some \( i_\beta \in N \setminus A_\beta \), such that \(|\{\alpha_\xi: d \in D_{i_\beta}\}| = \kappa \). \( D_{i_\beta} \subset F_\beta \), so \( \overline{D_{i_\beta}} \subset F_\beta \).

We let \( F_\alpha = \overline{D_{i_\beta}} \setminus D_{i_\beta} \). Thus \( F_\alpha \subset F_\beta \). By the same proof as the Claim, we may know that \( F_\alpha \) is a non-empty closed set of \( X \) and \( F_\alpha \cap X_{i_\beta} = \emptyset \), and \( \mathcal{U} \) has no subfamily \( \mathcal{U}_\alpha \) of cardinality \( < \kappa \), such that \( F_\alpha \subset \cup \mathcal{U}_\alpha \). Let \( A_\alpha = A_\beta \cup \{i_\beta\} \).

We know that \( i_\beta \neq A_\beta \). Thus we have \( i_\beta \neq i_\beta \), if \( \beta_1 \neq \beta_2 \) and \( \beta_1, \beta_2 < \alpha \).

Secondly, \( \alpha \) is a limit ordinal.

We let \( F_\alpha = \cap \{F_\beta: \beta < \alpha\} \). Since \( X \) is countably compact, and \( F_\beta \) is a non-empty closed set of \( X \), and \( \alpha < \omega_1 \).

Then we know that \( F_\alpha \) is a non-empty closed set of \( X \). Let \( V \) be any open set of \( X \). If \( F_\alpha \subset V \), then there is some \( \beta < \alpha \), such that \( F_\alpha \subset V \) following from the countably compact property of \( X \). Suppose \( \mathcal{U} \) has a subfamily \( \mathcal{U}_\alpha \) of cardinality \( < \kappa \), such that \( F_\alpha \subset \cup \mathcal{U}_\alpha \). Then there is some \( \beta < \alpha \), such that \( F_\beta \subset F_\beta \subset \cup \mathcal{U}_\beta \). This contradicts with condition (1). So \( \mathcal{U} \) has no subfamily \( \mathcal{U}_\alpha \) of cardinality \( < \kappa \), such that \( F_\alpha \subset \cup \mathcal{U}_\alpha \).

We let \( A_\alpha = \cup \{A_\beta: \beta < \alpha\} \). Then \( F_\alpha = \cup \{F_\beta \cap X_i: i \in N \setminus A_\alpha\} \). For each \( i \in N \setminus A_\alpha \), \( F_\beta \cap X_i \) is a \( D \)-space. We use the same method as the first step, we can get a relative closed discrete set \( D_{i_\alpha} \) of \( F_\alpha \cap X_{i_\alpha} \) for some \( i_\alpha \in N \setminus A_\alpha \), such that \( \overline{D_{i_\alpha}} \setminus D_{i_\alpha} = F_{\alpha+1} \) is a non-empty closed set of \( X \), and \( \mathcal{U} \) has no subfamily \( \mathcal{U}_{\alpha+1} \) of cardinality \( < \kappa \), such that \( F_{\alpha+1} \subset \cup \mathcal{U}_{\alpha+1} \). Let \( A_{\alpha+1} = A_\alpha \cup \{i_\alpha\} \).

From the proof we know that \( F_\alpha, A_\alpha \) and \( D_\beta \) for each \( \beta < \alpha \), satisfying the conditions (1)–(5).

Thus for each \( \alpha < \omega_1 \), we have a \( i_\alpha \in N \). If \( \beta_1, \beta_2 < \omega_1 \), and \( \beta_1 \neq \beta_2 \), then there is some \( \alpha \in \omega_1 \), such that \( \beta_1, \beta_2 < \alpha \) and \( i_\beta_1, i_\beta_2 \subset A_\alpha \). Thus we have that \( i_\beta_1 \neq i_\beta_2 \) by the condition (2). This contradicts with \(|\{i_\alpha: \alpha < \omega_1\}| \leq \omega \).

So \( X \) is linearly Lindelöf. Thus \( X \) is compact. \( \square \)

Next we will discuss the \( D \)-property of a regular regular space with a point-countable \( k \)-network. Let’s recall some definitions. A family \( \mathcal{F} \) of a space \( X \) is a \( k \)-network for \( X \), if for any open set \( U \) in \( X \) and any compact set \( C \subset U \), there exists a finite \( \mathcal{F}' \subset \mathcal{F} \), such that \( C \subset \cup \mathcal{F}' \subset U \) (cf. [10]).

A subset \( U \) of \( X \) is called sequentially open if each sequence in \( X \) converging to a point in \( U \) is eventually in \( U \). A space is called sequential if every sequentially open set of \( X \) is open in \( X \) (cf. [11]).

**Theorem 3.** If a regular space \( X \) is sequential and has a point-countable \( k \)-network, then \( X \) is a \( D \)-space.

