Available online at www.sciencedirect.com
SCIENCE DIRECT
J. Math. Anal. Appl. 299 (2004) 563–577
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Initial boundary value problem of the generalized cubic double dispersion equation ☆

Guowang Chen *, Yanping Wang, Shubin Wang

Department of Mathematics, Zhengzhou University, Zhengzhou 450052, People's Republic of China

Received 26 December 2002
Available online 16 September 2004
Submitted by G. Chen

Abstract

In this paper, the existence and the uniqueness of the global generalized solution and the global classical solution for the initial boundary value problem of the generalized cubic double dispersion equation are proved. The nonexistence of global solution for the initial boundary value problem of the generalized cubic double dispersion equation is discussed.

Keywords: Generalized cubic double dispersion equation; Initial boundary value problem; Existence and uniqueness of global solution; Nonexistence of global solution

1. Introduction

In this paper, we study the following initial boundary value problem:

$$u_{tt} - u_{xx} - au_{xxtt} + bu_{x^4} - du_{xxt} = f(u)_{xx}, \quad x \in \Omega,\ t > 0, \tag{1.1}$$

$$u(0,t) = u(l,t) = 0, \quad u_{xx}(0,t) = u_{xx}(l,t) = 0, \quad t \geqslant 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \bar{\Omega}, \tag{1.3}$$

☆ This project is supported by the National Natural Science Foundation of China (Grant No. 10371111) and by the Natural Science Foundation of Henan Province.
* Corresponding author.
E-mail address: chen_gw@371.net (G. Chen).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.05.044

where $u(x,t)$ denotes the unknown function, $u_{x^4} = u_{xxxx}$, $f(s)$ is a given nonlinear function, $\Omega = (0,l)$, $a > 0$, $b > 0$ and d are constants, $u_0(x)$ and $u_1(x)$ are given functions and satisfy the boundary condition (1.2).

We also consider the following equation:

$$u_{tt} - u_{xx} - au_{xxtt} + bu_{x^4} = f(u)_{xx}, \quad x \in \Omega,\ t > 0, \tag{1.4}$$

with the initial boundary value conditions (1.2), (1.3).

There are several examples of physical problems, which can be formulated as Eq. (1.1). Indeed, in some problems of nonlinear wave propagation in waveguide, the interaction of the waveguides and the external medium and, therefore, the possibility of energy exchange through lateral surfaces of the waveguide cannot be neglected. If the model of interaction between the surface of a nonlinear elastic rod, whose material is hyperelastic (e.g., the Murnaghan material), and a medium, proposed by Winkler or by Pasternak [1] is considered, then the longitudinal displacement $u(x,t)$ of the rod satisfies the following double dispersion equation (DDE):

$$u_{tt} - u_{xx} = \frac{1}{4}(6u^2 + au_{tt} - bu_{xx})_{xx}, \tag{1.5}$$

which is obtained by means of the Hamiltonian principle (see [2,3]). Similarly, the general cubic DDE (CDDE)

$$u_{tt} - u_{xx} = \frac{1}{4}(cu^3 + 6u^2 + au_{tt} - bu_{xx} + du_t)_{xx} \tag{1.6}$$

can be obtained (see [2,3]). Here $u(x,t)$ is the longitudinal displacement and is proportional to strain $\frac{\partial U}{\partial x}$, $U(x,t)$ is the transversal displacement, $a > 0$, $b > 0$, $c > 0$ and $d \neq 0$ are some constants depending on the Young modulus, the shear modulus μ, density of waveguide ρ and the Poisson coefficient ν. Obviously, if $f(u) = \frac{c}{4}u^3 + \frac{3}{2}u^2$ and a, b and d are replaced by $\frac{a}{4}$, $\frac{b}{4}$ and $\frac{d}{4}$, respectively, Eq. (1.1) becomes Eq. (1.6). If $d = 0$, $f(u) = \frac{3}{2}u^2$ and a and b are replaced by $\frac{a}{4}$ and $\frac{b}{4}$, respectively, Eq. (1.1) becomes Eq. (1.5).

For the classical problem of condensed matter physics of waves in stratified liquid the following equation was obtained in [4]:

$$u_{tt} - u_{xx} = (cu^3 + du^2 - bu_{xx})_{xx} + (fu)_{xt} + gu_{xxt}, \tag{1.7}$$

whilst wave propagation in a different medium, namely, in a one-dimensional nonlinear elastic solid wave guide, is described in [5] by the DDE with dissipation

$$u_{tt} - u_{xx} = \varepsilon(cu^2 + au_{tt} - bu_{xx} + gu_t)_{xx} + O(\varepsilon^2), \tag{1.8}$$

where u is the longitudinal strain, $a, b, c > 0$, $f, g \neq 0$ and $\varepsilon \leqslant 1$ are constants.

In [6], the Boussinesq equation obtained from the Euler equation for surface waves in irrotational motion reads (see [7])

$$\phi_{tt} - \phi_{xx} - \frac{\varepsilon}{2}\phi_{xxtt} + \frac{\varepsilon}{6}\phi_{x^4} - 3\varepsilon\phi_x\phi_{xx} = 0.$$

If we let $\phi_x = w$, then the above equation becomes

$$u_{tt} - u_{xx} - \frac{\varepsilon}{2}u_{xxtt} + \frac{\varepsilon}{6}u_{x^4} - \frac{3\varepsilon}{2}(u^2)_{xx} = 0. \tag{1.9}$$

When one omits $O(\varepsilon^2)$ and takes $\varepsilon = 1$ in (1.8), Eqs. (1.8) and (1.9) are then the special cases of Eq. (1.1).

In [8], when the nonlinear longitudinal strain solitary waves were studied inside cylindrical elastic rod with microstructure, the related problem was solved by using the pseudocontinuum Cosserat model and the Le Roux continuum model. A procedure was developed for derivation of a governing equation for longitudinal nonlinear strain waves. The equation governing this process is of Boussinesq type, namely, a double dispersive equation

$$u_{tt} - \alpha_1 u_{xx} - \alpha_2 (u^2)_{xx} + \alpha_3 u_{xxtt} - \alpha_4 u_{xxxx} = 0, \tag{1.10}$$

where the coefficients α_i depend upon the elastic parameters of the rod material. If $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_3 < 0$ and $\alpha_4 < 0$, then Eq. (1.10) is also the special case of Eq. (1.1).

