On the Exponent of a Primitive, Minimally Strong Digraph

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ABSTRACT

We consider $n\times n$ primitive nearly reducible matrices for $n\geqslant 5$. As defined by Ross, let $e(n)$ be the least integer such that no such $n\times n$ matrix has this integer as its exponent. The investigation of $e(n)$ is the first open problem in Ross's paper. Here we offer a method to compute $e(n)$ for small n. Then we generalize Ross's estimate, which is $e(n)>n+1$, to

$$e(n)>(p+1)(n-p)$$

for $n\geqslant 2p+1$, and $p\geqslant 11$ a prime less than 100,000. There are extant various estimates and conjectures concerning the difference of successive primes. If one of the most hopeful of these conjectures be true, then our lower bound for $e(n)$ holds for all $p\geqslant 11$ and in fact we obtain

$$e(n)>\frac{n^2}{4}-\left(\frac{n}{2}\right)^{3/2}.$$

1. INTRODUCTION

The exponent set of primitive matrices is a very interesting subject to which many papers have contributed (see, for example, [3], [4], [6], [7], [8]).

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R. A. Brualdi and J. A. Ross investigated the exponent of a primitive, nearly reducible matrix, and gave many important results ([3] and [8]). Ross [8] defined $e(n)$ to be the least integer greater than or equal to 6 such that no $n \times n$ $(n \geqslant 5)$ primitive, nearly reducible matrix has this integer as its exponent. Finding $e(n)$ for each $n \geqslant 5$ is an interesting problem. It is the first open problem of Ross [8].

In Section 2 we shall give a method to compute $e(n)$ for small n. This is, of course, a partial answer for the open problem. Since it is probably very difficult to answer the problem completely, we turn to the question: How to estimate $e(n)$? Ross's results for this question is $e(n) \geqslant n+2$ (see Theorem 1.1). In Section 3 we shall give some better estimates for $e(n)$, including a quadratic estimate if Conjecture 3.7 for the distribution of primes holds.

Here we list all the known results that we shall need. We always use the same notation as in Ross's paper; for instance, we use $\gamma(D)$ for the exponent of a digraph D. We also use PMSD to abbreviate "primitive, minimally strong digraph."

THEOREM 1.1 (Theorem 3.3 of [8]). *Given integers $n \geqslant 5$ and $k \geqslant 6$ with $k \leqslant n+1$, there exists a PMSD D on n vertices with $\gamma(D) = k$.*

REMARKS (cf. [4, pp 645–646]). Suppose D is a primitive digraph in which the circuit lengths are $p_1, p_2, \dots, p_u$. For any ordered pair of vertices (i, j) we define the nonnegative integer r_{ij} as follows. If $i = j$ and if for $s = 1, 2, \dots, u$ there is a circuit through vertex i of length p_s, then $r_{ij} = 0$. Otherwise r_{ij} is the length of the shortest path from i to j which has at least one vertex on some circuit of length p_s for $s = 1, 2, \dots, u$. Let $r = \max r_{ij}$, where the maximum is taken over all ordered pairs (i, j). In [4] Dulmage and Mendelsohn discuss the unique path property and a weaker condition. This weaker condition is the one we employ. Since the assumption is that certain path lengths look like affine combinations of the p_s and r_{ij}, we shall refer to this as the affine sum property.

DEFINITION 1.2. An ordered pair of vertices (k, m) is said to satisfy the *affine sum property* iff for every path from k to m of length $w \geqslant r_{km}$ there are nonnegative integers $a_1, a_2, \dots, a_n$ for which

$$w = r_{km} + a_1 p_1 + \cdots + a_u p_u.$$

If the ordered pair (k, m) has the affine sum property and $r_{km} = r$, then $\gamma(D) = F(p_1, p_2, \dots, p_u) + r + 1$. Here $F(p_1, p_2, \dots, p_u)$ is the largest integer which is not expressible as a nonnegative integral combination of $p_1, p_2, \dots, p_u$ (cf. [4]).

We close by summarizing properties of F and $\gamma(D)$ which we shall need.

THEOREM 1.3. *The function $F(p_1, p_2, \dots, p_u)$ has the following properties:*

(i) (*Cf.* [5, *p.* 6].) *If p and q are relatively prime, then $F(p,q) = pq - p - q$.*

(ii) (*Cf.* [4, *p.* 644].) *If $a_j = a_0 + jd$ $(j = 0,1,\dots,s)$ and $a_0 \geqslant 2$, then*

$$F(a_0,\dots,a_s) = \left(\left[\frac{a_0 - 2}{s}\right] + 1\right)a_0 + (d-1)(a_0 - 1) - 1$$

where $[x]$ denotes the greatest integer less than or equal to x.

(iii) *If $p_i' \leqslant p_i$ for $i = 1,2$, then $F(p_1', p_2') \leqslant F(p_1, p_2)$.*

(iv) *If the set $\{q_1, q_2, \dots, q_v\}$ is a subset of the set $\{p_1, p_2, \dots, p_u\}$, then*

$$F(q_1, q_2, \dots, q_v) \geqslant F(p_1, p_2, \dots, p_u).$$

THEOREM 1.4 (Theorem 4.9 of [8]). *Let D be a PMSD on n vertices, and let s be the length of a shortest circuit in D. Then $s \neq 1$, $s \neq n-1$, and $\gamma(D) \leqslant n + s(n-3)$.*

THEOREM 1.5 (Theorem 4.2 of [3]). *Let D be a PMSD on n vertices. Then $\gamma(D) \leqslant n^2 - 4n + 6$, and there exists a primitive minimally strong digraph of exponent $n^2 - 4n + 6$ for each $n \geqslant 5$.*

THEOREM 1.6 (Corollary 5.2 of [8]). *Let n be an integer at least six. Then there exists no PMSD D on n vertices such that either*

$$n^2 - 5n + 9 < \gamma(D) < n^2 - 4n + 6$$

or

$$n^2 - 6n + 12 < \gamma(D) < n^2 - 5n + 9.$$

There always exists a PMSD on n vertices of exponent $n^2 - 6n + 12$; and there is a PMSD on n vertices of exponent $n^2 - 5n + 9$ iff n is even.

2. $e(n)$ FOR A SMALL n

Using the results listed in Section 1, we are able to find $e(n)$ for a small n. Our arguments are based on directed graphs. And we shall use the other equivalent definition of $e(n)$, that $e(n)$ is the least integer greater than or equal to six such that no PMSD on $n \geqslant 5$ vertices has the integer as its exponent.

