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A successful concept for measuring non-planarity of graphs: the crossing number

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Abstract

This paper surveys how the concept of crossing number, which used to be familiar only to a limited group of specialists, emerges as a significant graph parameter. This paper has dual purposes: first, it reviews foundational, historical, and philosophical issues of crossing numbers, second, it shows a new lower bound for crossing numbers. This new lower bound may be helpful in estimating crossing numbers.

Keywords: Graph drawing; Crossing number

1. Foundational issues

Pach and Tóth [48] noted that although researchers seem to agree in what they understand under the concept of “crossing number”, “drawings” are defined in a variety of ways in the literature, and the possibility is there that some definitions might not be equivalent. Pach and Tóth [48] introduced two new versions of the crossing number problem, and there is a fourth version, implicitly present in [63]. First, I give a careful definition of three classes of drawings, in which all four kinds of crossing numbers can be conveniently set.

A *drawing* D of a finite graph G on the plane is an injection ϕ from the vertex set $V(G)$ into the plane, and a mapping of the edge set $E(G)$ into the set of *simple plane curves*, i.e. homeomorphic images of the interval $[0,1]$, such that the curve

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corresponding to the edge $e=uv$ has endpoints $\phi(u)$ and $\phi(v)$, and contains no more points from the image of ϕ.

We also speak about the images of vertices as vertices, and about the curves as edges. We say that two edges in a drawing *cross* in a certain point of the plane, or the point is a *crossing point* of the two edges, if this point belongs to the interiors of the curves representing the edges. The *number of crossings* $\mathrm{cr}(D)$ in the drawing D is the sum of the number of crossing points for all unordered pairs of edges.

A *drawing D* is *normal* if it satisfies (i) and (ii):

(i) any two of the curves have finitely many points in common; and
(ii) no two curves have a point in common in a tangential (touching) way, i.e. we can define locally the "left side" and the "right side" of the curves at the common point, both curves are present at both sides of each in every small neighborhood of that point.

We should have defined above *intersection points* instead of *crossing point*, but assumption (ii) allows for speaking about crossing instead of intersection. For normal drawings we will also assume:

(iii) no point of the plane belongs to the interior of three curves, each representing an edge of the graph.

Requirement (iii) is convenient, since using it one can simplify the definition of $\mathrm{cr}(D)$ to the number of points, where crossing happens in the drawing. Also, some proof techniques about crossing numbers derive a planar graph with a drawing from a drawing D by introducing new vertices of degree 4 in the points of crossing, and those proof techniques require assumption (iii). However, many drawings in applications, especially straight line drawings, do not satisfy (iii). Notice that if (iii) fails and some k curves cross each other in an otherwise normal drawing in a single point, then this situation can easily be transformed locally into a normal drawing where any two of the k curves cross each other locally once, and the number of crossings in the drawing does not change. Therefore, we assume (iii) for normal drawings and it will not cause any problem that some drawings that we use fail (iii). We take a similar approach to (ii), since some conveniently defined drawings—see the last section—will contain tangential (touching) type of intersections. We take the view that those are easily *removable*, and we simply do not count them if they are present.

A *drawing D* is *nice*, if it is normal, and in addition satisfies

(iv) no two adjacent edges (i.e. edges sharing an endpoint) cross; and
(v) any two edges cross at most once.

The *crossing number* $\mathrm{CR}(G)$ of the graph G is the minimum of $\mathrm{cr}(D)$ over all drawings of G. We call a drawing D *optimal* (for CR) if it realizes $\mathrm{cr}(D)=\mathrm{CR}(G)$. It is easy to see that an optimal drawing must satisfy (i) and (ii), and a little work shows that it also must satisfy (iv) and (v). Therefore, we have an equivalent definition of $\mathrm{CR}(G)$: the minimum of $\mathrm{cr}(D)$ over all normal, nice drawings of G.

We show (v) first. Assume that the curves p and q corresponding to edges uz and xy cross in points r and t. Call p_1, p_2, p_3 and q_1, q_2, q_3 the pieces of p and q determined by r and t, with p_2 and q_2 denoting the rt sections. Redefine the curves as

$$p' = p_1 \cup q_2 \cup p_3 \quad \text{and} \quad q' = q_1 \cup p_2 \cup q_3. \tag{1}$$

Now we can eliminate the tangential intersections of p' and q' at r and t. A problem is that p' and q' may not be simple curves (i.e we may have created self-crossings), but we can shortcut them, and this does not increase the number of crossings in the drawing. (Although we may have generated new crossing edge pairs, the total number of crossings decreased, contradicting the optimality of the original drawing. Since this step may create new crossing edge pairs, one cannot show by using this step that the crossing number is equal to the pairwise crossing number (see below).) The proof of (iv) is similar, use the shared endvertex for r, and t for a crossing point. p_1 or p_3 (q_1 or q_3) degenerates for a point. Surgery (1) works again.

Pach and Tóth [48] introduced two new variants of the crossing number problem:

the pairwise crossing number CR-PAIR(G) is equal to the minimum number of unordered pairs of edges that cross each other at least once (i.e. they are counted once instead of as many times they cross), over all normal drawings of G; and

the odd crossing number CR-ODD(G) is equal to the minimum number of unordered pairs of edges that cross each other odd times, over all normal drawings of G.

In Tutte's work [63] another kind of crossing number is implicit:

the independent-odd crossing number CR-IODD(G) is equal to the minimum number of unordered pairs of non-adjacent edges that cross each other odd times, over all normal drawings of G.

The following chain of inequalities is obvious from the definitions:

$$\text{CR-IODD}(G) \leqslant \text{CR-ODD}(G) \leqslant \text{CR-PAIR}(G) \leqslant \text{CR}(G). \tag{2}$$

No example of strict inequality is known. Pach [44] considers the problem if all these numbers are always equal as the most important open problem on crossing numbers. Mohar [42] independently posed the problem whether

$$\text{CR-PAIR}(G) = \text{CR}(G).$$

The smallest graphs with CR(G)=1 are K_5 and $K_{3,3}$. For these graphs the following stronger result holds:

Theorem 1.1 (Chojnacki [10]).

$$\text{CR-IODD}(K_5) = 1 \quad \textit{and} \quad \text{CR-IODD}(K_{3,3}) = 1.$$

(For other proofs and generalizations, see [48,63].)

Fáry's theorem [19] telling that planar graphs can be drawn using straight line segments for edges and Zarankiewicz's crossing number conjecture (Section 2.1) may suggest that optimal drawings can be done using straight line segments for edges. This is not the case. Guy [25] showed that first for K_9, and later Bienstock and Dean [6,7] constructed graphs with crossing number four for any number k, such that drawings of those graphs using straight line segments for edges have more crossings than k. Several authors study CR-LIN(G), which is the minimum number of crossings if all edges are drawn by straight line segments [12,29,48,57]; since $\mathrm{CR}(G) \leqslant \text{CR-LIN}(G)$, CR-LIN($G$) is the fifth (and largest) item in line (2).

It is clear that if similar crossing number problems are posed for the sphere instead of the plane, stereographic projection shows that the corresponding planar and spheric crossing numbers are always equal. Crossing number problems can be posed on orientable and non-orientable surfaces of higher genus, and many of the results discussed in this paper generalizes for them, see [46,54,55,56].