There are also several equations with principal term $u_{tt} - u_{xxtt}$, which are closely related to Eq. (1.1). For example, it is known that the equation (called the Bq equation)

$$u_{tt} - u_{xx} - u_{xxxx} = (u^2)_{xx} \tag{1.11}$$

was derived by J. Boussinesq in 1872 to describe shallow water waves. On the other hand, the improved Bq equation (called the IBq equation) reads

$$u_{tt} - u_{xx} - u_{xxtt} = (u^2)_{xx}. \tag{1.12}$$

A modification of the IBq equation analogous of the MKdV equation yields

$$u_{tt} - u_{xx} - u_{xxtt} = (u^3)_{xx}, \tag{1.13}$$

which is called the IMBq equation (see [9]).

References [2,3,5] have studied the strain solutions of Eq. (1.5), while [10–12] investigated the traveling wave solutions of Eqs. (1.5), (1.6). In [3,11] the exact explicit traveling wave solutions to Eqs. (1.8) and (1.9) were obtained, while in [8], the author gave the solitary wave solution of Eq. (1.10). To our best knowledge, however, there has not been any discussion on global solutions of the initial boundary value problem for Eq. (1.1) in the literature.

In [13] the authors proved that the initial boundary value problem for Eq. (1.13) has a unique global generalized or classical solution. The basic steps of the proof in [13] can be summarized as follows: First, the initial boundary value problem for Eq. (1.13) is reduced to an equivalent integral equation by using Green's function for a boundary value problem of a second-order ordinary differential equation, and then the existence and uniqueness of generalized and classical local solutions to this integral equation is obtained by applying the contraction mapping principle, and finally, the extension of the solution to the whole interval $[0, T]$ is guaranteed by the extension theorem.

The paper [13] gave the sufficient conditions of the nonexistence of global solution to the initial boundary value problem for Eq. (1.12), too.

The aim of the present paper is to prove that under certain conditions, the problem (1.1)–(1.3) possesses a unique global generalized and classical solutions by using different methods from [13], and to give sufficient conditions of the nonexistence of global solutions to the problem (1.1)–(1.3). Moreover, as applications of our abstract theorems, we shall prove that the problem (1.6), (1.2), (1.3) has a unique global generalized solution, while the problem (1.5), (1.2), (1.3) does not possess global generalized and classical solutions under certain assumptions.

In order to obtain the global generalized and classical solutions of the problem (1.1)–(1.3), we shall consider the following auxiliary problem:

$$v_{tt} - v_{xx} - av_{xxtt} + bv_{x^4} - dv_{xxt} = f(v_x)_x, \quad x \in \Omega,\ t > 0, \tag{1.14}$$

$$v_x(0,t) = v_x(l,t) = 0, \quad v_{xxx}(0,t) = v_{xxx}(l,t) = 0, \quad t \geqslant 0, \tag{1.15}$$

$$v(x,0) = \varphi(x), \quad v_t(x,0) = \psi(x), \quad x \in \bar{\Omega}. \tag{1.16}$$

We first show that there is a smooth global classical solution of the problem (1.14)–(1.16), and then by setting $v_x(x,t) = u(x,t)$, $\varphi_x(x) = u_0(x)$ and $\psi_x(x) = u_1(x)$, we obtain the global existence of the generalized and classical solutions to the problem (1.1)–(1.3).

This paper is organized as follows. In Section 2, we prove the existence and uniqueness of global generalized and classical solutions to the problem (1.14)–(1.16). The existence and uniqueness of global generalized and classical solutions of the problem (1.1)–(1.3) are given in Section 3. In Section 4, the nonexistence of global solutions to the problem (1.1)–(1.3) is discussed, and in Section 5 we study the problems (1.6), (1.2), (1.3) and (1.5), (1.2), (1.3).

2. Global solution of the problem (1.14)–(1.16)

Let $\{y_s(x)\}$ be the orthonormal base in $L^2(\Omega)$ composed of the eigenfunctions of the eigenvalue problem

$$y'' + \lambda y = 0, \quad x \in \Omega, \quad y'(0) = y'(l) = 0,$$

corresponding to eigenvalue λ_i $(i = 1, 2, \ldots)$, where "$'$" denotes the derivative with respect to x. Let $v_N(x,t) = \sum_{i=1}^{N} \alpha_{Ni}(t) y_i(x)$ be the Galerkin approximate solution of the problem (1.14)–(1.16), where $\alpha_{Ni}(t)$ $(i = 1, 2, \ldots, N)$ are functions to be determined, N is a natural number. Suppose that the initial data $\varphi(x)$ and $\psi(x)$ can be expressed by $\varphi(x) = \sum_{i=1}^{\infty} \beta_i y_i(x)$, $\psi(x) = \sum_{i=1}^{\infty} \gamma_i y_i(x)$, respectively, where β_i, γ_i $(i = 1, 2, \ldots)$ are constants. Then, substituting the approximate solution $v_N(x,t)$ into (1.14)–(1.16), we obtain that $v_N(x,t)$ solves the following problem:

$$v_{Ntt} - v_{Nxx} - av_{Nxxtt} + bv_{Nx^4} - dv_{Nxxt} = f(v_{Nx})_x, \tag{2.1}$$

$$v_{Nx}(0,t) = v_{Nx}(l,t) = 0, \quad v_{Nx^3}(0,t) = v_{Nx^3}(l,t) = 0, \tag{2.2}$$

$$v_N(x,0) = \varphi_N(x), \quad v_{Nt}(x,0) = \psi_N(x). \tag{2.3}$$

Multiplying both sides of (2.1) and (2.3) by $y_s(x)$, respectively, and integrating on Ω, we get

$$(v_{Ntt} - v_{Nxx} - av_{Nxxtt} + bv_{Nx^4} - dv_{Nxxt}, y_s) = \big(f(v_{Nx})_x, y_s\big), \tag{2.4}$$

$$\alpha_{Ns}(0) = \beta_s, \quad \alpha_{Nst}(0) = \gamma_s, \quad s = 1, 2, \ldots, N, \tag{2.5}$$

where $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega)$.

Lemma 2.1. *Suppose that $f \in C^1(R)$, and there is a constant C_0 such that $f'(s) \geqslant C_0$ for any $s \in R$, $\varphi \in H^2(\Omega)$, $\psi \in H^1(\Omega)$ and $\varphi(x), \psi(x)$ satisfy the boundary conditions* (1.15). *Then for any N, the Cauchy problem* (2.4), (2.5) *has a global classical solution $\alpha_{Ns} \in C^2[0,T]$ $(s = 1, 2, \dots, N)$. Moreover, the following estimate holds*:

$$\left\|v_N(\cdot,t)\right\|^2_{H^2(\Omega)} + \left\|v_{Nt}(\cdot,t)\right\|^2_{H^1(\Omega)} \leqslant C_1(T), \quad t \in [0,T], \tag{2.6}$$

where and in the sequel $C_1(T)$ and $C_i(T)$ $(i = 1, 2, \dots)$ are constants which depend on T, but not on N.

