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Existence of a solution for the fractional differential equation with nonlinear boundary conditions☆

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Using the method of upper and lower solutions and its associated monotone iterative, we present an existence theorem for a nonlinear fractional differential equation with nonlinear boundary conditions.

1. Introduction

We will devote to considering the existence of a solution of the following nonlinear fractional differential equation with nonlinear boundary conditions, using the monotone iterative

$$D^\alpha u = f(t, u), \quad t \in (0, T], \tag{1.1}$$

$$g(u(0), u(T)) = 0, \tag{1.2}$$

where D^α is a regularized fractional derivative (the Caputo derivative) of order $0 < \alpha < 1$ (see [1]) defined by

$$D^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{\partial}{\partial t}\int_0^t (t-\tau)^{-\alpha} u(\tau)d\tau - t^{-\alpha}u(0)\right], \quad 0 < \alpha < 1, \tag{1.3}$$

here

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha} u(\tau)d\tau = I^{1-\alpha}u(t)$$

is the Riemann–Liouville fractional integral of order $1-\alpha$; see [1].

Differential equations of fractional order occur more frequently in different research and engineering areas, such as physics, chemistry, dynamical control, etc. Recently, many people have paid attention to the existence results of solutions of initial value problems for fractional differential equations; see [2–10].

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In [3], authors present an existence theorem for the following nonlinear ordinary differential equations of first order with nonlinear boundary conditions

$$\begin{cases} u'(t) = F(t, u(t)), & t \in I = [0, T], T > 0, \\ g(u(0), u(T)) = 0, \end{cases}$$

where $F : [0, T] \times R \to R$ and $g : R^2 \to R$ are continuous functions.

Motivated by [3], in this paper, we will investigate the existence of a solution of the nonlinear fractional differential equation (1.1) with nonlinear boundary conditions (1.2).

Definition 1.1. In this paper, we call a function $u(t)$ a solution of problem (1.1)–(1.2), if $u(t) \in C([0, T])$ ($C([0, T])$ is the space of functions which are continuous on $[0, T]$), and satisfies equation (1.1) and boundary value condition (1.2).

With regards to equation (1.1), we have the following definitions of upper and lower solutions.

Definition 1.2. A function $\varphi \in C([0, T])$ is called an upper solution of equation (1.1), if it satisfies

$$D^\alpha \varphi(t) \geq f(t, \varphi), \quad t \in (0, T]. \tag{1.4}$$

Analogously, function $\phi \in C([0, T])$ is called a lower solution of equation (1.1), if it satisfies

$$D^\alpha \phi(t) \leq f(t, \phi), \quad t \in (0, T]. \tag{1.5}$$

Similarly to [3], we introduce concepts of coupled lower and upper solutions, coupled quasisolutions for problem (1.1)–(1.2).

Definition 1.3. Functions $\widehat{u}, \widetilde{u} \in C([0, T])$ are called coupled lower and upper solutions of problem (1.1)–(1.2), if $\widehat{u}$ is a lower solution and $\widetilde{u}$ is an upper solution of equation (1.1), and they satisfy the relations $\widehat{u}(t) \leq \widetilde{u}(t)$, $t \in [0, T]$ and

$$g(\widehat{u}(0), \widetilde{u}(T)) \leq 0 \leq g(\widetilde{u}(0), \widehat{u}(T)). \tag{1.6}$$

In what follows we assume that

$$\widetilde{u}(t) \geq \widehat{u}(t), \quad t \in [0, T], \tag{1.7}$$

and define that sector

$$\langle \widehat{u}, \widetilde{u} \rangle = \langle u \in C([0, T]); \widehat{u} \leq u \leq \widetilde{u} \rangle.$$

Definition 1.4. Functions $v, w \in C([0, T])$ are called coupled quasisolutions of problem (1.1)–(1.2), if v and w are solutions of equation (1.1), and

$$\widehat{u}(t) \leq v(t) \leq w(t) \leq \widetilde{u}(t), \quad t \in [0, T], \tag{1.8}$$

$$g(v(0), w(T)) = 0 = g(w(0), v(T)), \tag{1.9}$$

where $\widehat{u}, \widetilde{u} \in C([0, T])$ are coupled lower and upper solutions of problem (1.1)–(1.2).