It is not the purpose of the present paper to give a comprehensive survey of the literature of crossing numbers. Much of the literature falls into one of two categories: the first investigates exact values of crossing numbers or makes lower bounds on crossing numbers based on information on the crossing number of a certain small graph, the second tries to prove bounds based on structural properties of the graph. We call the first the theory of small graphs, the second the theory of large graphs. During the early history of crossing numbers the theory of small graphs existed only. For more information on the early history and the theory of small graphs, see [66], for the modern history and the theory of large graphs, see [56], and for the most recent results see [44]. A bibliography of papers on crossing numbers by Vrťo [65] is available online.

2. Theory of small graphs

2.1. Turán's brick factory problem

It was Paul Turán who introduced the concept of crossing numbers. Turán [62] tells about how he posed the problem, while in a forced labor camp in World War II: "There were some kilns where the bricks were made and some open storage yards where the bricks were stored. All the kilns were connected by rail with all storage yards. … the trouble was only at crossings. The trucks generally jumped the rails there, and the bricks fell out of them; in short this caused a lot of trouble and loss of time … the idea occurred to me that this loss of time could have been minimized if the number of crossings of the rails had been minimized. But what is the minimum number of crossings?"

Put in technical terms, Turán's Brick Factory Problem is: *what is the crossing number* $\mathrm{CR}(K_{n,m})$ *of the complete bipartite graph* $K_{n,m}$?

Place $\lfloor n/2 \rfloor$ vertices to negative positions on the x-axis, $\lceil n/2 \rceil$ vertices to positive positions on the x-axis, $\lfloor m/2 \rfloor$ vertices to negative positions on the y-axis, $\lceil m/2 \rceil$ vertices to positive positions on the y-axis, and draw nm edges by straight line segments to obtain a drawing of $K_{n,m}$. It is not hard to check that the following formula gives the number of crossings in this particular drawing:

$$\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor. \tag{3}$$

Zarankiewicz's crossing number conjecture is that the drawing described above is optimal.

The conjectured crossing number of the complete graph K_n is

$$\frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor. \tag{4}$$

We show a drawing with this number of crossings for even n, the construction is due to Guy [24] and Blažek and Koman [8]: take a soup can, which is homeomorphic to a sphere, place $n/2$ vertices equidistantly on the perimeter of the top disk and on the perimeter of the bottom disk, respectively. Draw a $K_{n/2}$ with straight line segments on the top disk and on the bottom disk, respectively. From one point of the bottom disk, draw shortest helical curves to all vertices of the top disk. Repeat this for all $n/2$ vertices on the bottom disk. Although this is not a straight line drawing of K_n, interestingly, the curves that we use are "geodetic" on the soup can.

It is usually not hard to come up with drawings of graph whose optimality is intuitively clear. The difficulty lies in proving matching lower bounds for the crossing numbers.

2.2. Euler's formula

The simplest lower bound for the crossing number of a simple graph with $n \geqslant 3$ vertices and m edges is

$$m - 3n + 6. \tag{5}$$

This immediately follows from Euler's polyhedral formula, and already gives $\mathrm{CR}(K_5) \geqslant 1$. A counterpart of this formula for triangle-free graphs $\mathrm{CR}(G) \geqslant m - 2n + 4$, which proves $\mathrm{CR}(K_{3,3}) \geqslant 1$. Formula (5) can give interesting lower bounds for small graphs only, since the magnitude of the crossing number can be as large as m^2. It was Pach and Tóth who observed that (5) sets a lower bound for *all four* crossing numbers in (2), and this extends to all lower bounds which solely depend on (5). We present their argument for the smallest crossing number, CR-IODD(G). If $m \leqslant 3n-6$, then there is nothing to prove. If $m \geqslant 3n-5$, then G is non-planar, and hence contains by Kuratowski's Theorem a subdivision of a K_5 or a $K_{3,3}$ (in fact both). Hence, in any normal drawing of G there is a normal subdrawing of a K_5 or a $K_{3,3}$. By Theorem 1.1, there are two vertex disjoint paths of G which cross each other an odd number of times. Hence, there is an edge e from the first path and an edge f from the second path that cross each other odd times. If formula (5) holds for $G-e$, then it holds for G, and the base case for this induction proof is $m = 3n-5$. If G has girth g, then the lower bound (5) can be strengthened to (6):

$$m - \frac{g}{g-2}(n-2). \tag{6}$$

2.3. Known crossing numbers

Kleitman showed that (3) holds for $m \leqslant 6$ [30] and also proved that the smallest counterexample to the Zarankiewicz's conjecture must occur for odd n and m. Woodall

[67] used elaborate computer search to show that (3) holds for $K_{7,7}$ and $K_{7,9}$. Thus, the smallest unsettled instances of Zarankiewicz's conjecture are $K_{7,11}$ and $K_{9,9}$. $\mathrm{CR}(K_n)$ is known to be equal to (4) for $n \leqslant 10$ [25].

There are some infinite families of graphs whose crossing numbers are known. Exoo et al. [18] started the investigation of the crossing number of generalized Petersen graphs. A *generalized Petersen graph* $G(n,k)$ has vertex set $\{u_i, v_i: 1 \leqslant i \leqslant n\}$ and edge set $\{u_iv_i,\ u_iu_{i+1},\ v_iv_{i+k}: 1 \leqslant i \leqslant n\}$, where subscripts are taken modulo n. Exoo et al. [18] showed that $\mathrm{CR}(G(n,2)) = 0$, if $n = 3$ or n is even; $\mathrm{CR}(G(n,2)) = 2$, when $n=5$; and $\mathrm{CR}(G(n,2))=3$, when n is odd, $n \geqslant 7$. Fiorini [20] showed $\mathrm{CR}(G(9,2))=2$, $\mathrm{CR}(G(3h,3))=h$ for $h \geqslant 4$, $\mathrm{CR}(G(3h+2,3))=h+2$, $\mathrm{CR}(G(4h,4))=2h$; and claimed $\mathrm{CR}(G(10,3)) = 4$. McQuillan and Richter [41] corrected the last claim by proving $\mathrm{CR}(G(10,3)) > 4$. Lovrečič Saražin [39] showed $\mathrm{CR}(G(10,4)) = 4$.

Let C_n, P_n, S_n denote the cycle, path, and star with n edges, respectively. The crossing number of the Cartesian product of any graph G of order 4 with the cycle C_n, $\mathrm{CR}(G \times C_n)$, has been determined by Beineke and Ringeisen [5], and by Jendrol' and Ščerbová [28]; and for any graph G of order 4, $\mathrm{CR}(G \times P_n)$ and star $\mathrm{CR}(G \times S_n)$ has been determined by Klešč [31]. Recently, Klešč [32] determined $\mathrm{CR}(G \times P_n)$ for all graphs of order 5; see those and the known values of $\mathrm{CR}(G \times C_n)$ and $\mathrm{CR}(G \times S_n)$ in a table on p. 358 in [32].