First we establish the following useful lemma.

LEMMA 2.1. *For any integer $m \geqslant 5$ we have*

$$e(m+1) \geqslant e(m).$$

Proof. By the definition of $e(m)$ there exists a PMSD D_m on m vertices with $\gamma(D_m) = e(m) - 1$. Since D_m is not an elementary circuit, D_m contains a branch $\pi = (x_0, x_1, \ldots, x_k)$ ($k \geqslant 2$) by Lemma 2.3 of [8]. From the digraph D_m we can make a PMSD D_{m+1} of $m+1$ vertices by adding the $(m+1)$th vertex x_{m+1} and two arcs $(x_0, x_{m+1}), (x_{m+1}, x_2)$. It is clear that $\gamma(D_{m+1}) = \gamma(D_m)$ and thus $e(m+1) \geqslant e(m)$. ∎

PROPOSITION 2.2. *There exists no PMSD on* 5 *vertices with exponent different from* 6, 8, *or* 11. *In other words,* $e(5) = 7$.

Proof. By Theorem 1.4, s is 2 or 3. And any PMSD should be isomorphic to one of the digraphs in Figure 1:

$$\gamma(D_1) = \gamma(D_2) = \gamma(D_3) = \gamma(D_4) = F(2,3) + 4 + 1 = 6,$$

$$\gamma(D_5) = F(2,3) + 6 + 1 = 8,$$

$$\gamma(D_6) = F(3,4) + 5 + 1 = 11.$$

[In each digraph D_i the ordered pair (a, a) or (a, b) of vertices has the affine sum property, and r_{aa} or r_{ab} is equal to $\max r_{ij}$ for all ordered pairs (i, j) in D_i.] ∎

PROPOSITION 2.3. *For $n = 6$ and $n = 7$, the two gaps shown in Theorem* 1.6 *are the only two gaps of PMSD on n vertices. Therefore* $e(6) = 13$ *and* $e(7) = 20$.

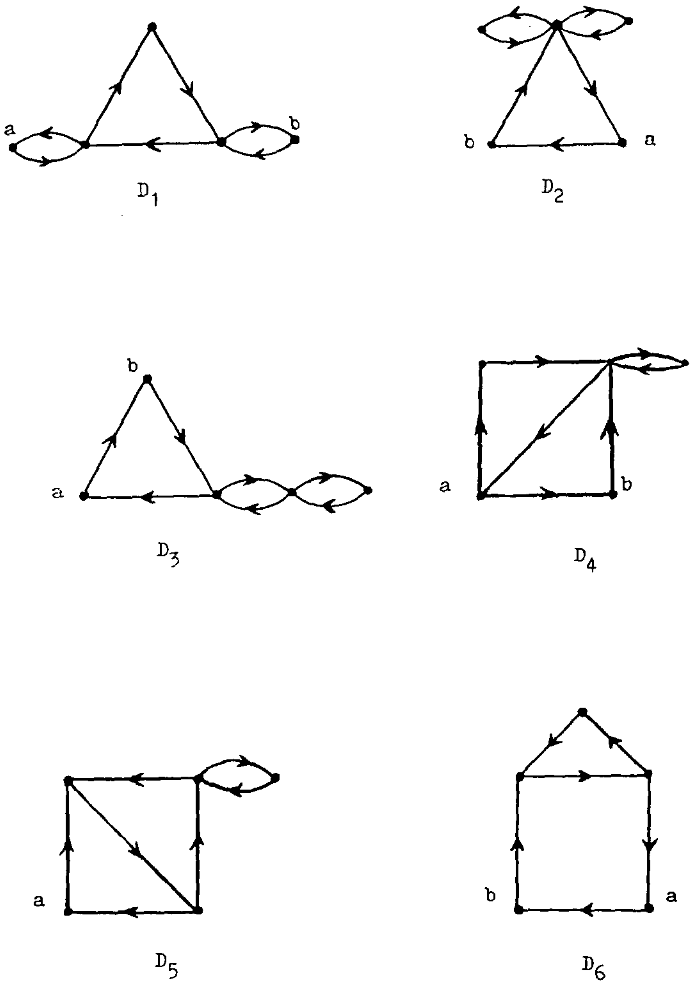

FIG. 1.

Proof.

Case 1: $n = 6$. By Theorem 1.1 $e(6) \geqslant 8$, and by Theorem 1.6 $e(6) \leqslant 13$. It suffices to prove that there exist PMSD on 6 vertices of exponent k such that $k = 8, 9, 10, 11$. (By Theorem 1.6, $k \neq 12$.) Figure 2 shows that there really exist these kinds of digraphs.

Case 2: $n = 7$. By Theorem 1.6 $e(7) \leqslant 20$, and by Lemma 2.1 $e(7) \geqslant e(6) = 13$. It suffices to prove that there exist PMSD on 7 vertices of exponent

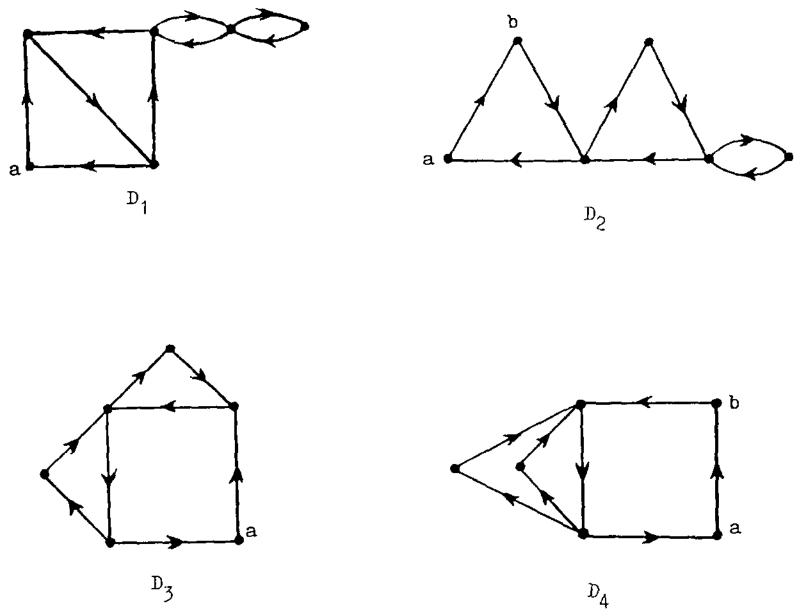

FIG. 2.

k such that $k = 13, 14, \ldots, 18$. And the digraphs in Figure 3 are the digraphs desired. [As in Figure 1, in both Figure 2 and Figure 3 the marked ordered pair (a, a) or (a, b) of vertices in each digraph D_i has the affine sum property and r_{aa} or r_{ab} is equal to $\max r_{ij}$ for all ordered pairs (i, j) in D_i.] Thus we have

$$\gamma(D_i) = 7 + i, \qquad i = 1, 2, 3, 4,$$

and Figure 2, and

$$\gamma(D_i) = 12 + i, \qquad i = 1, 2, \ldots, 6,$$

for Figure 3.