Lemma 1.1 (*Lemma 2.22 [1]*)**.** *If $f(t) \in C([0, T])$ and $0 < \alpha < 1$, then*

$$I^\alpha D^\alpha f(t) = f(t) - f(0).$$

The following are the existence results of the solution for the linear initial value problem for the fractional differential equation, which are important to us in obtaining the existence results of solutions for problem (1.1)–(1.2).

Lemma 1.2 (*[1]*)**.** *The linear initial value problem*

$$\begin{cases} D^\alpha u + du = q(t), & t \in (0, T], \\ u(0) = u_0, \end{cases} \tag{1.10}$$

where d is a constant and $q \in C([0, T])$, has the following integral representation of solution

$$u(t) = u_0 E_{\alpha,1}(-dt^\alpha) + \int_0^t (t - s)^{\alpha-1} E_{\alpha,\alpha}(-d(t - s)^\alpha) q(s) ds, \tag{1.11}$$

where $E_{\alpha,\alpha}(-dt^\alpha)$, $E_{\alpha,1}(-dt^\alpha)$ are Mittag-Leffler functions [1].

Remark 1.1. In particular, when $d = 0$, then initial value problem (1.10) has a solution

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} q(s) ds.$$

2. Main result

The following result will play a very important role in our next analysis.

Lemma 2.1. *If* $w \in C([0, T])$ *and satisfies the relations*

$$\begin{cases} D^\alpha w + d_0 w \geq 0, & t \in (0, T], \\ w(0) \geq 0, \end{cases} \tag{2.1}$$

where $d_0 > -\frac{\Gamma(1+\alpha)}{T^\alpha}$ *is a constant. Then* $w \geq 0$ *for* $t \in [0, T]$.

Proof. Firstly, we verify the result for $d_0 \geq 0$. We assume by contradiction that $w(t) \geq 0, t \in [0, T]$ is false. Then, from $w(0) \geq 0$, there exists points $t_0 \in [0, T)$, $t_1 \in (0, T]$ such that, $w(t_0) = 0$, $w(t_1) < 0$, $w(t) \geq 0$ for $t \in [0, t_0]$, and $w(t) < 0$ for $t \in (t_0, t_1]$. Then we have $D^\alpha w(t_1) + d_0 w(t_1) \geq 0$, therefore,

$$D^\alpha w(t_1) \geq 0. \tag{2.1$'$}$$

Since Riemann–Liouville fractional integral I^α is a monotone operator, we apply the fractional integral I^α on both sides of inequality (2.1′); and by Lemma 1.1, we have

$$w(t_1) - w(0) \geq 0 \tag{2.1$''$}$$

which implies that $w(t_1) \geq 0$, which is a contradiction. So, for $d_0 \geq 0$, we verify that $w \geq 0$ for $t \in [0, T]$.

Now, we will verify the result for $-\frac{\Gamma(1+\alpha)}{T^\alpha} < d_0 < 0$. Since Riemann–Liouville fractional integral I^α is a monotone operator, we apply the fractional integral I^α on both sides of inequality (2.1), and by Lemma 1.1, we have

$$w(t) - w(0) + d_0 I^\alpha w(t) \geq 0, \quad t \in (0, T]. \tag{2.1$'''$}$$

Let $v(t) = \mathrm{e}^{-\gamma t} w(t)$ for any positive constant $1 < \gamma < \frac{-\Gamma(1+\alpha)}{d_0 T^\alpha}$, thus, by (2.1‴), we obtain that the transformation $\mathrm{e}^{-\gamma t} w(t)$ transforms inequality (2.1‴) into relations

$$\begin{cases} \mathrm{e}^{\gamma t} v(t) - v(0) + d_0 I^\alpha \mathrm{e}^{\gamma t} v(t) \geq 0, & t \in (0, T], \\ v(0) \geq 0. \end{cases} \tag{2.2}$$