There is a longstanding conjecture of Harary et al. [27], which states that for $n \geqslant m \geqslant 3$, the crossing number of the Cartesian product of two cycles,

$$\mathrm{CR}(C_n \times C_m) = n(m-2). \tag{7}$$

There is a simple drawing with this number of crossings, the difficulty lies in proving that $n(m-2)$ crossings are, in fact, needed. Proving the conjecture for different small values of n and m took separate, highly technical papers; and the case $n=m=8$ is still open [3,4,5,33,50,52]. Richter and Thomassen [50] introduced here the most general approach so far: consider n red closed curves and m blue closed curves, where each may cover certain points twice, such that every blue curve intersects every red curve, and no point of the plane is covered three times. What is then the minimum number of intersection points of curves? This problem is rather geometric than graph theoretic, and is a better subject to inductive arguments than the Cartesian product of two cycles. In a recent breakthrough paper, Glebsky and Salazar [22] proved (7) for every m for all sufficiently large n, but this already belongs to the theory of large graphs.

2.4. The standard counting method

A basic technique to obtain a lower bound for the crossing number of a larger graph from that of a sample graph is the *standard counting method*. Take a hypothetical {normal, nice, optimal} drawing of the large graph, find many copies of the sample graph in it, each exhibiting as many crossings as its crossing number. Add up those numbers, and divide by the largest multiplicity with which a crossing may have been counted in different copies of the sample graph. Make this argument more tangible by the following example: $\text{CR-IODD}(K_n) \geqslant (1 + o(1))n^4/120$. Take a normal drawing of K_n. Any five vertices span a normal subdrawing of a K_5, which exhibit at least one pair of non-adjacent edges crossing odd times. We find at least total of $\binom{n}{5}$ such edge pairs, and every such edge pair occurs in exactly in $(n-4)$ 5-tuples of vertices. The claim follows.

Applying the standard counting argument for K_{n+1} with sample graph K_n, or for $K_{n+1,n+1}$ with sample graph $K_{n,n}$, one obtains that

$$\frac{\mathrm{CR}(K_n)}{24\binom{n}{4}} \quad \text{and} \quad \frac{\mathrm{CR}(K_{n,n})}{4\binom{n}{2}^2} \tag{8}$$

are non-decreasing and bounded. Therefore, the sequences in (8) have a limit which provides asymptotic formulae $\mathrm{CR}(K_n) \sim c_1 n^4$ and $\mathrm{CR}(K_{n,n}) \sim c_2 n^4$ [51,66]. However, the values of c_1 and c_2 are not known. The drawings shown above imply $c_1 \leqslant \frac{1}{64}$ and $c_2 \leqslant \frac{1}{16}$, and if the drawings are optimal, equalities hold.

Woodall's result [67], which showed Zarankiewicz's conjecture for $K_{7,9}$, implies $\frac{1}{21} \leqslant c_2$ by a standard counting argument. Kleitman's [30] cited result allows us to use $K_{n-6,6}$ as a sample graph to count crossings in K_n, and one obtains $\frac{1}{80} \leqslant c_1$. Applying the standard counting argument to K_n with sample graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ [51] shows that if $c_2 = \frac{1}{16}$ then $c_1 = \frac{1}{64}$. The converse of this implication is not known.

2.5. Graph minors

The graph minor community also has an interest in crossing numbers. Their usual approach is characterization in terms of excluded minors. Robertson and Seymour [53] calls a graph H *singly crossing* provided H is a minor of a graph that can be drawn on the sphere with at most one crossing. They show that a graph is singly crossing if and only if it does not have one of 41 explicitly given graphs as a minor.

3. Theory of large graphs

The modern history started with Leighton's thesis [36]. Leighton introduced methods to set lower bounds for crossing numbers which instead of crossing numbers of small graphs, depended on certain parameters of the large graphs. He introduced three methods that become classic: lower bounds in terms of number of edges, bisection width, and graph embedding.

3.1. Number of edges

Ajtai et al. [2] and Leighton [36] independently discovered that for graphs with $m \geqslant cn$ edges, the crossing number is at least

$$\mathrm{CR}(G) \geqslant \frac{c-3}{c^3} \frac{m^3}{n^2}. \tag{9}$$

The maximum constant factor in (9) is $\frac{4}{243}$, achieved at selecting $c = 4.5$. It follows from the argument after (5) that (9) holds for all four crossing numbers in (2). The original proofs of (9) went by induction, a folklore probabilistic proof can be found in [56] and also made it to the book [1].

For $c=4$, Pach and Tóth [47] improved $\frac{1}{64}$ to $\frac{1}{33.75}$, but this improved lower bound is not known to extend for all kinds of crossing numbers.

Erdös and Guy [16] conjectured (9) (although those who proved it were not aware of it), and even more. If $\kappa(n,m)$ denotes the minimum crossing number of graphs with n vertices and m edges, they conjectured that $\lim\kappa(n,m)n^2/m^3$ has a limit if $m/n \to \infty$ and $n^2/m \to \infty$. Recently, Pach et al. [46] proved this conjecture, but the value of this limit is not known.

3.2. Bisection width and graph embedding

For this type of results, see the survey [56].

3.3. Random graphs

Pach and Tóth [49] showed that for the random graph model $G(n,p)$ with $m = p\binom{n}{2} > 10n$,

$$\mathrm{CR}(G(n,p)) > \frac{m^2}{4000}$$

almost surely. Spencer and Tóth [57] studied this problem for CR-PAIR, and were able to show that for every $\varepsilon > 0$ and $p = n^{\varepsilon-1}$,

$$\text{CR-PAIR}(G(n,p)) = \Omega(m^2).$$

Using a martingale inequality, Pach and Tóth [49] showed that the following large deviation inequality holds: for every $(m/4)^3 \mathrm{e}^{-m/4} \leqslant \alpha \leqslant \sqrt{m}$,

$$P[|\mathrm{CR}(G(n,p)) - E[\mathrm{CR}(G(n,p))]| > 3\alpha m^{3/2}] < 3\mathrm{e}^{-\alpha^2/4}.$$

3.4. Computational complexity

Garey and Johnson [21] proved that testing $\mathrm{CR}(G) \leqslant k$ is NP-complete, Pach and Tóth [48] extended this to CR-ODD, and also proved that CR-PAIR is NP-hard. The reduction uses the NP-completeness of linear arrangement. Testing planarity, and therefore testing $\mathrm{CR}(G) \leqslant k$ for any fixed k can be done in polynomial time—introduce at most k new vertices for crossing points in all possible ways and test planarity. Leighton and Rao [37] designed the first provably good approximation algorithm for crossing numbers. This algorithm approximates $n + \mathrm{CR}(G)$ within a factor of $\log^4 n$ for degree bounded graphs (and, therefore, provides little information on small crossing numbers). A recent paper of Even et al. [17] reduced the factor to $\log^3 n$. We know nothing that would exclude the possibility of approximation within a constant multiplicative factor.