If we notice that Theorem 1.6 implies $e(n) \leqslant n^2 - 6n + 13$, then Proposition 2.3 implies that the equality holds when $n = 6$ or 7. ■

Unfortunately, this nice result does not hold for $n > 7$. In fact, when $n = 8$ we have $n^2 - 6n + 12 = 28$ and $e(8) = 21$. This can be proved by an argument which we outline as follows. Let t denote the length of a maximal

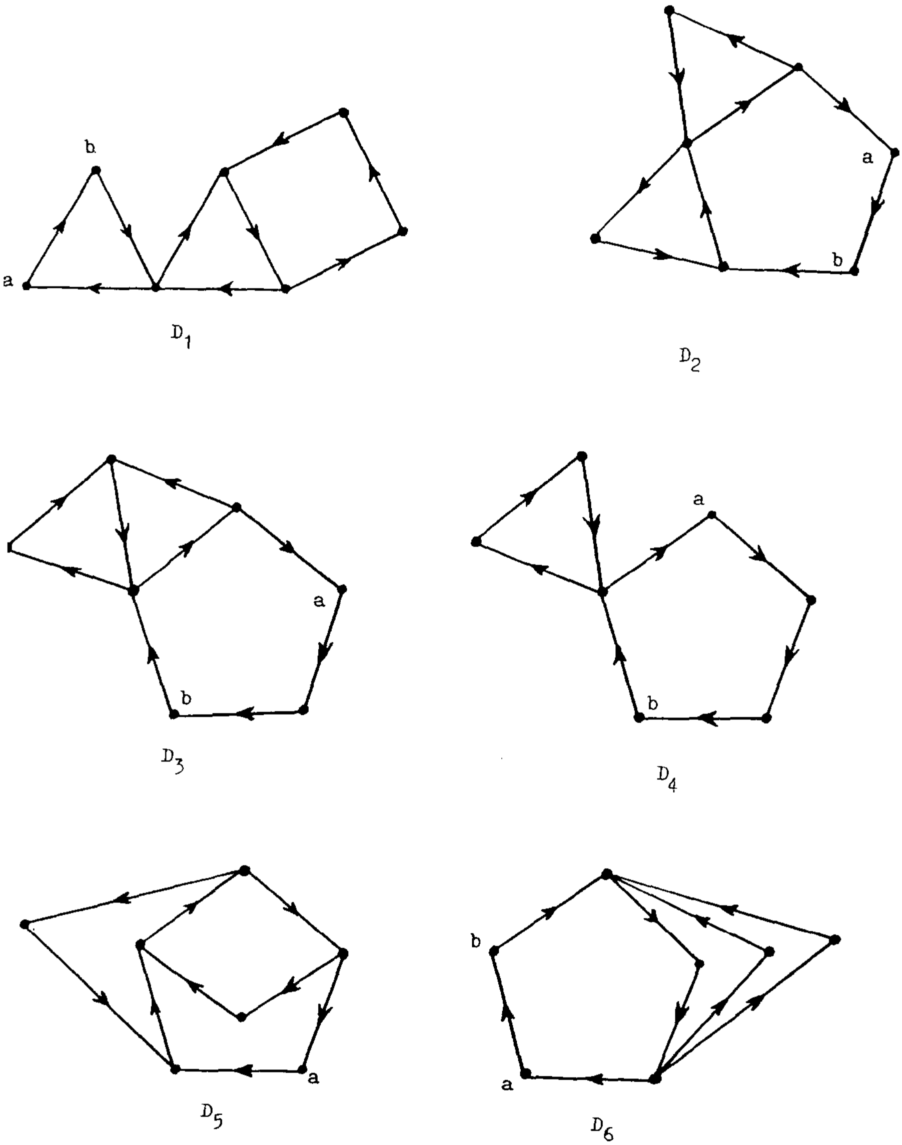

FIG. 3.

circuit in a PMSD D with 8 vertices. Since t can not be 8, we have four cases:

Case I: $t = 7$. In this case the digraph D has only two kinds of circuits of different lengths and the digraph shown in Figure 4(a) has the lowest exponent 23 by Theorem 1.3(iii). [The pair (a, b) has the affine sum property and $r_{ab} = r = 11$, $F(3,7) + r + 1 = 11 + 11 + 1 = 23$.] Therefore there is no PMSD with $n = 8$ and $t = 7$ which has exponent 21.

Case II: $t = 6$. There are two subcases:

(II_1) D has only two kinds of circuits of different lengths. In this subcase the digraph shown in Figure 4(b) has the lowest exponent 26 by Theorem 1.3. [The pair (a, b) has the unique path property, and $r_{ab} = r = 6$, $F(5,6) + r + 1 = 19 + 6 + 1 = 26$.]

(II_2) D has more than two kinds of circuits of different lengths. In this subcase the digraph shown in Figure 4(c) has the highest exponent 16 by Theorem 1.3(ii)–(iv). [The pair (a, b) has the affine sum property, and $r_{ab} = r = 8$, $F(3,5,6) + r + 1 = 7 + 8 + 1 = 16$.]

Therefore there are no PMSD with $n = 8$ and $t = 6$ which has exponent 21.

Case III: $t = 5$. In this case the digraph shown in Figure 4(d) has the highest exponent 20, so there is no PMSD with $n = 8$ and $t = 5$ which has exponent 21.

Case IV: $t \leqslant 4$. It is obvious that there is no PMSD with $n = 8$ and $t \leqslant 4$ which has exponent 21.

Therefore we have $e(8) = 21$.

LEMMA 2.4. *Suppose s and t are any positive integers such that $2 < s < t < n$, $(s, t) = 1$, where (a, b) denotes the greatest common divisor of a and b. Then there exists a PMSD on $n \geqslant 2s + 1$ vertices with exponent k for every k such that*

$$m < k < M = \min\{st + n - 2s,\ st + t - s - 1\} \tag{2.1}$$

where m is determined by s, t, and n, and the length of the interval $[m, M]$ is greater than 1 if $t < n - 1$.