Proof. Let $f_0(s) = f(s) - \delta s - f(0)$ with $\delta = \min\{C_0, 0\}$ $(\leqslant 0)$, then $f_0(0) = 0$, $f_0'(s) = f'(s) - \delta \geqslant 0$ and $f_0(s)$ is a monotonically increasing function. Thus $F(s) = \int_0^s f_0(\tau)\,d\tau \geqslant 0$. Clearly, Eq. (1.14) is equivalent to the following equation:

$$v_{tt} - v_{xx} - av_{xxtt} + bv_{x^4} - dv_{xxt} - \delta v_{xx} = f_0(v_x)_x. \tag{2.7}$$

Obviously, Eq. (2.4) is equivalent to the following system:

$$\begin{aligned}&(v_{Ntt} - v_{Nxx} - av_{Nxxtt} + bv_{Nx^4} - dv_{Nxxt} - \delta v_{Nxx}, y_s) = \big(f_0(v_{Nx})_x, y_s\big),\\ &s = 1, 2, \dots, N.\end{aligned} \tag{2.8}$$

Multiplying both sides of Eq. (2.8) by $2\alpha_{Nst}(t)$, summing up for $s = 1, 2, \dots, N$, adding $2(v_N, v_{Nt})$ to the both sides, integrating by parts and using Gronwall's inequality, we have

$$\begin{aligned}&\left\|v_N(\cdot,t)\right\|^2_{H^2(\Omega)} + \left\|v_{Nt}(\cdot,t)\right\|^2_{H^1(\Omega)}\\ &\quad\leqslant e^{C_2(|\delta|+2|d|+1)T}\left(\|\varphi\|^2_{H^2(\Omega)} + \|\psi\|^2_{H^1(\Omega)} + 2\int_\Omega F\big(\varphi_x(x)\big)\,dx + 1\right),\\ &\quad t \in [0,T],\end{aligned} \tag{2.9}$$

where $\|\cdot\|$ denotes the norm of the space $L^2(\Omega)$. Thus (2.6) follows from (2.9) immediately.

Similarly to [14], we can prove by applying (2.9) and the Leray–Schauder fixed point theorem [15] that the Cauchy problem (2.4), (2.5) has a solution $\alpha_{Ns} \in C^2[0,T]$ $(s = 1, 2, \dots, N)$. The proof is complete. □

Lemma 2.2. *Suppose that the conditions of Lemma* 2.1 *hold. If $f \in C^3(R)$, $\varphi \in H^5(\Omega)$ and $\psi \in H^4(\Omega)$, then, the approximate solution of the problem* (1.14)–(1.16) *satisfies the following estimate*:

$$\|v_N\|^2_{H^5(\Omega)} + \|v_{Nt}\|^2_{H^4(\Omega)} + \|v_{Ntt}\|^2_{H^3(\Omega)} \leqslant C_2(T), \quad 0 \leqslant t \leqslant T. \tag{2.10}$$

Proof. Multiplying Eq. (2.4) by $2\lambda_s^2\alpha_{Nst}$, summing up for $s = 1, 2, \dots, N$, integrating by parts with respect to x, utilizing (2.6), and recalling that the space $H^2(\Omega)$ is continuously imbedded into $C^1(\bar{\Omega})$, we infer that

$$\begin{aligned}&\frac{d}{dt}\big(\|v_{Nxxt}\|^2 + \|v_{Nx^3}\|^2 + a\|v_{Nx^3t}\|^2 + b\|v_{Nx^4}\|^2\big)\\ &\quad\leqslant C_3(T)\big(\|v_{Nxx}\|^2_{L^4(\Omega)} + \|v_{Nx^3}\|^2\big) + 2\big(|d|+1\big)\|v_{Nx^3t}\|^2.\end{aligned} \tag{2.11}$$

Using the Gagliardo–Nirenberg interpolation theorem, Young's inequality and Gronwall's inequality, we conclude

$$\|v_{Nxxt}\|^2+\|v_{Nx^3}\|^2+\|v_{Nx^3t}\|^2+\|v_{Nx^4}\|^2 \leqslant C_4(T)\big(\|\varphi\|^2_{H^4(\Omega)}+\|\psi\|^2_{H^3(\Omega)}+1\big),\quad t\in[0,T]. \tag{2.12}$$

Similarly, multiplying Eq. (2.4) by $-2\lambda_s^3\alpha_{Nst}$, summing up for $s=1,2,\dots,N$, and integrating with respect to t, we get

$$\begin{aligned}
&\|v_{Nx^3t}\|^2+\|v_{Nx^4}\|^2+a\|v_{Nx^4t}\|^2+b\|v_{Nx^5}\|^2\\
&\quad\leqslant 2|d|\int_0^t\|v_{Nx^4\tau}\|^2\,d\tau-2\int_0^t f''(0)v^2_{Nxx}(l,\tau)v_{Nx^4\tau}(l,\tau)\,d\tau\\
&\qquad+2\int_0^t f''(0)v^2_{Nxx}(0,\tau)v_{Nx^4\tau}(0,\tau)\,d\tau\\
&\qquad+2\int_0^t\int_\Omega\big(f'''(v_{Nx})v^3_{Nxx}+3f''(v_{Nx})v_{Nxx}v_{Nx^3}+f'(v_{Nx})v_{Nx^4}\big)v_{Nx^4\tau}\,dx\,d\tau\\
&\qquad+\|\psi_{x^3}\|^2+\|\varphi_{x^4}\|^2+a\|\psi_{x^4}\|^2+b\|\varphi_{x^5}\|^2.
\end{aligned} \tag{2.13}$$