**Proof.** Let \( \mathcal{F} \) be the point-countable \( k \)-network of \( X \), and let \( \mathcal{F}_\gamma = \{F: x \in F \in \mathcal{F}\} \). Let \( \phi \) be any neighborhood assignment of \( X \). We may assume \( X = \{x_\gamma: \alpha < \gamma\} \), where \( \gamma \) is a cardinal number. Suppose for each \( \alpha < \beta < \gamma \), we have chosen a closed discrete set \( D_\alpha \) satisfying:

1. \( x_\alpha \in \cup \{\phi(d): d \in \cup \{D_i: i < \alpha\}\} \);
(2) \( \bigcup\{D_\eta; \eta < \alpha\} \) is a closed discrete set of \( X \);
(3) \( D_\alpha \cap (\bigcup \{\phi(d); d \in \bigcup\{D_i; i < \alpha\}\}) = \emptyset \);
(4) For any \( x \in X \setminus \bigcup \{\phi(d); d \in \bigcup\{D_\eta; \eta \leq \alpha\}\} \) and for any \( F \in \mathcal{F} \), if there exists a sequence \( C \subset F \) such that \( C \) converges to \( x \) and \( \overline{F} \subset \phi(x) \), then \( F \cap D_\eta = \emptyset \) for any \( \eta \leq \alpha \).

Before we construct \( D_\beta \), let us show that \( D_\beta^* = \bigcup\{D_\alpha; \alpha < \beta\} \) is a closed discrete set of \( X \).

By condition (2) we have known that for any \( \alpha < \beta \), \( D_\alpha^* = \bigcup\{D_\eta; \eta < \alpha\} \) is a closed discrete set of \( X \). We now show that \( D_\beta^* = \bigcup\{D_\alpha; \alpha < \beta\} \) is a closed discrete set of \( X \).

Firstly, \( \beta \) is a limit ordinal number.

We first show that \( D_\beta^* \) is a closed set of \( X \). Suppose \( D_\beta^* \) is not a closed set of \( X \). Since \( X \) is sequential there exists a sequence \( C \subset D_\beta^* \), such that \( C \) converges to some point \( x \), and \( x \in X \setminus D_\beta^* \). For any \( y \in \phi(D_\beta^*) = \bigcup \{\phi(d); d \in D_\beta^*\} \), let \( \alpha_y \) be the minimal ordinal, such that \( y \in \bigcup \{\phi(d); d \in D_{\alpha_y}\} \). Let \( V_y = (\bigcup \{\phi(d); d \in D_{\alpha_y}\}) \setminus \bigcup \{D_\eta; \eta < \alpha_y\} \), and \( V_y' = (\bigcup \{\phi(d); d \in D_{\alpha_y}\}) \setminus \bigcup \{V_x; x \in V_y\} \), where \( V_y' \) is an open set of \( X \), such that \( y \in V_y' \) and \( |V_y' \cap D_{\alpha_y}| \leq 1 \).

So \( y \in V_y' \), and \( V_y' \) is an open set of \( X \), such that \( |V_y' \cap D_\beta| \leq 1 \). So \( x \neq y \). Thus \( x \in X \setminus \phi(D_\beta^*) \). Since \( C \) converges to \( x \), then there is a subsequence \( C_1 \subset C \), such that \( C_1 \setminus \{x\} \subset \phi(x) \). \( C_1 \setminus \{x\} \) is compact and \( \mathcal{F} \) is a \( k \)-network of \( X \), so there is a finite family \( \mathcal{F}_1 \subset \mathcal{F} \), such that \( C_1 \setminus \{x\} \subset \bigcup \mathcal{F}_1 \subset \phi(x) \). So there exists some \( F \in \mathcal{F}_1 \), such that \( F \) contains a subsequence of \( C_1 \) which converges to \( x \). Thus \( F \cap C_1 \) converges to \( x \) and \( \overline{F} \subset \phi(x) \). This contradicts with condition (4).

So \( D_\beta^* \) is a closed set of \( X \). By the proof we know that \( D_\beta^* \) is a closed discrete set of \( X \).

Secondly, \( \beta = \alpha + 1 \) for some \( \alpha \). So \( D_\beta^* = \bigcup\{D_\alpha; \eta < \beta\} = \bigcup\{D_\eta; \eta \leq \alpha\} = \bigcup\{D_\eta; \eta < \alpha\} \cup D_\alpha \). By assumption, we know that \( \bigcup\{D_\eta; \eta < \alpha\} \) is a discrete closed set of \( X \), so is \( D_\beta^* \).

Let \( U_\beta = \bigcup \{\phi(d); d \in \bigcup\{D_\eta; \eta < \beta\}\} \). Now we will construct \( D_\beta \).