In the course of the proof we shall use a certain type of subpath sufficiently often that we give it a name.

DEFINITION 2.5. Let α be a circuit of the digraph D, and let β be a subpath of α. We call β a pure subpath of α iff no vertex of β belongs to a circuit of D other than α.

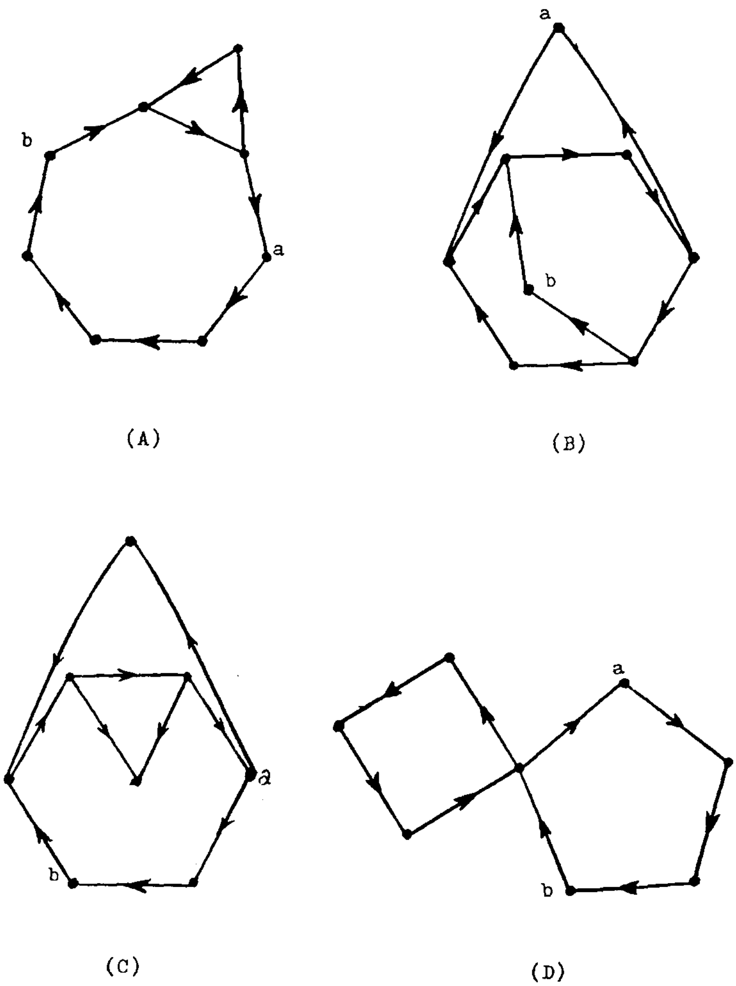

Fig. 4.

Proof. We shall construct D, a PMSD on n vertices, with only one elementary circuit of length t, several elementary circuits of length s, and no other elementary circuits. Then Theorem 1.3 allows us to compute the exponent of D, that is

$$\gamma(D) = F(s,t) + r + 1 = st - s - t + r + 1. \tag{2.2}$$

To begin, construct a digraph D with two circuits as follows. The first circuit, α, passes through vertices $1,2,\dots,t$, and the subpath β passing through $1,2,\dots,d$ is pure, where

$$d = \min\{n-s, t-1\}. \tag{2.3}$$

(see Figure 5). The other circuit passes through

$$t+1,\dots,n,d+1,\dots,t.$$

Now the pair of vertices $(1,d)$ has the affine sum property and

$$r_{1,d} = r = t + (d-1).$$

Thus

$$\gamma(D) = st - s - t + r + 1 = st - s + d.$$

By (2.3) we have

$$\gamma(D) = \min\{st + n - 2s, st + t - s - 1\}.$$

If $n - t = 1$, there is only one PMSD on n vertices with a circuit of length t and a circuit of length s, whence $m = M = st - 2s - 1$. If $n - t > 1$, first modify D so that it has a circuit length s in the portion of D complementary to β and so that one circuit of length s passes through d. Now change D so that α remains the same, β becomes $(1,2,\dots,d-1)$, and a circuit of length s passes through d (see Figure 5 again). The exponent of this digraph is $M-1$. Continuing in this fashion we obtain digraphs with pure subpaths $(1,2,\dots,d-2),(1,2,\dots,d-3),\dots,$ and respective exponents $M-2, M-3,\dots$. ■

As an example, let us consider the case that $n = 9$, $s = 3$, and $t = 7$. In this case $d = t - 1 = 6$, $M = st + t - s - 1 = 24$, $n - t = 2 > 1$, and the digraphs in Figure 6 have exponents $24 = M$, 23, 22, 21, and $20 = m = M - 4$.

For the given n, if a pair (s,t) of positive integers satisfies the conditions of Lemma 2.4, then by the lemma we have an interval $[m, M]$ such that there is a PMSD with only two kinds of elementary circuits of length s and t, whose exponent can be any integer $k \in [m, M]$. For convenience we use the notation $(s,t)_{\min}$ and $(s,t)_{\max}$ for m and M.

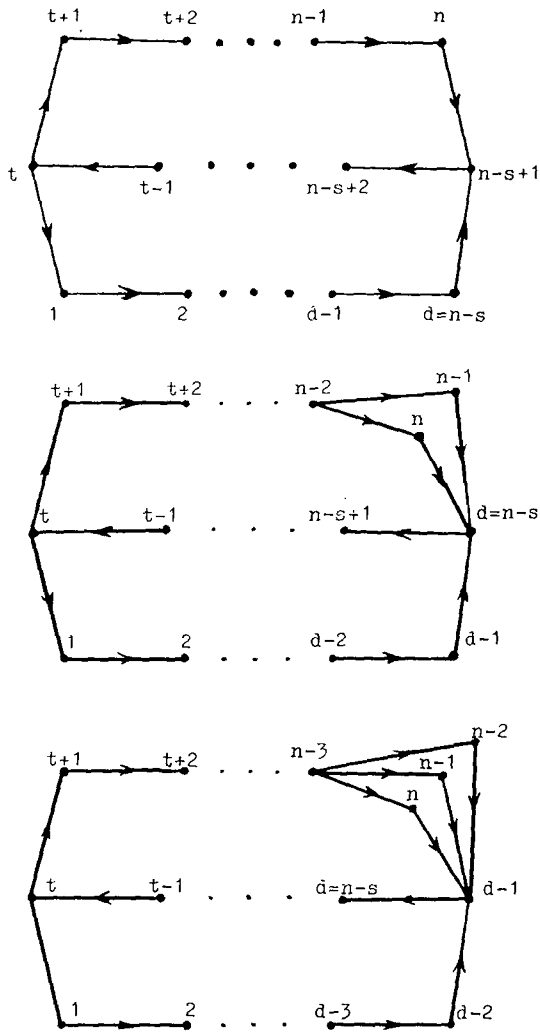
t+1
t+2
n−1
n
t
t−1
n−s+2
n−s+1
1
2
d−1
d=n−s
t+1
t+2
n−2
n−1
n
t
t−1
n−s+1
d=n−s
1
2
d−2
d−1
t+1
t+2
n−3
n−2
n−1
n
t
t−1
d=n−s
d−1
1
2
d−3
d−2