4. Biplanar crossing number

Recall that a graph G is *biplanar*, if one can write $G = G_1 \cup G_2$, where G_1 and G_2 are planar graphs (a graph is understood here as a set of edges). Although planarity can be tested in polynomial time, testing biplanarity is NP-complete [40]. Owens [43] introduced the *biplanar crossing number* of a graph G, that we denote by $\mathrm{CR}_2(G)$. By definition $\mathrm{CR}_2(G)=\min_{G_1\cup G_2=G}\{\mathrm{CR}(G_1)+\mathrm{CR}(G_2)\}$, where CR is the planar crossing number. Biplanar crossing number problems have a Ramsey flavour, and are even more difficult than ordinary crossing number problems. Although (5) and (9) have their natural analogues for $\mathrm{CR}_2(G)$, the embedding method or the bisection width method do not seem to generalize to biplanar crossing numbers. Even worse, as Tutte noted, the biplanar crossing number is not an invariant for homeomorphic graphs; in fact, the edges of every graph can be subdivided such that the subdivided graph is biplanar!

Recent work of Sýkora et al. [59,60] focuses on the biplanar crossing number. They showed that for all graphs G, $\mathrm{CR}_2(G) \leqslant \frac{3}{8}\,\mathrm{CR}(G)$. However, one cannot give an upper bound for $\mathrm{CR}(G)$ in terms of $\mathrm{CR}_2(G)$, since there are graphs G of order n and size m, with crossing number $\mathrm{CR}(G)=\Theta(m^2)$ (i.e. as large as possible) and biplanar crossing number $\mathrm{CR}_2(G)=\Theta(m^3/n^2)$ (i.e. as small as possible), for any $m=m(n)$, where m/n exceeds a certain absolute constant.

Sýkora et al. also showed that

$$\mathrm{CR}_2(K_{5,n})=\left\lfloor\frac{n}{12}\right\rfloor\left(n-6\left\lfloor\frac{n}{12}\right\rfloor-6\right) \tag{10}$$

for $n \geqslant 12$. (Note that for $n \leqslant 11$, $K_{5,n}$ is biplanar.)

Sýkora et al. [58] refuted in a strong sense Halton's conjecture, which asserted (among other things) that any graph of maximum degree 6 is biplanar.

5. Corroborating Lakatos

Zarankiewicz [68] and Urbaník [64] independently claimed and published that $\mathrm{CR}(K_{n,m})$ was actually equal to (3), their proof was reprinted in a book [9], cited, and used in follow-up papers. Kainen and Ringel discovered a flaw in the proof and the flaw withstood all attempts for correction. Guy [24] deserves much credit for rectifying this confused state of art and also for pointing out "much more sweeping assumptions than the overt hypotheses of the theorem" in some other crossing number papers [26].

Lakatos [35], who applied the Popperian epistemology to mathematics, carried out his arguments on the paradigmatic example of Euler's polyhedral formula. Actually, crossing numbers, closely connected to Euler's polyhedral formula by (5), could also have served as his paradigmatic example.

In a recent paper, Pach and Tóth [48] scrutinize the very definition of crossing numbers! They point out that some authors might have thought of CR-PAIR instead of CR.

How is it possible that decades in research of crossing numbers passed by and no major confusion resulted from these foundational problems? The answer is the following: the conjectured optimal drawings are usually normal and nice, and the lower bounds—as (5) and (9)—usually also apply for all kinds of crossing numbers.

6. Applications of crossing numbers

Many concepts have been introduced in the literature which measure quantitatively “how far” a non-planar graph is from being a planar graph: genus, crossing number, thickness, splitting number, skewness, vertex deletion number, etc. [56,66]. Computing these quantities (or their slight variations) is known or conjectured to be NP-hard [21], and apart from this, with the exception of genus and crossing number, there is not much to tell about them.

So far, only familiarity with the genus was a must for every discrete mathematician. Now the crossing number aligns with the genus, since it has applications and is connected to other areas of mathematics.

Ringel discovered that the Turán number $T(n,5,4)$ sets a lower bound for the crossing number of the complete graph on n vertices. Consider an optimal (normal, nice) drawing of the complete graph. Define a 4-uniform hypergraph on the vertex set of the complete graph by the quadruplets of vertices of pairs of crossing edges. Since K_5 is non-planar, any five-element subset of vertices does contain an edge of the 4-uniform hypergraph.

Leighton’s interest in crossing numbers was motivated by VLSI, and he used the crossing number to set lower bound for the VLSI layout area of the graph. In fact, the relevance of crossing number for engineering was well known already in the pre-VLSI “transistor age” [9].

Székely [61] used the cited theorem of Ajtai et al. [2] and Leighton [36] to give a new proof for the Szemerédi–Trotter theorem, which tells how many incidences can be among n points and m straight lines in the plane. The proof consists of comparing lower bound (9) to an upper bound, coming from a given drawing, for a certain graph. This crossing number method also yielded simple proofs [61] for the best available results regarding two classic Erdős problems: Given n points in the plane, how many unit distances can be among them? Given n points in the plane, what is the least number of distinct distances among them? Just in a couple of years, the crossing number method gave a number of other applications to discrete geometry [11,45,47], etc. Surprisingly, this crossing number method is also cited in number theory, see [13–15,23,34,38]. Some applications, for example [34], actually need a more general version of the Szemerédi–Trotter theorem, for the number of incidences among points and pseudolines [61], which also follows from the crossing number method.

Pach et al. [46] proved a conjecture of Simonovits, improving the bound of (9). If G has girth $>2r$ and $m \geqslant 4n$, then

$$\mathrm{CR}(G) = \Omega\left(\frac{m^{r+2}}{n^{r+1}}\right) \tag{11}$$

and proved an even more general theorem for graphs G satisfying a monotone graph property. Since $m^2 > \mathrm{CR}(G)$, (11) immediately implies that a graph with girth $>2r$ has at most $\mathrm{O}(n^{1+(1/r)})$ edges, which is the best-known result, tight within a constant multiplicative factor for $r=2,3,5$. This may be thought of as an “artificial” application, since the proof in [46] uses these corollaries from extremal graph theory, but this is a new genuine connection between crossing numbers and extremal graph theory.

7. Formulae for CR-IODD

To have a graph parameter that we cannot even asymptotically evaluate for complete graphs is rather annoying. In addition, knowing the crossing numbers of complete graphs would immediately imply improved lower bounds on the crossing numbers of many other graphs, either by the standard counting argument or by graph embedding.

The present section yields formulae for CR-IODD, which are far from obvious how to evaluate, but give a hope to evaluate CR-IODD for complete graphs.

7.1. Tutte's theory

Earlier, Tutte [63] introduced an algebraic theory of crossing numbers and proved Chojnacki's Theorem 1.1 from this theory. Tutte's theory is very complicated, since it tries to follow closely not just crossing numbers but drawings. Tutte studies normal drawings. Denoting the vertex set by $V=\{1,2,\ldots,n\}$, he defines two orientation for every edge, connecting vertices i and j, ij and ji. The orientation ij defines locally a left side and a right side of the curve, as if we were facing j on the curve. Tutte denotes by $\lambda(ij,kl)$, for two non-adjacent oriented edges ij and kl, the difference of the following two numbers: number of left-to-right crossings that oriented edge ij does on kl and the number of right-to-left crossings that oriented edge ij does on kl. He observes that $\lambda(ij,kl)$ has the same parity as the number of crossings of ij and kl, and fixing an orientation for every edge, he suggests the lower bound

$$\min_{\text{normal drawings}} \sum |\lambda(ij,kl)| \tag{12}$$

(where summation goes for unordered pairs of non-adjacent edges) for the crossing number, and poses the question if equality holds. It is clear that CR-ODD $\leqslant$ (12) $\leqslant$ CR. There is an enigmatic sentence of Tutte: "We are taking the view that crossings of adjacent edges are trivial, and easily got rid of." We interpret this sentence as a philosophical view and not a mathematical claim.