Using the integration by parts, the Sobolev imbedding theorem, (2.6) and (2.12), we find that

$$\begin{aligned}
&-2\int_0^t f''(0)v^2_{Nxx}(l,\tau)v_{Nx^4\tau}(l,\tau)\,d\tau\\
&\quad=-2f''(0)\big[v^2_{Nxx}(l,t)v_{Nx^4}(l,t)-v^2_{Nxx}(l,0)v_{Nx^4}(l,0)\big]\\
&\qquad+2f''(0)\int_0^t\big(v^2_{Nxx}(l,\tau)\big)_\tau v_{Nx^4}(l,\tau)\,d\tau\\
&\quad\leqslant 2\big|f''(0)\big|\Big\{\sup_{0\leqslant t\leqslant T}\big(\big\|v_{Nxx}(\cdot,t)\big\|_{C(\bar\Omega)}\big)^2\big\|v_{Nx^4}(\cdot,t)\big\|_{C(\bar\Omega)}+\|\varphi_{xx}\|^2_{C(\bar\Omega)}\|\varphi_{x^4}\|_{C(\bar\Omega)}\\
&\qquad+4\int_0^t\big\|v_{Nxx}(\cdot,\tau)\big\|_{C(\bar\Omega)}\big\|v_{Nxx\tau}(\cdot,\tau)\big\|_{C(\bar\Omega)}\big\|v_{Nx^4}(\cdot,\tau)\big\|_{C(\bar\Omega)}\,d\tau\Big\}\\
&\quad\leqslant C_5(T)+C_6\|\varphi\|_{H^3(\Omega)}\|\varphi\|_{H^5(\Omega)}+\frac{b}{4}\big\|v_{Nx^5}(\cdot,t)\big\|^2+\int_0^t\big\|v_{Nx^5}(\cdot,\tau)\big\|^2\,d\tau,
\end{aligned} \tag{2.14}$$

where $\|\cdot\|_{C(\bar\Omega)}$ denotes the norm in the space $C(\bar\Omega)$.

Similarly to (2.14), we can prove

$$2\int_0^t f''(0)v_{Nxx}^2(0,\tau)v_{Nx^4\tau}(0,\tau)\,d\tau \leqslant C_7(T)+C_8\|\varphi\|_{H^3(\Omega)}^2\|\varphi\|_{H^5(\Omega)}+\frac{b}{4}\big\|v_{Nx^5}(\cdot,t)\big\|^2+\int_0^t\big\|v_{Nx^5}(\cdot,\tau)\big\|^2\,d\tau. \tag{2.15}$$

By the Sobolev embedding theorem, (2.6) and (2.12), we conclude

$$2\int_0^t\int_\Omega \big(f'''(v_{Nx})v_{Nxx}^3+3f''(v_{Nx})v_{Nxx}v_{Nx^3}+f'(v_{Nx})v_{Nx^4}\big)v_{Nx^4t}\,dx\,d\tau \leqslant C_9(T)+\int_0^t\|v_{Nx^4\tau}\|^2\,d\tau. \tag{2.16}$$

Inserting (2.14)–(2.16) into (2.13) and using Gronwall's inequality, we obtain

$$\|v_{Nx^3t}\|^2+\|v_{Nx^4}\|^2+\|v_{Nx^4t}\|^2+\|v_{Nx^5}\|^2\leqslant C_{10}(T),\quad t\in[0,T]. \tag{2.17}$$

Multiplying both sides of Eq. (2.4) by $\alpha_{Nstt}(t)+\lambda_s^2\alpha_{Nstt}(t)$, summing up for $s=1,2,\ldots,N$ and using the Cauchy inequality, the estimates (2.6), (2.17) and the Sobolev embedding theorem, we arrive at

$$\|v_{Ntt}\|_{H^3(\Omega)}^2\leqslant C_{11}(T),\quad 0\leqslant t\leqslant T. \tag{2.18}$$

This completes the proof. □

Theorem 2.1. *Under the conditions of Lemma* 2.2, *the problem* (1.14)–(1.16) *has a unique global generalized solution*

$$v\in C\big([0,T];H^5(\Omega)\big)\cap C^1\big([0,T];H^4(\Omega)\big)\cap C^2\big([0,T];H^3(\Omega)\big)=A. \tag{2.19}$$

Proof. From (2.10), the Sobolev imbedding theorem and the compactness principle, we see that the problem (1.14)–(1.16) has a global generalized solution $v\in A$. The uniqueness of solutions is obvious. The proof is complete. □

Lemma 2.3. *Suppose that the conditions of Lemma* 2.2 *hold. If* $\varphi\in H^7(\Omega)$, $\psi\in H^6(\Omega)$, $f\in C^4(R)$ *and* $f^{(i)}(0)=0$ $(i=2,4)$, *then, the approximate solution of the problem* (1.14)–(1.16) *satisfies the following estimate*:

$$\|v_N\|_{H^7(\Omega)}^2+\|v_{Nt}\|_{H^6(\Omega)}^2+\|v_{Ntt}\|_{H^5(\Omega)}^2+\|v_{Nt^3}\|_{H^4(\Omega)}^2\leqslant C_{12}(T),\quad t\in[0,T]. \tag{2.20}$$

Proof. Multiplying Eq. (2.4) by $-2\lambda_s^5\alpha_{Nst}(t)$, summing up for $s=1,2,\ldots,N$, integrating by parts and using Gronwall's inequality, we obtain

$$\begin{aligned}&\|v_{Nx^5t}\|^2+\|v_{Nx^6}\|^2+\|v_{Nx^6t}\|^2+\|v_{Nx^7}\|^2\\&\quad\leqslant C_{13}(T)\big(\|\varphi\|^2_{H^7(\Omega)}+\|\psi\|^2_{H^6(\Omega)}+1\big),\quad t\in[0,T].\end{aligned}\tag{2.21}$$

In the same manner, we have by multiplying Eq. (2.4) by $\lambda_s^4\alpha_{Nstt}(t)$ that

$$\begin{aligned}\|v_{Nx^4tt}\|^2+a\|v_{Nx^5tt}\|^2&=\big(-v_{Nx^5}+bv_{Nx^7}-dv_{Nx^5t}-f(v_{Nx})_{x^4},v_{Nx^5tt}\big)\\&\leqslant\frac{a}{4}\|v_{Nx^5tt}\|^2+C_{14}\big(\|v_{Nx^5}\|^2+\|v_{Nx^7}\|^2+\|v_{Nx^5t}\|^2+\big\|f(v_{Nx})_{x^4}\big\|^2\big).\end{aligned}\tag{2.22}$$

It follows from (2.10), (2.21) and (2.22) that

$$\|v_{Nx^4tt}\|^2+\|v_{Nx^5tt}\|^2\leqslant C_{15}(T),\quad t\in[0,T].\tag{2.23}$$

Similarly, we obtain

$$\|v_{Nx^3t^3}\|^2+\|v_{Nx^4t^3}\|^2\leqslant C_{16}(T),\quad 0\leqslant t\leqslant T.\tag{2.24}$$

Combining the estimates (2.6), (2.10), (2.21), (2.23) with (2.24), we get the estimate (2.20). This completes the proof. □

Using Lemma 2.3 and following the same procedure as in the proof of Theorem 2.1, we have