FIG. 5.

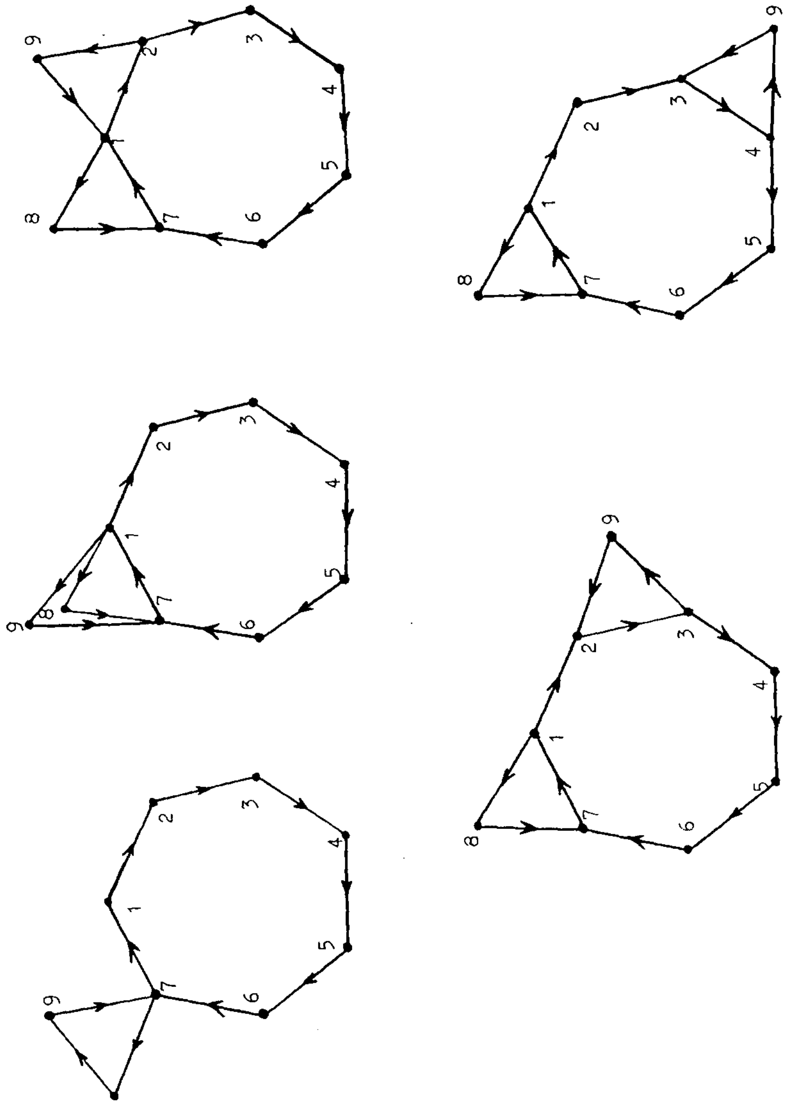

FIG. 6.

TABLE 1

n	$e(n) \geqslant$	n	$e(n) \geqslant$
10	36	17	126
11	47	18	151
12	49	19	176
13	63	20	203
14	87	21	232
15	106	22	233
16	107	23	248

Now there is a general procedure for computing $e(n)$ for small n which is based on Lemmas 2.1 and 2.4:

1. Find a pair (s_1, t_1) satisfying the conditions of Lemma 2.4 and

$$(s_1, t_1)_{\min} \leqslant e(n-1) \leqslant (s_1, t_1)_{\max}.$$

2. Among all the pairs (s,t) such that s and t satisfy the conditions of Lemma 2.4 and $s+t = s_1 + t_1$ or $s + t = s_1 + t_1 + 1$, choose a pair (s_2, t_2) such that $(s_2, t_2)_{\min} \leqslant (s_1, t_1)_{\max} + 1$ and $(s_2, t_2)_{\max} > (s_1, t_1)_{\max}$. Similarly choose $(s_3, t_3), (s_4, t_4), \dots$.
3. If step 2 stops at (s_u, t_u), and there is no PMSD on n vertices having exponent $(s_u, t_u)_{\max} + 1$, then $e(n) = (s_u, t_u)_{\max} + 1$.

For instance, when $n = 9$ we have $(3,7)_{\min} = 20 < e(8)$, $(3,7)_{\max} = 24$ (see Figure 6); $(4,7)_{\min} = 26 = (5,6)_{\min}$. We might use an argument similar to the one we used for $e(8)$ to prove that there is no PMSD on 9 vertices having exponent 25. So we claim $e(9) = 25$. When $n = 10$, we have $(3,8)_{\min} \leqslant 23$, $(3,8)_{\max} \geqslant 28$; $(4,7)_{\min} \leqslant 26$, $(4,7)_{\max} \geqslant 30$; $(5,7)_{\min} \leqslant 31$, $(5,7)_{\max} \geqslant 35$; and hence $e(10) \geqslant 36$. When $n = 11$, we have $(5,7)_{\max} \geqslant 36$, $(6,7)_{\min} \leqslant 37$, $(6,7)_{\max} \geqslant 41$, $(5,9)_{\min} \leqslant 42$, $(5,9)_{\max} \geqslant 46$. Therefore $e(11) \geqslant 47$. In this way we may compute a lower bound of $e(n)$ one by one for small n. We list a few of them in Table 1 for future use.