Pach and Tóth [48] had some formulae, or rather discrete integer programs, for the value of CR-ODD, which involved pairs of edges. Tutte, and Pach and Tóth described how their respective formulae transform when an edge is "pulled over" a vertex, a generic step to move from one drawing to another. I am not aware of any paper which draws further conclusions on crossings numbers from Tutte's theory. There seems to be a trade-off between getting tangible results and following faithfully the drawing. Our result, presented next, is a mod 2 version of Tutte's theory. This requires only maintaining information on (edge, vertex) type pairs, which simplifies everything. We show how these results can be used to prove Chojnacki's Theorem 1.1.

7.2. The new results

Let us be given an arbitrary cyclic order $\mathscr{C}=v_1,v_2,\ldots,v_n$ of the vertices of a simple graph G. We say that two non-adjacent edges of G, say xy and uz are in *acyclic order*, if the cyclic order $\mathscr{C}$ restricted to these four vertices is x,u,y,z or x,z,y,u. Otherwise, two non-adjacent edges are in *cyclic order*. These two relations are clearly symmetric. Under a *bipartition* of a set we understand its unordered partition into two subsets, one of which may be empty. We use the notation $\|$ for a bipartition, and write $u\|v$ to express that u and v belong to different classes, and $-\|uv$ to express that u and v belong to the same class. For every edge $xy \in E(G)$, consider an arbitrary bipartition $\|_{xy}$ of $V(G)\setminus\{x,y\}$, and let

$$\mathscr{B} = \{\|_{xy}\colon xy \in E(G)\} \tag{13}$$

denote the set of bipartitions.

We define now the relations $O_{\mathscr{C}}$ and $P_{\mathscr{B}}$ of non-adjacent edges of G as follows:

$$O_{\mathscr{C}}(xy,uz) = \begin{cases} 1 & \text{if } xy \text{ and } uz \text{ are in cyclic order,} \\ 0 & \text{otherwise,} \end{cases} \tag{14}$$

$$P_{\mathscr{B}}(xy,uz) = \begin{cases} 1 & \text{if } u\|_{xy}z, \\ 0 & \text{otherwise, i.e. if } -\|_{xy}uz \text{ holds.} \end{cases} \tag{15}$$

Note that $O_{\mathscr{C}}(xy,uz)=O_{\mathscr{C}}(xy,zu)=O_{\mathscr{C}}(uz,xy)$, and $P_{\mathscr{B}}(xy,uz)=P_{\mathscr{B}}(yx,uz)=P_{\mathscr{B}}(xy,zu)$, but it is possible that $P_{\mathscr{B}}(xy,uz) \neq P_{\mathscr{B}}(uz,xy)$. Define

$$\begin{aligned} \mathrm{forced}_{\mathscr{B},\mathscr{C}}(xy,uz) = {} & [1 - O_{\mathscr{C}}(xy,uz)][1 - P_{\mathscr{B}}(xy,uz)][1 - P_{\mathscr{B}}(uz,xy)] \\ & + [1 - O_{\mathscr{C}}(xy,uz)]P_{\mathscr{B}}(xy,uz)P_{\mathscr{B}}(uz,xy) \\ & + O_{\mathscr{C}}(xy,uz)[1 - P_{\mathscr{B}}(xy,uz)]P_{\mathscr{B}}(uz,xy) \\ & + O_{\mathscr{C}}(xy,uz)P_{\mathscr{B}}(xy,uz)][1 - P_{\mathscr{B}}(uz,xy)]. \end{aligned} \tag{16}$$

Note that $\mathrm{forced}_{\mathscr{B},\mathscr{C}}(xy,uz)$ does not change if we interchange x and y, u and z, or xy and uz. The concepts $\mathscr{C}$, $\mathscr{B}$, $P_{\mathscr{B}}$, $O_{\mathscr{C}}$, and $\mathrm{forced}_{\mathscr{B},\mathscr{C}}$ were introduced in *abstract graphs*, not in graph drawing, although their function will be to grasp some properties of graph drawings.

Theorem 7.1. *For every simple graph G and every cyclic order $\mathscr{C}$ of $V(G)$, we have*

$$\text{CR-IODD}(G) = \min_{\mathscr{B}} \frac{1}{2} \sum_{xy \in E(G)} \sum_{\substack{uz \in E(G) \\ \{x,y\}\cap\{u,z\}=\emptyset}} \mathrm{forced}_{\mathscr{B},\mathscr{C}}(xy,uz), \tag{17}$$

where the minimization goes for all possible sets of bipartitions of form (13).

Since $\mathrm{forced}_{\mathscr{B},\mathscr{C}}(xy,uz)$ does not change if we interchange xy and uz, therefore the objective function of the minimization in (17) can be understood as a summation for unordered pairs of non-adjacent edges of G, without the factor of $\frac{1}{2}$.

We show an equivalent reformulation of Theorem 7.1. Perhaps this reformulation can be evaluated analytically for certain classes of graphs, e.g. complete graphs, and certainly can be evaluated using computer for some particular graphs. This is just a quartic expression evaluated on ± 1 values, which is highly symmetric in the case of complete graphs. For every $ab \in E(G)$ let $Q_{ab}: V(G)\setminus\{a,b\} \to \{-1,+1\}$ be an arbitrary function, such that $Q_{ab}=Q_{ba}$. Observe that every Q_{ab} function gives rise to a bipartition of $V(G)\setminus\{a,b\}$ through the full inverse image partition, and every bipartition of $V(G)\setminus\{a,b\}$ can be obtained by exactly two such functions, where one is the negative of the other. Also note that the value of the product $Q_{ab}(u)Q_{ab}(z)$ does not change, if we write $-Q_{ab}$ to the place of Q_{ab}. Define

$$\mathscr{Q}=\{Q_{ab}: ab\in E(G)\}. \tag{18}$$

Theorem 7.2. *For every simple graph G and every cyclic order $\mathscr{C}$ of $V(G)$, we have*

$$\text{CR-IODD}(G)=\frac{N}{2}-\max_{\mathscr{Q}}\frac{1}{2}\sum_{xy\in E(G)}\sum_{\substack{uz\in E(G)\\ \{u,z\}\cap\{x,y\}=\emptyset}}\left\{O_{\mathscr{C}}(xy,uz)-\frac{1}{2}\right\}$$
$$\times Q_{xy}(u)Q_{xy}(z)Q_{uz}(x)Q_{uz}(y), \tag{19}$$

where the maximization goes for a set of functions $\mathscr{Q}$ as in (18), and N denotes the number of unordered pairs of non-adjacent edges in G.