Theorem 2.2. *Under the conditions of Lemma* 2.3*, the problem* (1.14)–(1.16) *has a unique global classical solution*

$$v\in C\big([0,T];C^5(\bar\Omega)\big)\cap C^1\big([0,T];C^4(\bar\Omega)\big)\cap C^2\big([0,T];C^3(\bar\Omega)\big)=B.$$

3. Global solutions of the problem (1.1)–(1.3)

Theorem 3.1. *Suppose that $u_0\in H^4(\Omega)$, $u_1\in H^3(\Omega)$, $f\in C^3(R)$ and $f'(s)$ is bounded below. Then, the problem* (1.1)–(1.3) *has a unique global generalized solution*

$$u\in C\big([0,T];H^4(\Omega)\big)\cap C^1\big([0,T];H^3(\Omega)\big)\cap C^2\big([0,T];H^2(\Omega)\big)=D.$$

Proof. Differentiating (2.1) with respect to x, one gets

$$v_{Nxtt}-v_{Nx^3}-av_{Nx^3tt}+bv_{Nx^5}-dv_{Nx^3t}=f(v_{Nx})_{xx}.\tag{3.1}$$

Let

$$v_{Nx}(x,t)=u_N(x,t).\tag{3.2}$$

Substituting (3.2) into (3.1), (2.2) and (2.3), one obtains

$$u_{Ntt}-u_{Nxx}-au_{Nxxtt}+bu_{Nx^4}-dv_{Nxxt}=f(u_N)_{xx},\tag{3.3}$$

$$u_N(0,t)=u_N(l,t)=0,\quad u_{Nxx}(0,t)=u_{Nxx}(l,t)=0,\tag{3.4}$$

$$u_N(x,0)=u_{0N}(x),\quad u_{Nt}(x,0)=u_{1N}(x),\tag{3.5}$$

where in (3.5), $u_{0N}(x)=\sum_{i=1}^{N} a_i y_i(x)$ and $u_{1N}(x)=\sum_{i=1}^{N} b_i y_i(x)$ are the approximations of the

$$u_0(x)=\sum_{i=1}^{\infty} a_i y_i(x), \qquad u_1(x)=\sum_{i=1}^{\infty} b_i y_i(x),$$

a_i, b_i constants, respectively.

From (3.2) and (2.10) it follows that

$$\|u_N\|_{H^4(\Omega)}+\|u_{Nt}\|_{H^3(\Omega)}+\|u_{Ntt}\|_{H^2(\Omega)} \leqslant C_{17}(T), \quad 0\leqslant t\leqslant T. \tag{3.6}$$

From (3.6) and the Sobolev imbedding theorem, we find that

$$\|u_N\|_{C^{3,\lambda}(\bar\Omega)}+\|u_{Nt}\|_{C^{2,\lambda}(\bar\Omega)}+\|u_{Ntt}\|_{C^{1,\lambda}(\bar\Omega)} \leqslant C_{18}(T), \quad 0\leqslant t\leqslant T, \tag{3.7}$$

where $0<\lambda\leqslant\frac{1}{2}$. It follows from (3.7) and the Ascoli–Arzelá theorem that there exist a function $u(x,t)$ and a subsequence of $\{u_N(x,t)\}$, still denoted by $\{u_N(x,t)\}$, such that as $N\to\infty$, $\{u_{Nx^i}(x,t)\}$ $(i=0,1,2)$ and $\{u_{Nx^it}(x,t)\}$ $(i=0,1)$ converge uniformly to $u_{x^i}(x,t)$ $(i=0,1,2)$ and $u_{x^it}(x,t)$ $(i=0,1)$ on $\bar Q_T$, respectively. The subsequences $\{u_{Nx^i}(x,t)\}$ $(i=0,1,2,3,4)$, $\{u_{Nx^it}(x,t)\}$ $(i=0,1,2,3)$ and $\{u_{Nx^itt}(x,t)\}$ $(i=0,1,2)$ converge to $u_{x^i}(x,t)$ $(i=0,1,2,3,4)$, $u_{x^it}(x,t)$ $(i=0,1,2,3)$ and $u_{x^itt}(x,t)$ $(i=0,1,2)$ weakly in $L^2(Q_T)$, respectively. Thus the initial boundary value problem (1.1)–(1.3) has a global generalized solution $u\in D$.

Now, we prove the uniqueness of the solution for the problem (1.1)–(1.3).

Let $u_1(x,t)$ and $u_2(x,t)$ be two generalized solutions of the problem (1.1)–(1.3). Thus, $u(x,t)=u_1(x,t)-u_2(x,t)$ satisfies the following problem:

$$u_{tt}-u_{xx}-au_{xxtt}+bu_{x^4}-du_{xxt}=f(u_1)_{xx}-f(u_2)_{xx}, \quad x\in\Omega,\ t>0, \tag{3.8}$$

$$u(0,t)=u(l,t)=0, \quad u_{xx}(0,t)=u_{xx}(l,t)=0, \quad t\geqslant 0, \tag{3.9}$$

$$u(x,0)=0, \quad u_t(x,0)=0, \quad x\in\Omega. \tag{3.10}$$

Multiplying Eq. (3.8) by $2u_t$, integrating over Ω, adding $2\int_\Omega uu_t\,dx$ to the resulting equation, and integrating by parts, we obtain

$$\begin{aligned}
&\frac{d}{dt}\big(\|u\|^2+\|u_t\|^2+\|u_x\|^2+a\|u_{xt}\|^2+b\|u_{xx}\|^2\big)\\
&\quad=-2d\|u_{xt}\|^2-2\int_\Omega\big\{f''\big(u_1+\theta(u_2-u_1)\big)uu_{1x}+f'(u_2)u_x\big\}u_{xt}\,dx+2\int_\Omega uu_t\,dx\\
&\quad\leqslant 2|d|\|u_{xt}\|^2+2\max_{0\leqslant t\leqslant T,\ x\in\Omega}\big|f''\big(u_1+\theta(u_2-u_1)\big)u_{1x}\big|\int_\Omega|u||u_{xt}|\,dx\\
&\qquad+2\max_{0\leqslant t\leqslant T,\ x\in\Omega}\big|f'(u_2)\big|\int_\Omega|u_x||u_{xt}|\,dx+\big(\|u\|^2+\|u_t\|^2\big)\\
&\quad\leqslant C_{19}(T)\big(\|u\|^2+\|u_t\|^2+\|u_x\|^2+\|u_{xt}\|^2\big),
\end{aligned}$$

which together with Gronwall's inequality yields

$$\|u\|^2+\|u_t\|^2+\|u_x\|^2+\|u_{xt}\|^2+\|u_{xx}\|^2=0.$$

Hence, we have the uniqueness. □

Theorem 3.2. *Suppose that* $u_0 \in H^6(\Omega)$, $u_1 \in H^5(\Omega)$, $f \in C^4(\Omega)$, $f^{(i)}(0) = 0$ $(i = 2, 4)$ *and* $f'(s)$ *is bounded below. Then, the problem* (1.1)–(1.3) *has a unique global classical solution* $u(x,t)$.