3. LOWER BOUNDS FOR $e(n)$

Theorem 1.1 implies

$$e(n) > n + 1, \tag{3.1}$$

and Theorem 1.6 implies

$$e(n) < n^2 - 6n + 13. \tag{3.2}$$

These are the known estimates of lower and upper bounds for $e(n)$ given by Ross [8]. As we pointed out before, the equality in (3.2) holds when $n = 6$ or 7. So (3.2) gives a very good estimate of upper bounds for $e(n)$. But (3.1) does not offer a good estimate of lower bounds for $e(n)$. From now on, improving the inequality (3.1) is our task in this paper. First of all, we need the following theorem:

THEOREM 3.1. *Let $p \geqslant 5$ be any prime number and $n \geqslant 2p+1$, $t^* = \min\{n - [n/p], n - p + 1\}$. If*

$$e(2p+1) \geqslant p(p+1)+1 \tag{3.3}$$

and

$$e(3p) > 2p^2 + 1, \tag{3.4}$$

then

$$e(n) > \begin{cases} \min\{(p+1)(n-[n/p]-2),(p+1)(n-p-1)\} & \text{if } p \mid t^*, \\ \min\{(p+1)(n-[n/p]-1),(p+1)(n-p)\} & \text{if } p \nmid t^*. \end{cases} \tag{3.5}$$

Proof. For any integer t such that $p \nmid t$ and

$$p + 2 \leqslant t \leqslant \min\{n - [n/p], n - p + 1\} = t^*$$

there exists a PMSD on n vertices with exponent k for each k satisfying

$$b_t \leqslant k \leqslant B_t,$$

where

$$\begin{aligned} b_t &= F(p,t) + t + 1 = pt - p - t + t + 1 \\ &= p(t-1) + 1 \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} B_t &= F(p,t)+(2t-2)+1 = pt-p-t+2t-1 \\ &= (p+1)(t-1). \end{aligned} \tag{3.7}$$

In fact, we could form, as shown in Figure 7, a PMSD on n vertices with only one elementary circuit α of length t whose vertices are named $1,2,\dots,t$, and $n-t$ ($\geqslant [n/p]$) circuits $\beta_1,\beta_2,\dots,\beta_{n-t}$ of length p such that β_i contains vertices

$$t+i,(i-1)p+1,(i-1)p+2,\dots,ip-1 \qquad \text{for} \quad i=1,2,\dots,[t/p];$$

and $\beta_{[t/p]+1},\dots,\beta_{n-t}$ contain the same $p-1$ vertices of α:

$$t-1,t-2,\dots,t-p+1.$$

It is clear that the digraph in Figure 7(a) has exponent

$$b_t = F(p,t)+t+1 = pt-p-t+t+1 = p(t-1)+1,$$

for the ordered pair of vertices (t,t) has the affine sum property and $t_{t,t}=r=t$ ($r_{i,j}\leqslant t$ for any two vertices i and j outside α).

Set $\beta_1=\{2,3,\dots,p-1,p,t+1\}$, and keep the remaining portion of the digraph in Figure 7(a) the same to obtain a new digraph with exponent $F(p,t)+(t+1)+1=p(t-1)+2$, since in this case the pair $(t,1)$ has the affine sum property and $r_{t,1}=r=t+1$. Now construct a new digraph by shifting the vertices of β_1 on α one step clockwise, that is, set

$$\beta_1=\{3,4,\dots,p+1,t+1\},$$

to obtain a pure subpath $\{t,1,2\}$ of α no vertex of which belongs to any circuit of length p. This modification to β_1 results in a digraph of exponent $F(p,t)+(t+2)+1=p(t-1)+3$. [The pair $(t,2)$ has the affine sum property, and $r_{t,2}=r=t+2$.] Continue to shift β_1 in this way step by step toward β_2. After β_1 and β_2 have $p-1$ common vertices on α, shift their common vertices toward β_3; carry this out until all the circuits of length p have $p-1$ common vertices: $t-1,t-2,\dots,t-p+1$. This gives exponents $p(t-1)+1,\dots,p(t-1)+(t-p)$. We construct yet another digraph with the same circuit α as before and one fewer circuit of length p. The remaining circuits are called $\beta_2,\beta_3,\dots,\beta_{n-t}$ and are constructed as follows.

We connect $t+1$ to $t-2$ and have $\{t-p+1,\dots,t-2,t+1\}$ as the common subpath of $\beta_2,\beta_3,\dots,\beta_{n-t}$. However, each new circuit β_j will contain the $p-1$ common vertices: $t+1, t-2, t-3, \dots, t-p+1$, for $j = 2,3,\dots,n-t$ [see Figure 7(b)]. We call this process "canceling β." Analogously, we may cancel $\beta_2, \beta_3, \dots, \beta_{p-2}$. Now cancel β_2 and let $\{t+1, t+2, t+3, t-p+1, \dots, t-3\}$ be the common subpath of $\beta_3, \beta_4, \dots, \beta_{n-t}, \dots$. Finally, cancel β_{p-2} and let $\{t+1, t+2, \dots, t+p-2, \dots, n, t-p+1\}$ be the common subpath of all remaining circuits of length p [Figure 7(c)].

In this way we get a series of pure subpaths of α:

$$\{t,1\}, \{t,1,2\}, \dots, \{t,1,2,\dots,t-p\},$$

$$\{t-1,t,1,\dots,t-p\}, \dots, \{t-p+2,\dots,t,1,2,\dots,t-p\},$$

such that the two ends of each pure path form a pair (i, j) which has the affine sum property and $r_{ij} = r$. Therefore there are a series of PMSD on n vertices with exponents

$$\begin{aligned} b_t &= p(t-1)+1, p(t-1)+2, \dots, \\ B_t &= F(p,t) + (2t-2) + 1 \\ &= (p+1)(t-1). \end{aligned}$$

Since $t \geqslant p+2$, we have

$$\begin{aligned} B_{t-1} &= (p+1)(t-2) = p(t-1) + t - p - 2 \\ &\geqslant p(t-1) = b_t - 1. \end{aligned}$$

Therefore if $e(2p+1) \geqslant p(p+1)+1 = b_{p+2}$, then let $t = 2p-1$ and we have, by Lemma 2.1, a PMSD on $n \geqslant 2p+1$ vertices with exponent k from 6 to B_{2p-1}.