Proof. We are going to show that (17) and (19) are equal by algebraic manipulation. Take a minimizing $\mathscr{B}$ in (17). We put the RHS of (17) into the form of (19). Write $P_{xy}(uz)=1$, if $u\|_{xy}z$, i.e. $P_{\mathscr{B}}(xy,uz)=1$ and $P_{xy}(uz)=-1$ otherwise, namely if $-\|_{xy}uz$, i.e. $P_{\mathscr{B}}(xy,uz)=0$. It is easy to see from (16) that

$$\text{forced}_{\mathscr{B},\mathscr{C}}(xy,uz)=[1-O_C(xy,uz)]\frac{1+P_{xy}(uz)P_{uz}(xy)}{2}$$
$$+O_C(xy,uz)\frac{1-P_{xy}(uz)P_{uz}(xy)}{2}. \tag{20}$$

Observe that P_{xy}, which is defined on pairs of vertices, can be written in terms of Q_{xy}, which is defined on vertices, such that $Q_{xy}=1$ on one class of the bipartition $\|_{xy}$, and $Q_{xy}=-1$ on the other class, since then $P_{xy}(uz)=Q_{xy}(u)Q_{xy}(z)$. There is a bijective correspondence between $\mathscr{B}$'s and equivalence classes of $\mathscr{Q}$'s, where $\mathscr{Q}\sim\mathscr{Q}'$ if and only for all edges ab, $Q_{ab}=\pm Q'_{ab}$. Rewriting (20) in terms of $\mathscr{Q}$, we obtain (19). Conversely, if $\mathscr{Q}$ is given, define $P_{xy}(uz)=Q_{xy}(u)Q_{xy}(z)$, and $P_{\mathscr{B}}(xy,uz)=1$ if $P_{xy}(uz)=1$, otherwise $P_{\mathscr{B}}(xy,uz)=0$. □

7.3. *Proof to Chojnacki's Theorem 1.1*

In order to show that Theorem 7.1 is a promising combinatorial approach to crossing numbers, we give an unexpectedly purely combinatorial proof to Chojnacki's Theorem 1.1.

Proof. Consider the equivalent of (17), which goes for unordered pairs of non-adjacent edges (see the remark after Theorem 7.1). Assume that for some graph G, cyclic order $\mathscr{C}$ and set of bipartitions $\mathscr{B}$, the summation is zero. Expand summation (17) which has zero value, by substituting four terms into every forced$_{\mathscr{B},\mathscr{C}}$ formula from (16). Since all terms after the substitution are non-negative, the summation has to be termwise zero. Split the summation for two parts, where $O_{\mathscr{C}}(xy,uz)=0$, but the coefficient $1-O_{\mathscr{C}}(xy,uz)$ is present; and where $O_{\mathscr{C}}(xy,uz)=1$ and the coefficient $O_{\mathscr{C}}(xy,uz)$ is present. Hence, we are left with

$$\begin{aligned}0=&\sum_{\substack{\{xy,uz\}\\ O_{\mathscr{C}}(xy,uz)=0}}([1-P_{\mathscr{B}}(xy,uz)][1-P_{\mathscr{B}}(uz,xy)]\\ &+P_{\mathscr{B}}(xy,uz)P_{\mathscr{B}}(uz,xy))\\ &+\sum_{\substack{\{xy,uz\}\\ O_{\mathscr{C}}(xy,uz)=1}}([1-P_{\mathscr{B}}(xy,uz)]P_{\mathscr{B}}(uz,xy)\\ &+P_{\mathscr{B}}(xy,uz)[1-P_{\mathscr{B}}(uz,xy)]),\end{aligned} \tag{21}$$

where the summations still go for unordered pairs of non-adjacent edges. Expanding the terms after the summations in (21), we see that the generic term in the first summation is $1+2P_{\mathscr{B}}(xy,uz)P_{\mathscr{B}}(uz,xy)-P_{\mathscr{B}}(xy,uz)-P_{\mathscr{B}}(uz,xy)$, and the generic term in the second summation is $-2P_{\mathscr{B}}(xy,uz)P_{\mathscr{B}}(uz,xy)+P_{\mathscr{B}}(xy,uz)+P_{\mathscr{B}}(uz,xy)$. If the number of unordered, non-adjacent edge pairs in G with $O_{\mathscr{C}}=0$ is q, then taking (21) mod 2 we obtain:

$$\begin{aligned}0&\equiv q+\sum_{\substack{\{xy,uz\}\\ \{x,y\}\cap\{u,z\}=\emptyset}}[P_{\mathscr{B}}(xy,uz)+P_{\mathscr{B}}(uz,xy)]\\ &\equiv q+\sum_{xy}\sum_{\substack{uz\\ \{x,y\}\cap\{u,z\}=\emptyset}}P_{\mathscr{B}}(xy,uz) \mod 2.\end{aligned} \tag{22}$$

We only need to prove CR-IODD$(K_5)\neq 0$, since CR$(K_5)\leqslant 1$ is well known. We do indirect proof. Consider the vertices of K_5 in the cyclic order $\mathscr{C}=1,2,3,4,5$, and we have (22) for $G=K_5$ and a $\mathscr{B}$ set of bipartitions realizing this zero. There are 15 unordered pairs of non-adjacent edges. Note that 10 of the 15 unordered pairs of non-adjacent edges $\{xy,uz\}$ yields $O_{\mathscr{C}}(xy,uz)=1$, and five of them yields $O_{\mathscr{C}}(xy,uz)=0$, i.e. $q=5$. We are going to show that the double summation in the right-hand side of (22) is even which yields a contradiction. Observe that any bipartition $\|_{xy}$ of three elements results either in a 3:0 distribution or in a 2:1 distribution; and in both cases the number of separated unordered pairs of elements, 3×0 or 2×1, is even. Recall that $P_{\mathscr{B}}(xy,uz)=1$ iff $\|_{xy}$ has u and z in different classes. Therefore, for an arbitrary xy, $\sum_{uz}P_{\mathscr{B}}(xy,uz)=2k_{xy}$, even. From here,

$$\sum_{xy}\sum_{\substack{uz\\ \{x,y\}\cap\{u,z\}=\emptyset}}P_{\mathscr{B}}(xy,uz)=2k \tag{23}$$

yields the needed contradiction by $0\equiv 5+2k \bmod 2$. For the other graph, we only need to prove CR-IODD$(K_{3,3})\neq 0$, since CR$(K_{3,3})\leqslant 1$ is well known. We start with a copy of $K_{3,3}$ in which the colorclasses are $\{1,3,5\}$ (red vertices) and $\{2,4,6\}$ (white vertices). We use the cyclic order of vertices $\mathscr{C}$=1,2,3,4,5,6. $K_{3,3}$ has nine edges, and 18 unordered pairs of non-adjacent edges. It is easy to see that three unordered pairs of non-adjacent edges yields $O_{\mathscr{C}}=0$ and 15 unordered pairs of non-adjacent edges yields $O_{\mathscr{C}}=1$, i.e. $q=3$. We proceed as we did for K_5, and repeat a slight variation of the counting argument above. We face formula (22) again, but keep in mind that the graph is different, $\mathscr{C}$ is different, $\mathscr{B}$ is different, and now xy, uz denote non-adjacent edges of $K_{3,3}$. We will have a contradiction again by showing that the double sum in (22) is even. It suffices to show that for an arbitrary $xy\in E(K_{3,3})$,

$$\sum_{uz\in E(K_{3,3})} P_{\mathscr{B}}(xy,uz)=2k_{xy}. \tag{24}$$

To prove (24), we study how many red–white vertex pairs a bipartition $P_{\mathscr{B}}$ can separate. The possibilities are RR‖WW, RW‖RW, R‖RWW, W‖WRR, −‖RRWW; and in each case the number of separated red–white vertex pairs is even. We showed (24) and have the needed contradiction by $0\equiv 3+2k \bmod 2$. □