Proof. By virtue of Theorem 2.2, $v(x,t) \in B$ satisfies Eq. (1.14) and the initial boundary conditions (1.15) and (1.16). Differentiating Eq. (1.14) with respect to x and substituting $v_x(x,t) = u(x,t)$ into the resulting equation, we see that $u(x,t)$ is the global classical solution of the problem (1.1)–(1.3). The uniqueness of the solution is obvious. The theorem is proved. □

4. Nonexistence of global solutions of the problem (1.1)–(1.3)

Theorem 4.1. *Let* $u(x,t)$ *be a generalized solution or a classical solution of the problem* (1.1)–(1.3). *Suppose that the following conditions are satisfied*:

(1) $$-\frac{\pi}{2l}\int_\Omega u_0(x)\sin\frac{\pi x}{l}\,dx = \alpha > 0, \qquad -\frac{\pi}{2l}\int_\Omega u_1(x)\sin\frac{\pi x}{l}\,dx = \beta > 0;$$

(2) (i) $f(s) \in C^2(R)$ *is an even and convex function satisfying*

$$f(0) = 0 \quad \text{and} \quad f(\alpha) - \frac{l^2 + b\pi^2}{l^2}\alpha \geqslant 0;$$

(ii) $f(s)$ *grows fast enough as* $s \to \infty$, *so that the integral*

$$\mathcal{B} = \frac{d\pi^2}{l^2 + a\pi^2}\int_\alpha^\infty \left[\beta^2 + \frac{2\pi^2}{l^2 + a\pi^2}\int_\alpha^y \left(f(s) - \frac{l^2 + b\pi^2}{l^2}s\right)ds\right]^{-\frac{1}{2}} dy \tag{4.1}$$

converges when $d > 0$, *moreover,* $\mathcal{B} < 1$; *the integral*

$$\bar{T}_2 = \int_\alpha^\infty \left[\beta^2 + \frac{2\pi^2}{l^2 + a\pi^2}\left(\int_\alpha^y f(s)\,ds - \frac{l^2 + b\pi^2}{2l^2}y^2\right) + \frac{\pi^2(l^2 + b\pi^2)}{l^2(l^2 + a\pi^2)}\alpha^2\right]^{-\frac{1}{2}} dy \tag{4.2}$$

converges for $d \leqslant 0$.

Then, when $d > 0$,

$$\lim_{t \to t_0^-} \sup_{x \in \Omega} |u(x,t)| = \infty \tag{4.3}$$

for some finite time $t_0 \leqslant \bar{T}_1 = -\frac{l^2 + a\pi^2}{d\pi^2}\ln(1 - \mathcal{B})$; *when* $d \leqslant 0$,

$$\lim_{t \to t_0^-} \sup_{x \in \Omega} |u(x,t)| = \infty \tag{4.4}$$

for some finite time $t_0 \leqslant \bar{T}_2$, *where* $\bar{T}_2$ *is given by* (4.2).

Proof. Let

$$\phi(t) = -\frac{\pi}{2l}\int_{\Omega} u(x)\sin\frac{\pi x}{l}\,dx.$$

Multiplying both sides of Eq. (1.1) by $\frac{\pi}{2l}\sin\frac{\pi x}{l}$ and integrating by parts, we obtain

$$\left(1+\frac{a\pi^2}{l^2}\right)\ddot{\phi} + \left(\frac{\pi^2}{l^2}+\frac{b\pi^4}{l^4}\right)\phi + \frac{d\pi^2}{l^2}\dot{\phi} = -\frac{\pi}{2l}\int_{\Omega} f(u)_{xx}\sin\frac{\pi x}{l}\,dx. \tag{4.5}$$

Since $f(s)$ is even and convex, we have by using integration by parts and the Jensen inequality that

$$\begin{aligned}
-\frac{\pi}{2l}\int_{\Omega} f(u)_{xx}\sin\frac{\pi x}{l}\,dx &= \frac{\pi^3}{2l^3}\int_{\Omega} f(u)\sin\frac{\pi x}{l}\,dx \\
&\geqslant \frac{\pi^2}{l^2} f\left(-\frac{\pi}{2l}\int_{\Omega} u\sin\frac{\pi x}{l}\,dx\right) = \frac{\pi^2}{l^2} f(\phi).
\end{aligned} \tag{4.6}$$

Substituting (4.6) into (4.5), one gets

$$\ddot{\phi} + \frac{d\pi^2}{l^2+a\pi^2}\dot{\phi} + \frac{\pi^2(l^2+b\pi^2)}{l^2(l^2+a\pi^2)}\phi \geqslant \frac{\pi^2}{l^2+a\pi^2} f(\phi) \tag{4.7}$$

with $\phi(0) = \alpha > 0$, $\dot{\phi}(0) = \beta > 0$.

In order to prove $\dot{\phi}(t) > 0$ for any $t > 0$, we first show that $f(s) - \frac{l^2+b\pi^2}{l^2}s \geqslant 0$ for all $s \geqslant \alpha$. In fact, since $f \in C^2(R)$ is even and convex function, we have $f''(s) \geqslant 0$ and $f'(0) = 0$. Denote $F(s) = f(s) - \frac{l^2+b\pi^2}{l^2}s$, then $F''(s) = f''(s) \geqslant 0$. Thus $F'(s)$ is a monotonically increasing function. By virtue of

$$F(0) = f(0) = 0, \qquad F'(0) = f'(0) - \frac{l^2+b\pi^2}{l^2} = -\frac{l^2+b\pi^2}{l^2} < 0$$

and $F(\alpha) \geqslant 0$, we see that $F(s)$ takes its minimum at some point s_0 in $(0,\alpha)$ and $F'(s_0) = 0$. Thanks to the monotone increase of $F'(s)$ we find that $F'(s) \geqslant F'(s_0) = 0$, for $s \geqslant s_0$, i.e., when $s \geqslant s_0$, $F(s)$ is a monotonically increasing function. In particular, $F(s)$ is monotonically increasing in $[\alpha,\infty)$ and $F(s) \geqslant F(\alpha) \geqslant 0$. Thus $f(s) - \frac{l^2+b\pi^2}{l^2}s \geqslant 0$ for all $s \geqslant \alpha$.