When $t \geqslant 2p+1$, we have $n \geqslant t+p-1 \geqslant 3p$ and hence

$$e(n) \geqslant e(3p) \geqslant 2p^2 + 1 = b_{2p+1}.$$

And for $s \geqslant 3$, $B_{sp-1} = (p+1)(sp-2) \geqslant sp^2 + 1 = b_{sp+1}$. Therefore there exists a PMSD on n vertices with exponent k such that

$$6 \leqslant k \leqslant B_{t^*-1} = \min\{(p+1)(n-[n/p]-2), (p+1)(n-p-1)\} \qquad \text{if} \quad p \mid t^*,$$

$$b \leqslant k \leqslant B_{t^*} = \min\{(p+1)(n-[n/p]-1), (p+1)(n-p)\} \qquad \text{if} \quad p \nmid t^*.$$

∎

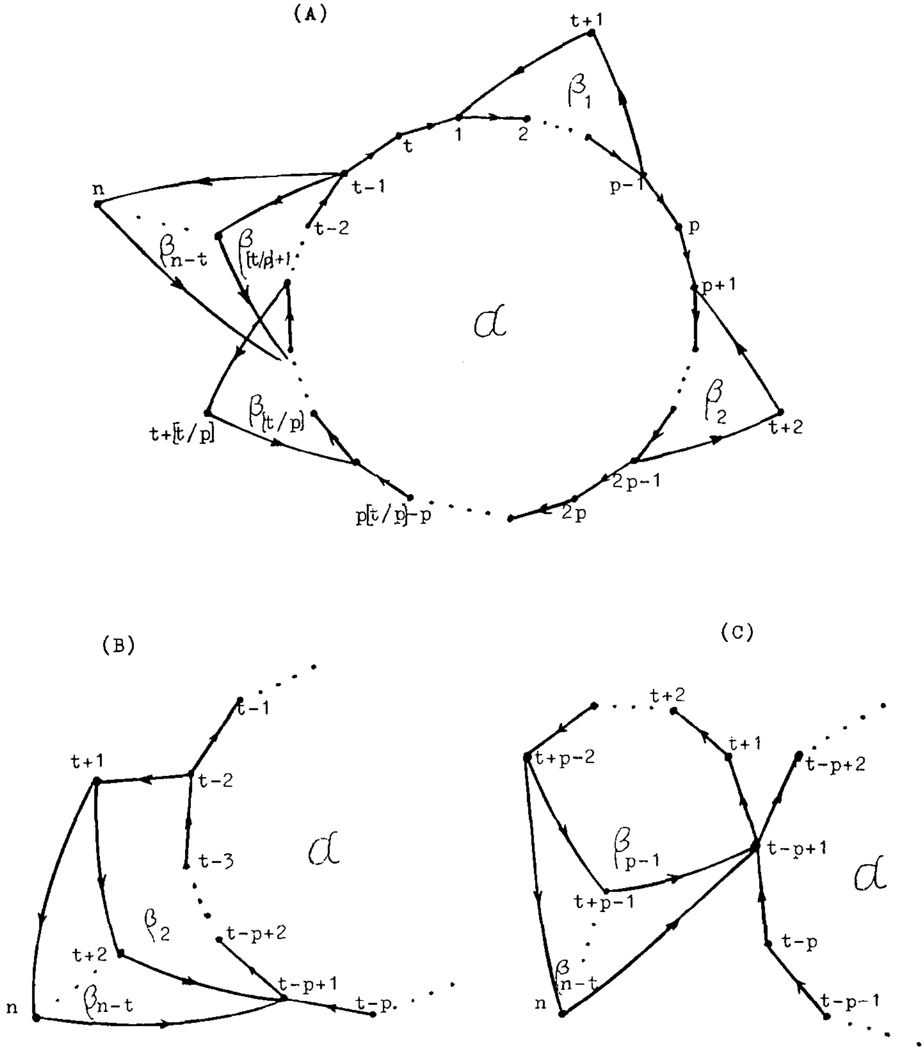
(A)
t+1
β_1
t
1
2
n
t-1
p-1
p
t-2
β_{n-t}
$\beta_{[t/p]+1}$
p+1
α
β_2
t+[t/p]
$\beta_{[t/p]}$
t+2
2p-1
2p
p[t/p]-p
(B)
t-1
t+1
t-2
t-3
α
β_2
t-p+2
t+2
t-p+1
n
β_{n-t}
t-p.
(C)
t+2
t+1
t+p-2
t-p+2
β_{p-1}
t-p+1
α
t+p-1
t-p
β_{n-t}
n
t-p-1

FIG. 7.

COROLLARY 3.2. *We have*

$$e(n) > \min\{6(n-[n/5]-2), 6(n-4)\} \quad \textit{for} \quad n \geqslant 11, \tag{3.8}$$

$$e(n) > \min\{8(n-[n/7]-2), 8(n-6)\} \quad \textit{for} \quad n \geqslant 15, \tag{3.9}$$

$$e(n) > \min\{12(n-[n/11]-2), 12(n-10)\} \quad \textit{for} \quad n \geqslant 23. \tag{3.10}$$

Proof. By Table 1, we have

$$e(2(5)+1) \geqslant 47 > 5(5+1)+1,$$

$$e(2(7)+1) \geqslant 106 > 7(7+1)+1,$$

$$e(2(11)+1) \geqslant 248 > 11(11+1)+1,$$

$$e(3(5)) \geqslant 106 > 2(5^2)+1,$$

$$e(3(7)) \geqslant 232 > 2(7^2)+1,$$

$$e(3(11)) \geqslant e(23) \geqslant 248 > 2(11^2)+1.$$

Therefore the conditions (3.3) and (3.4) of Theorem 3.1 hold for $p = 5$, 7, and 11, and hence (3.8), (3.9), and (3.10) follow from (3.5). ∎

THEOREM 3.3. *Let $p_1 \geqslant 11$ be any prime such that*

$$e(n) > (p_1+1)(n-p_1) \quad \textit{for} \quad 2p_1+1 \leqslant n < 3p_1-1.$$

If $p > p_1$ is another prime satisfying

$$2p+1 < 3p_1-1 \tag{3.11}$$

and

$$(p_1+1)(2p+1-p_1) \geqslant p(p+1)+1, \tag{3.12}$$

then

$$e(n) > (p+1)(n-p) \quad \textit{for} \quad 2p+1 \leqslant n < 3p-1. \tag{3.13}$$