8. Proofs

In the forthcoming arguments, we are concerned with the one-point or Alexandrov compactification of the plane π, $\pi^*=\pi\cup\infty$. It is well known that π^* is homeomorphic to a sphere. We say that a closed curve c is *simply drawn* in π^*, if c has a finite number of self-intersections, no point of π^* is covered by c more than twice, and whenever c has a self-intersection, then at that point we have a crossing, and not a tangential (touching) situation. Take two points $a,b\in\pi^*\setminus c$. We say that a simple curve p connecting a and b is *regular* with respect to c, if p does not passes through any self-intersection point of c, p and c have finite intersection, and whenever c has a point of intersection with p, then at that point we have a crossing, and not a tangential (touching) situation.

Lemma 8.1. *If c is a simply drawn closed curve in π^*, then for any $a,b\in\pi^*\setminus c$, there exists a simple curve p connecting a and b, which is regular with respect to c.*

Lemma 8.2. *Let us be given a simply drawn closed curve c in the plane, and $a,b\in\pi^*\setminus c$. Assume that two simple curves, l_1 and l_2 connecting a and b are regular with respect to c. Then $|l_1\cap c|$ and $|l_2\cap c|$ have the same parity.*

Proof. We apply induction on the number of self-intersection points of c. If c is a simple closed curve, i.e. there are no self-intersections, then the conclusion follows from the Jordan curve theorem. If c has a self-intersection at a point a, then redraw c to c' by making change only in a small neighborhood of a, which is disjoint from l_1 and l_2, such that we reduce the number of self-intersections of c by 1. We have $|l_1\cap c'|=|l_1\cap c|$ and $|l_2\cap c|=|l_2\cap c'|$. Then use the inductive hypothesis $|l_1\cap c'|\equiv|l_2\cap c'| \bmod 2$. □

Let us be given a simply drawn closed curve c in π^*. Define two relations on $\pi^* \setminus c$ as follows:

$a \sim_c b$, if there is a simple curve connecting points a and b, which is regular with respect to c, and intersects c even number of times; and

$a|_c b$, if there is a simple curve connecting points a and b, which is regular with respect to c, and intersects c odd number of times.

Note that Lemma 8.2 implies that these relations are well defined.

Lemma 8.3. (i) $\sim_c$ *is an equivalence relation*; *and*

(ii) $\sim_c$ *and* $|_c$ *are complementary relations*, *i.e. for any two points* $a, b \in \pi^* \setminus c$, *exactly one of the relations* $a \sim_c b$ *and* $a|_c b$ *holds; and*

(iii) $\sim_c$ *has exactly two classes.*

Proof. For (i), only the transitivity of the relation is a problem. Assume that $a \sim_c b$ is shown by p_1 and $b \sim_c d$ is shown by p_2. Then, $|p_1 \cap c| + |p_2 \cap c|$ is even. If $p_1 \cup p_2$ is a simple curve, then it is also regular with respect to c, and $b \sim_c d$ is shown by $p_1 \cup p_2$. If $p_1 \cup p_2$ is not a simple curve, let v denote the first point of p_2 on p_1 (following p_1 from a to b). v splits both p_1 and p_2 into two parts, say $p_1 = a p_1' v p_1'' b$, $p_2 = b p_2' v p_2'' d$. Now $a p_1' v p_2'' d$ is a simple curve connecting a to d, and it is regular with respect to c, $v p_1'' b p_2' v$ is a closed curve; and these two new curves, $a p_1' v p_2'' d$ and $v p_1'' b p_2' v$ together, yield $p_1 \cup p_2$. If $v \notin c$, then

$$|p_1 \cap c| + |p_2 \cap c| = |a p_1' v p_2'' \cap c| + |v p_1'' b p_2' v \cap c|. \tag{25}$$

We know that the LHS of (25) is even, $|v p_1'' b p_2' v \cap c|$, which is the finite intersection of two closed curves without tangential intersection, is also even, and hence $|a p_1' v p_2'' d \cap c|$ is even. This is what we had to prove for the transitivity. There is a little more to do, if $v \in c$. Then v is a removable (tangential) intersection point of $a p_1' v p_2'' d$ and c, and also a removable (tangential) intersection point of $v p_1'' b p_2' v$ and c. Remove both tangential intersections by local change, creating a new simple curve p_3 in place of $a p_1' v p_2'' d$ connecting a and d, and a closed curve q in place of $v p_1'' b p_2' v$. Then,

$$|p_1 \cap c| + |p_2 \cap c| \equiv |p_3 \cap c| + 1 + |q \cap c| + 1 \mod 2 \tag{26}$$

and $|p_3 \cap c|$ is even again. It is easy to guarantee that p_3 is still regular with respect to c, and now p_3 is the evidence for $a \sim_c d$.

For (ii), one just needs that a and b can be connected by a simple curve p, which is regular with respect to c; this was Lemma 8.1. (iii) follows. $\square$

In the future we use the notation $-|_c uz$, if u and z are in the same equivalence class for c, i.e. $u \sim_c z$; this notation is suggesting that "c is not separating u and z", while $u|_c z$ is that suggesting that "c is separating u and z". The notation $-|_c uz$ would have been inconvenient for proving Lemma 8.3.

Construction 8.1. *Let us be given four points* a, b, c, d *in this cyclic order on a circle* S *in the plane* π. *Call* A, B, C, D *the four rays*, *perpendicular to* S, *which connect the* *points a,b,c,d to ∞ and stay outside the circle.* (*We follow the convention of using a lower case letter for a point on the circle, and the same upper case letter for the ray, which is perpendicular to circle, connecting this point to* ∞.) *Assume now that $x,y \in \{a,b,c,d\}$, $x \neq y$ are connected by a simple curve q'; and the other two points, $u,z \in \{a,b,c,d\}$, $u \neq z$ are connected by a simple curve p'. Define the closed curves $q = q' \cup X \cup \infty \cup Y$ and $p = p' \cup U \cup \infty \cup Z$.*

We are going to apply the relations from Lemma 8.3 to the closed curves p and q. For this end we have to make a few assumptions:

(i) p and q are simply drawn curves,
(ii) $x,y \notin p$ and $u,z \notin q$,
(iii) $\infty \notin p'$ and $\infty \notin q'$.