Now, we prove $\dot{\phi}(t) > 0$ for any $t > 0$. Suppose that this result is false. Then there is $t_0 > 0$, such that when $0 < t < t_0$, $\dot{\phi}(t) > 0$, but $\dot{\phi}(t_0) = 0$.

First of all, we consider the case $d > 0$. Multiplying (4.7) by $e^{\frac{d\pi^2}{l^2+a\pi^2}t}$ and integrating over $(0,t)$, we obtain

$$\int_0^t \frac{d}{d\tau}\left(\dot{\phi}e^{\frac{d\pi^2}{l^2+a\pi^2}\tau}\right)d\tau \geqslant \frac{\pi^2}{l^2+a\pi^2}\int_0^t \left[f(\phi) - \frac{l^2+b\pi^2}{l^2}\phi\right]e^{\frac{d\pi^2}{l^2+a\pi^2}\tau}\,d\tau. \tag{4.8}$$

By the definition of t_0, $\phi(t) \geqslant \alpha$ for $0 \leqslant t \leqslant t_0$. It follows from (4.8) that

$$\dot{\phi} \geqslant e^{-\frac{d\pi^2}{l^2+a\pi^2}t}\left\{\beta^2 + \frac{\pi^2}{l^2+a\pi^2}\int_0^t \left[f(\phi) - \frac{l^2+b\pi^2}{l^2}\phi\right] e^{\frac{d\pi^2}{l^2+a\pi^2}\tau}\,d\tau\right\} > 0,$$
$$t \in (0, t_0).$$

Therefore, $\dot{\phi}(t_0) > 0$. This contradicts the fact that $\dot{\phi}(t_0) = 0$. Hence $\dot{\phi}(t) > 0$ for $t > 0$.

It is easy to see that $\phi(t) > \alpha$ for $t > 0$. Hence, when $d > 0$, multiplying (4.7) by $2e^{\frac{2d\pi^2}{l^2+a\pi^2}t}\dot{\phi}$, integrating over $(0, t)$ and observing that $e^{\frac{2d\pi^2}{l^2+a\pi^2}t} > 1$, we get

$$\begin{aligned} e^{\frac{2d\pi^2}{l^2+a\pi^2}t}\dot{\phi}^2 &\geqslant \beta^2 + \frac{2\pi^2}{l^2+a\pi^2}\int_0^t e^{\frac{2d\pi^2}{l^2+a\pi^2}\tau}\left[f(\phi) - \frac{l^2+b\pi^2}{l^2}\phi\right]\dot{\phi}\,d\tau \\ &\geqslant \beta^2 + \frac{2\pi^2}{l^2+a\pi^2}\int_{\phi(0)}^{\phi(t)}\left[f(s) - \frac{l^2+b\pi^2}{l^2}s\right]ds. \end{aligned}$$

Thus

$$\dot{\phi} \geqslant e^{-\frac{d\pi^2}{l^2+a\pi^2}t}\left\{\beta^2 + \frac{\pi^2}{l^2+a\pi^2}\int_\alpha^{\phi(t)}\left[f(s) - \frac{l^2+b\pi^2}{l^2}s\right]ds\right\}^{\frac{1}{2}}, \quad t > 0. \tag{4.9}$$

By separation of variables from (4.9), we deduce

$$\frac{d\phi}{\left\{\beta^2 + \frac{2\pi^2}{l^2+a\pi^2}\int_\alpha^{\phi(t)}\left(f(s) - \frac{l^2+b\pi^2}{l^2}s\right)ds\right\}^{\frac{1}{2}}} \geqslant e^{-\frac{d\pi^2}{l^2+a\pi^2}t}\,dt. \tag{4.10}$$

Integrating (4.10) over $(0, t)$, we have

$$1 - e^{-\frac{d\pi^2}{l^2+a\pi^2}t} \leqslant \frac{d\pi^2}{l^2+a\pi^2}\int_\alpha^{\phi(t)}\left\{\beta^2 + \frac{2\pi^2}{l^2+a\pi^2}\int_\alpha^y\left(f(s) - \frac{l^2+b\pi^2}{l^2}s\right)ds\right\}^{-\frac{1}{2}}dy. \tag{4.11}$$

Thus $\phi(t)$ develops a singularity in finite time $t_0 \leqslant \bar{T}_1 = -\frac{l^2+a\pi^2}{d\pi^2}\ln(1 - \mathcal{B})$.

Finally, since $\phi(t) > 0$, we obtain

$$\phi(t) = \left|\phi(t)\right| = \left|-\frac{\pi}{2l}\int_\Omega u(x,t)\sin\frac{\pi x}{l}\,dx\right| \leqslant \sup_{x\in\Omega}\left|u(x,t)\right|,$$

which proves (4.3).

In the case of $d \leqslant 0$, we see from (4.7) that

$$\ddot{\phi} \geqslant \frac{\pi^2}{l^2+a\pi^2}\left(f(\phi) - \frac{l^2+b\pi^2}{l^2}\phi\right). \tag{4.12}$$

Similarly, we can prove $\dot\phi(t) > 0$ for $t > 0$. We multiply the differential inequality (4.12) by $2\dot\phi(t)$ to get

$$\frac{d}{dt}\left[\dot\phi^2 + \frac{\pi^2}{l^2+a\pi^2}\left(\frac{l^2+b\pi^2}{l^2}\phi^2 - 2\int_\alpha^\phi f(s)\,ds\right)\right] \geqslant 0.$$

Thus

$$\bigl(\dot\phi(t)\bigr)^2 \geqslant \beta^2 + \frac{2\pi^2}{l^2+a\pi^2}\left(\int_\alpha^{\phi(t)} f(s)\,ds - \frac{l^2+b\pi^2}{2l^2}\phi^2\right) + \frac{\pi^2(l^2+b\pi^2)}{l^2(l^2+a\pi^2)}\alpha^2. \quad (4.13)$$

Similarly, we conclude from (4.13) that

$$t \leqslant \int_\alpha^{\phi(t)} \left\{\beta^2 + \frac{2\pi^2}{l^2+a\pi^2}\left(\int_\alpha^{y} f(s)\,ds - \frac{l^2+b\pi^2}{2l^2}y^2\right) + \frac{\pi^2(l^2+b\pi^2)}{l^2(l^2+a\pi^2)}\alpha^2\right\}^{-\frac{1}{2}} dy.$$

Therefore, $\phi(t)$ develops a singularity in finite time $t_0 \leqslant \bar T_2$.