Proof. When $n < 3p-1$, we have

$$\min\{n-[n/p], n-p+1\} = n-p+1 < 2p.$$

According to the proof of Theorem 3.1, for any integer t such that $p+2 \leqslant t \leqslant 2p-1$, there exists a PMSD on n vertices with exponent k for each k satisfying

$$b_t \leqslant k \leqslant B_t,$$

where $b_t = p(t-1)+1$, $B_t = (p+1)(t-1) \geqslant b_{t+1}$. Since (3.11) and (3.12) imply $e(2p+1) > (p_1+1)(2p+1-p_1) \geqslant b_{p+2}$, there exists a PMSD on n vertices of exponent k for each k such that

$$6 \leqslant k \leqslant B_{2p-1} = (p+1)(2p-2).$$

But $n < 3p-1$ implies $(p+1)(2p-2) \geqslant (p+1)(n-p)$. Finally we have $e(n) > (p+1)(n-p)$. ■

LEMMA 3.4. *If the inequality*

$$(p-p_1)^2 < p_1 \tag{3.14}$$

holds for any two consecutive primes $p_1 \geqslant 127$ and p, and the inequalities (3.11) *and* (3.12) *hold.*

Proof. (3.14) implies

$$p_1(2p-p_1+1) > p^2, \tag{3.15}$$

and (3.15) obviously implies (3.12). Then (3.14) and $p > p_1 \geqslant 127$ imply

$$p - p_1 < \sqrt{p_1} < \frac{p_1-2}{2},$$

from which (3.11) follows. ■

THEOREM 3.5. *If the inequality* (3.14) *holds for any two consecutive primes $p_1 \geqslant 127$ and p_2, then for any prime $p > 11$ and $n \geqslant 2p+1$,*

$$e(n) > (p+1)(n-p). \tag{3.16}$$

Proof. We have, by Theorem 3.1, that $e(n) > (11+1)(n-11)$ for $2(11)+1 \leqslant n < 3(11)-1$. And (3.11), (3.12) hold for $p_1 = 11$, $p = 13$. Hence

Theorem 3.3 implies

$$e(n) > (13+1)(n-13) \qquad \text{for} \quad 2(13)+1 \leqslant n < 3(13)-1.$$

Set $p_1 = 13$, $p = 17$, and note that (3.11), (3.12) are true. Then Theorem 3.1 implies $e(n) > (17+1)(n-17)$ for $2(17)+1 \leqslant n < 3(17)-1$. It is easy to check that the inequalities (3.11) and (3.12) hold for any two consecutive primes p_1 and $p \leqslant 127$. Therefore the inequality (3.13) holds for $p \leqslant 127$.

If $p > p_1 \geqslant 127$ and (3.14) holds, then (3.11) and (3.12) hold by Lemma 3.4. In this case we may use Theorem 3.3 to complete the induction which implies

$$e(n) > (p+1)(n-p) \tag{3.17}$$

for any prime $p \geqslant 11$ and any n for which $2p+1 \leqslant n < 3p-1$.

If $n \geqslant 3p-1$, then there must be a prime $p' > p$ such that $2p'+1 \leqslant n < 3p'-1$, and thus $e(n) > (p'+1)(n-p')$. In fact if there is no such a prime p', then there should be two consecutive p' and p'' satisfying

$$p'' > p' \geqslant p$$

and

$$2p''+1 \leqslant 3p'-1,$$

or

$$p'' - p' > \frac{p'}{2} - 1. \tag{3.18}$$

The inequality (3.18) is not true for $11 \leqslant p' < p'' \leqslant 127$ and is contradictory to (3.14) for $127 \leqslant p' < p''$. Since $n > p'+p+1$, we have

$$n(p'-p) > (p'-p)(p'+p+1),$$

or

$$(p'+1)(n-p') > (p+1)(n-p). \tag{3.19}$$

Now (3.17) holds for any prime $p \geqslant 11$ and $n \geqslant 2p+1$. ∎

THEOREM 3.6. *If the inequality* (3.14) *holds for any two consecutive primes* $p_1 \geqslant 127$ *and* p, *then for any integer* $n \geqslant 23$ *we have*

$$e(n) > \frac{n^2}{4} - \left(\frac{n}{2}\right)^{3/2}. \tag{3.20}$$

Proof. By Theorem 3.5, we have

$$e(n) > (p_1 + 1)(n - p_1), \tag{3.21}$$

where p_1 is the greatest prime less than or equal to $(n-1)/2$. If $p_1 = (n-1)/2$, then (3.21) implies $e(n) > (n/2)^2$ and thus implies (3.20). So we may only consider the case that $p_1 < (n-1)/2$ and $p \geqslant n/2$, where p is the prime next to p_1. By (3.14) we have

$$p - p_1 < \sqrt{p_1}. \tag{3.22}$$

Now (3.21) and (3.22) imply that

$$\begin{aligned}
e(n) &> \frac{(p_1+1)(n+1)}{2} && \left(\text{since} \quad p_1 < \frac{n}{2} - \frac{1}{2}\right)\\
&> \left(\frac{n}{2}\right)p_1 \\
&\geqslant \frac{n}{2}\left(\frac{n}{2} - (p - p_1)\right) && \left(\text{since} \quad \frac{n}{2} - p \leqslant 0\right)\\
&> \frac{n}{2}\left(\frac{n}{2} - \sqrt{p_1}\right) && \left(\text{since} \quad (p-p_1)^2 < p_1\right)\\
&> \frac{n}{2}\left(\frac{n}{2} - \sqrt{\frac{n}{2}}\right) && \left(\text{since} \quad p_1 < \frac{n}{2}\right).
\end{aligned}$$

There the inequality (3.20) holds. ■

REMARK. The inequality (3.20) already holds for $n < 23$ by Table 1 and Lemma 2.1.

Theorem 3.6 gives a nice quadratic estimate for a lower bound of $e(n)$, by requiring (3.14) hold for any two consecutive primes $p_1 \geqslant 127$ and $p < n$. (3.14) can be described as the following conjecture for distribution of primes.

CONJECTURE 3.7. Let p_n denote the nth prime and $d_n = p_{n+1} - p_n$. Then

$$d_n < \sqrt{p_n} \qquad \text{for any} \quad n > 30. \tag{3.23}$$

Conjecture 3.7 can be verified from tables of primes (see for instance [2]) for small values such as $127 \leqslant p_n < 100{,}000$. Estimates of d_n have been given. There is a discussion of these estimates in [9, p. 100], where the best bound is given as

$$d_n = O\left(p_n^{\frac{11}{20}+\varepsilon}\right).$$

The conjecture represents a hoped-for best possible case.

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