Observe that if the cyclic order induced by S on the four points is x,u,y,z or x,z,y,u, then p and q crosses each other at ∞; and if the induced cyclic order is different, p and q intersect in a tangential (touching) situation at ∞. If the touching situation occurs, we could pull p and q slightly apart at ∞ and remove this point of intersection (however, for notational convenience, we just do not count ∞ among the crossing points, if touching happens there). We add

(iv) q' and $X \cup \infty \cup Y$ are regular with respect to p, and
(v) p' and $U \cup \infty \cup Z$ are regular with respect to q.

The following Lemma is crucial:

Lemma 8.4. *Assume conditions* (i)–(v) *above for Construction* 8.1. *If the cyclic order induced by S on the four points is x,u,y,z or x,z,y,u, and either*

$$-|_q uz \quad \textit{and} \quad -|_p xy \quad \textit{or}$$
$$u|_q z \quad \textit{and} \quad x|_p y,$$

then p' and q' crosses odd many times; *and either*

$$-|_q uz \quad \textit{and} \quad x|_p y \quad \textit{or}$$
$$u|_q z \quad \textit{and} \quad -|_p xy,$$

then p' and q' crosses even many times.

If the cyclic order induced by S on the four points is x,y,u,z or x,y,z,u or x,z,u,y or x,u,z,y, and either

$$-|_q uz \quad \textit{and} \quad x|_p y \quad \textit{or}$$
$$u|_q z \quad \textit{and} \quad -|_p xy,$$

then p' and q' crosses odd many times; and either

$$-|_q uz \quad \text{and} \quad -|_p xy \quad \text{or}$$

$$u|_q z \quad \text{and} \quad x|_p y,$$

then p' and q' crosses even many times.

Proof. We prove the first statement from the list of eight statements. All other proofs are similar and left to the reader. By definition, $-|_q uz$ means $u \sim_q z$, i.e. $|p' \cap q|$ is even. Using the definition of q,

$$|p' \cap q'| + |p' \cap Y| + |p' \cap X| \equiv 0 \mod 2. \tag{27}$$

Similarly, $-|_p xy$ means $x \sim_p y$, i.e. $|q' \cap p|$ is even. Using the definition of p,

$$|q' \cap p'| + |q' \cap U| + |q' \cap Z| \equiv 0 \mod 2. \tag{28}$$

Since p and q are closed curves, we have that $|p \cap q|$ is even. Spelling out all terms in $p \cap q$ we obtain

$$1 + |p' \cap q'| + |p' \cap Y| + |p' \cap X| + |q' \cap U| + |q' \cap Z| \equiv 0 \mod 2, \tag{29}$$

where the term 1 stands for the crossing at ∞, which is not removable in the case of cyclic order that we are in. Adding up (27)–(29), we obtain that $|p' \cap q'| \equiv 1 \mod 2$, as required. □

Note that the conditions in the two parts of Lemma 8.4 read in terms of Section 7 *as if* we had xy and uz edges in a graph, and they were in acyclic order and cyclic order, respectively. We introduce the O_S relation—analogously to (14)—for unordered pairs of unordered pairs of points on S, where all four points are distinct, as follows: $O_S(xy, uz) = 0$ for the following cyclic orders on S: x, u, y, z or x, z, y, u; and $O_S(xy, uz) = 1$ for all the other cyclic orders on S. We introduce a relation P^* by

$$P^*(xy, uz) = \begin{cases} 1 & \text{if } u|_q z, \\ 0 & \text{otherwise, i.e. if } -|_q uz \text{ holds,} \end{cases} \tag{30}$$

$$P^*(uz, xy) = \begin{cases} 1 & \text{if } x|_p y, \\ 0 & \text{otherwise, i.e. if } -|_p xy \text{ holds.} \end{cases} \tag{31}$$

Lemma 8.4 immediately implies the next Lemma:

Lemma 8.5. *Assume conditions* (i)–(v) *for Construction* 8.1. *The value of the quantity*

$$\begin{aligned} \mathrm{CR}(p', q') = {} & [1 - O_S(xy, uz)][1 - P^*(xy, uz)][1 - P^*(uz, xy)] \\ & + [1 - O_S(xy, uz)]P^*(xy, uz)P^*(uz, xy) \\ & + O_S(xy, uz)[1 - P^*(xy, uz)]P^*(uz, xy) \\ & + O_S(xy, uz)P^*(xy, uz)][1 - P^*(uz, xy)] \end{aligned} \tag{32}$$

is 1, *if p' and q' crosses odd times, and* 0 *otherwise.*

Proof. Lemma 8.4 tells if $|p' \cap q'|$ is odd or even in certain cases. The first four cases of Lemma 8.4 are defined by $O_S(xy, uz) = 0$, the last four cases of Lemma 8.4 are defined by $O_S(xy, uz) = 1$. By (30) and (31) the conditions in Lemma 8.4 turn into values of P^*. Checking all eight cases of Lemma 8.4 finishes the proof. □

Proof to Theorem 7.1. Let D denote a CR-IODD-optimal normal drawing of a graph G in the plane. Transform this drawing, using a homeomorphism of the plane to itself, into a new drawing, such that the vertices $v_1, v_2, \ldots, v_n$ are in this cyclic order on a circle S. The transformation does not change which edges cross and how many times. From the point v_i extend a ray V_i, outside the circle, to ∞, such that V_i is perpendicular to the circle. Using another homeomorphism of the plane to itself, which fixes all the vertices v_i, we can guarantee that carrying out for any two edges of the drawing Construction 8.1, *assumptions* (i)–(v) hold. From now on we call this drawing D. We have CR-IODD(G)= number the of non-adjacent, unordered edge pairs crossing odd times in D, which is, by Lemma 8.5

$$\sum_{\{p', q'\}} \mathrm{CR}(p', q'),$$

where the summation goes for non-adjacent, unordered edge pairs in D. Observe that for non-adjacent, unordered edge pairs xy, uz, we have $O_S(xy, uz) = O_C(xy, uz)$; and setting for $\mathscr{B}'$ the set of $|_q$ bipartitions arising from q' representations of edges in the drawing D, we have

$$\mathrm{CR}(p', q') = \mathrm{forced}_{\mathscr{B}', \mathscr{C}}(xy, uz),$$

where p' represents the edge xy and q' represents the edge uz. Therefore,

$$\begin{aligned} \text{CR-IODD}(G) &= \frac{1}{2} \sum_{xy} \sum_{\substack{uz \\ \{x,y\} \cap \{u,z\}}} \mathrm{forced}_{\mathscr{B}', \mathscr{C}}(xy, uz) \\ &\geqslant \min_{\mathscr{B}} \frac{1}{2} \sum_{xy} \sum_{\substack{uz \\ \{x,y\} \cap \{u,z\}}} \mathrm{forced}_{\mathscr{B}, \mathscr{C}}(xy, uz). \end{aligned} \tag{33}$$

On the other hand, equality is easy to achieve in (33): given a minimizing bipartition $\mathscr{B}$, and a placement of the vertices of G on the circle S, for edge xy draw a simple curve p' connecting vertices x and y, which does not intersect the rays X and Y, but $p = p' \cup X \cup \infty \cup Y$ generates exactly the relation $\|_{xy} \in \mathscr{B}$ through $|_{p'}$. □

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