If \( x \in U_\beta \), we let \( D_\beta = \emptyset \). So we assume that \( x \notin U_\beta \). We let \( \mathcal{F}_{x_\beta}^* = \{F^*; x\beta \in F, F \in \mathcal{F}\} \). If \( x \beta \) is a countable family following from that \( \mathcal{F} \) is a point-countable family of \( X \). Enumerate it by prime numbers \( p \). Let \( y_1 = x_\beta \) and take the first member \( F^* \) of \( \mathcal{F}_{x_\beta}^* \) such that \( F^* \setminus (\phi(y_1) \cup U_\beta) \neq \emptyset \). We choose a point \( y_2 \) that \( y_2 \in F^* \setminus (\phi(y_1) \cup U_\beta) \). Then \( F^* \cap \phi(y_2) \) is a family \( \mathcal{F}_{y_2} \) is countable, and we denote \( \mathcal{F}_{y_2} \) by \( \{F^*; F \in \mathcal{F}_{y_2} \setminus \mathcal{F}_{y_1}\} \).

We enumerate \( \mathcal{F}_{y_2} \) by the squares \( p^2 \) of prime numbers.

Suppose we have finished \( n \) steps. We have \( \phi(y_1), \ldots, \phi(y_n) \), and \( \mathcal{F}_{y_1}, \ldots, \mathcal{F}_{y_n} \), \( i \leq n \). If \( \bigcup \{\phi(y_i); i \leq n\} \cup U_\beta = X \), then stop the induction, and let \( D_\beta = \{y_1 \setminus i \leq n\} \). So \( F^* \cap (\bigcup \{\phi(y); i \leq n\} \cup U_\beta) = \emptyset \), then we take the first member of \( \bigcup \{\phi(y_i); i \leq n\} \cup U_\beta \), such that \( F^* \setminus (\bigcup \{\phi(y_i); i \leq n\} \cup U_\beta) \neq \emptyset \). We choose a point \( y_{n+1} \) that \( y_{n+1} \in F^* \setminus (\bigcup \{\phi(y_i); i \leq n\} \cup U_\beta) \). Then \( F^* \cap \phi(y_{n+1}) \) is in \( \bigcup \{\phi(y_i); i \leq n\} \cup U_\beta \), then we let \( x_{n+1} = \min\{\phi(y_i); i \leq n\} \cup U_\beta \). Then \( x_{n+1} \notin \phi(y_i); i \leq n\} \cup U_\beta \), and denote \( x_{n+1} \) by \( y_{n+1} \). And we let \( \mathcal{F}_{y_{n+1}} = \{F^*; F \in \mathcal{F}_{y_{n+1}} \setminus \mathcal{F}_{y_n} \} \), and enumerate it by the \((n + 1)^{st}\) powers of prime numbers.

In this way, we get \( D_\beta = \{y_n; n \in N\} \). We have that \( D_\beta \cap U_\beta = \emptyset \). Let us show that \( D_\beta \) is closed. Suppose not, there exists a sequence \( C \subset D_\beta \), such that \( C \) converges to some point \( x \). We know that \( x \notin U_\beta \cup \bigcup \{\phi(d); d \in D_\beta\} \). Thus there exists some \( F \in \mathcal{F} \), such that \( F \) contains a subsequence which converges to \( x \), and \( \overline{F} \subset \phi(x) \) (by regularity property of \( X \)). So \( x \in F^* \). By the construction, we know that there is some \( n \), such that \( F^* \cap \bigcup \{\phi(y_i); i \leq n\} \neq \emptyset \). So \( x \in \bigcup \{\phi(d); d \in D_\beta\} \). Contradiction. So \( D_\beta \) is a closed set of \( X \). And also we know that it is a closed discrete set of \( X \) by its construction.

We have shown that \( \bigcup \{D_\eta; \eta < \beta\} \) is a closed discrete set of \( X \), so the set \( \bigcup \{D_\alpha; \alpha < \beta\} \cup D_\beta \) is also a closed discrete set. From the above discussion, we know that the conditions (1)–(4) hold.

Let \( D = \bigcup \{D_\alpha; \alpha < \gamma\} \), we may easily see that \( X = \bigcup \{\phi(d); d \in D\} \). Next we will prove that \( D \) is a closed discrete set of \( X \).

For any \( x \in X \), let \( \eta = \min\{\alpha; x \in \bigcup \{\phi(d); d \in D_\alpha\}\} \), we denote \( V_x = (\bigcup \{\phi(d); d \in D_\eta\} \setminus \bigcup \{D_\delta; \delta < \eta\}) \cap V'_{x} \), where \( V'_{x} \) is an open set of \( X \), and \( x \in V'_{x} \), satisfying that \( |V'_{x} \cap D_\eta| \leq 1 \). So \( V_x \) is an open set of \( X \), and \( x \in V_x \), \( |V_x \cap D_\eta| \leq 1 \). Thus \( D \) is a discrete closed set of \( X \). So \( X \) is a \( D \)-space.
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References