Finally, since $\phi(t) > 0$, we have

$$\phi(t) \leqslant \sup_{x\in\Omega} \bigl|u(x,t)\bigr|,$$

which proves (4.4). The theorem is proved. □

Corollary 4.1. *For each p, $1 \leqslant p \leqslant \infty$,*

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega \bigl|u(x,t)\bigr|^p\,dx\right)^{\frac{1}{p}}$$

blows-up in finite time.

5. The problems (1.6), (1.2), (1.3) and (1.5), (1.2), (1.3)

In this section we apply the above theory to the problem (1.6), (1.2), (1.3) and the problem (1.5), (1.2), (1.3).

Theorem 5.1. *Suppose that $u_0 \in H^4(\Omega)$, $u_1 \in H^3(\Omega)$. Then, the problem* (1.6), (1.2), (1.3) *has a unique global generalized solution $u \in C([0,T]; H^4(\Omega)) \cap C^1([0,T]; H^3(\Omega)) \cap C^2([0,T]; H^2(\Omega))$.*

Proof. By virtue of Theorem 3.1, it is enough to prove that $f'(u) = \frac{1}{4}(3cu^2 + 12u)$ is bounded from below. In fact,

$$f'(u) = \frac{1}{4}(3cu^2+12u) = \frac{1}{4}\left(\sqrt{3c}\,u + \frac{6}{\sqrt{3c}}\right)^2 - \frac{3}{c} \geqslant -\frac{3}{c}.$$

The theorem is proved. □

By the contraction mapping principle [16] or the Galerkin method [17] we can prove that the problem (1.5), (1.2), (1.3) has a unique local generalized solution and a unique local classical solution. The following theorem then follows from Theorem 4.1.

Theorem 5.2. *Let $u(x,t)$ be the generalized solution of the problem* (1.5), (1.2), (1.3). *Suppose that the following conditions are satisfied*:

(1) $$-\frac{\pi}{2l}\int_{\Omega} u_0(x)\sin\frac{\pi x}{l}\,dx = \alpha > 0, \qquad -\frac{\pi}{2l}\int_{\Omega} u_1(x)\sin\frac{\pi x}{l}\,dx = \beta > 0;$$

(2) $$\frac{3}{2}\alpha^2 - \frac{l^2 + b\pi^2}{l^2}\alpha \geqslant 0.$$

Then

$$\lim_{t\to t_0^-}\sup_{x\in\Omega}\bigl|u(x,t)\bigr| = \infty \tag{5.1}$$

for some finite time $t_0 \leqslant \bar{T}_2$.

Proof. Since

$$\begin{aligned}\bar{T}_2 &= \int_{\alpha}^{\infty}\left[\beta^2 + \frac{2\pi^2}{l^2 + a\pi^2}\left(\int_{\alpha}^{y}\frac{3s^2}{2}\,ds - \frac{l^2 + b\pi^2}{2l^2}y^2\right) + \frac{\pi^2(l^2 + b\pi^2)}{l^2(l^2 + a\pi^2)}\alpha^2\right]^{-\frac{1}{2}} dy \\ &= \int_{\alpha}^{\infty}\left[\beta^2 + \frac{\pi^2}{l^2 + a\pi^2}\left(y^3 - \alpha^3 - \frac{l^2 + b\pi^2}{l^2}y^2\right) + \frac{\pi^2(l^2 + b\pi^2)}{l^2(l^2 + a\pi^2)}\alpha^2\right]^{-\frac{1}{2}} dy\end{aligned}$$

converges, (5.1) follows from Theorem 4.1 immediately. □

Acknowledgment

The authors are very grateful to the referees' valuable suggestions for the improvement of this paper.

References

[1] N.L. Pasternak, New Method for Calculation of Foundation on the Elastic Basement, Gosstroiizdat, Moscow (in Russian).

[2] A.M. Samsonov, E.V. Sokurinskaya, Energy exchange between nonlinear waves in elastic waveguides and external media, in: Nonlinear Waves in Active Media, Springer, Berlin, 1989, pp. 99–104.

[3] A.M. Samsonov, Nonlinear strain waves in elastic waveguide, in: A. Jeffrey, J. Engelbrecht (Eds.), Nonlinear Waves in Solids, in: CISM Courses and Lecture, vol. 341, Springer, Wien, 1994.

[4] R. Dodd, et al., Solitons and Nonlinear Wave Equations, Academic Press, 1984.

[5] A.M. Samsonov, On existence of longitudinal strain solutions in an infinite nonlinearly elastic rod, Soviet Phys. Dokl. 4 (1988) 298–300.

[6] J. Boussinesq, Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en Communiquant au liquide contenu dans ce cannal des vitesses sensiblement pareilles de la surface au fond, J. Math. Pure Appl. 17 (1872) 55–108.
[7] T. Kano, T. Nishida, A mathematical justification for Korteweg–de Vries equation and Boussinesq equation of water surface waves, Osaka J. Math. 23 (1986) 389–413.
[8] A.V. Porubov, Strain solitary waves in an elastic rod with microstructure, Rend. Sem. Mat. Univ. Politec. Torino 58 (2000) 189–198.
[9] V.G. Makhankov, Dynamics of classical solitons (in non-integrable systems), Phys. Rep. 35 (1978) 1–128.
[10] G.A. Mangin, Physical and Mathematical Models of Nonlinear Wave in Solid, Elsevier, Amsterdam, 1994.
[11] A.M. Samsonov, On Some Exact Travelling Wave Solutions for Nonlinear Hyperbolic Equation, in: Pitman Research Notes in Mathematics Series, vol. 227, Longman, 1993, pp. 123–132.
[12] A.M. Samsonov, E.V. Sokurinskaya, On the excitation of a longitudinal deformation soliton in a nonlinear elastic rod, Soviet Phys. Tech. Phys. 33 (1988) 989–991.
[13] G. Chen, S. Wang, Existence and nonexistence of global solutions for the generalized IMBq equation, Nonlinear Anal. 36 (1991) 961–980.
[14] G. Chen, J. Xing, Z. Yang, Cauchy problem for generalized IMBq equation with several variables, Nonlinear Anal. 26 (1996) 1255–1270.
[15] A. Friedman, Partial Differential Equations of Parabolic Type, Prentice Hall, Englewood Cliffs, NJ, 1964.
[16] W.J. Hrusa, Global existence and asymptotic stability for a semilinear hyperbolic Volterra equation with large initial data, SIAM J. Math. Anal. 16 (1985) 110–134.
[17] J.L. Lions, Quelques méthods de résolution des probléms aux limites non linéaires, Dunod/Gauthier–Villars, Paris, 1969.