We assume by contradiction that $v(t) \geq 0$, $t \in [0, T]$ is false. Then, from $v(0) \geq 0$, there exists points $t_0 \in [0, T)$, $t_0' \in (0, T]$ such that, $v(t_0) = 0$, $v(t_0') < 0$, $v(t) \geq 0$ for $t \in [0, t_0]$, $v(t) < 0$ for $t \in (t_0, t_0']$, and assume that t_1 is the first minimal point of $v(t)$ on $[t_0, t_0']$. Then we have

$$\mathrm{e}^{\gamma t_1} v(t_1) - v(0) + d_0 I^\alpha \mathrm{e}^{\gamma t_1} v(t_1) \geq 0. \tag{2.3}$$

We see that

$$\begin{aligned} I^\alpha(\mathrm{e}^{\gamma t_1} v(t_1)) &= \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} \mathrm{e}^{\gamma s} v(s) ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{t_0} (t_1 - s)^{\alpha-1} \mathrm{e}^{\gamma s} v(s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} \mathrm{e}^{\gamma s} v(s) ds \\ &\geq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} \mathrm{e}^{\gamma s} v(s) ds \\ &\geq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} \mathrm{e}^{\gamma t_1} v(s) ds \\ &\geq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} \mathrm{e}^{\gamma t_1} v(t_1) ds \\ &= \frac{v(t_1)}{\Gamma(1+\alpha)} \mathrm{e}^{\gamma t_1} (t_1 - t_0)^\alpha \\ &\geq \frac{T^\alpha}{\Gamma(1+\alpha)} \mathrm{e}^{\gamma t_1} v(t_1) \\ &\geq \frac{\gamma T^\alpha}{\Gamma(1+\alpha)} \mathrm{e}^{\gamma t_1} v(t_1). \end{aligned}$$

That is

$$d_0 I^\alpha(\mathrm{e}^{\gamma t_1} v(t_1)) \leq \frac{d_0 \gamma T^\alpha}{\Gamma(1+\alpha)} \mathrm{e}^{\gamma t_1} v(t_1). \tag{2.4}$$

Thus, by (2.3) and (2.4), we obtain that

$$\left(1+\frac{d_0\gamma T^{\alpha}}{\Gamma(1+\alpha)}\right)v(t_1)\geq 0,$$

which contradicts the negative property of $v(t_1)$, because $1<\gamma<\frac{-\Gamma(1+\alpha)}{d_0T^{\alpha}}$. Therefore, $v\geq 0$, $t\in[0,T]$. Thus, it follows from $w(t)=e^{\gamma t}v(t)$ that $w\geq 0$ in $[0,T]$. Thus, we complete this proof. □

We assume that f satisfies the following condition

$$f(t,u_1)-f(t,u_2)\geq -\underline{d}(u_1-u_2),\quad \widehat{u}\leq u_2\leq u_1\leq \widetilde{u}, \tag{2.5}$$

where $\underline{d}>-\frac{\Gamma(1+\alpha)}{T^{\alpha}}$ is a constant and $\widehat{u},\widetilde{u}\in C([0,T])$ are coupled lower and upper solutions of problem (1.1)–(1.2). Clearly this condition is satisfied with $\underline{d}=0$, when f is monotone nondecreasing in u. In view of (2.5), the function

$$F(t,u)=\underline{d}u+f(t,u) \tag{2.6}$$

is monotone nondecreasing in u for $u\in\langle\widehat{u},\widetilde{u}\rangle$.

We also suppose that there exists a constant $\frac{\Gamma(1+\alpha)}{T^{\alpha}}>\overline{d}\geq-\underline{d}$, such that

$$f(t,u_1)-f(t,u_2)\leq \overline{d}(u_1-u_2),\quad \widehat{u}\leq u_2\leq u_1\leq \widetilde{u}, \tag{2.7}$$

where $\widehat{u},\widetilde{u}\in C([0,T])$ are coupled lower and upper solutions of problem (1.1)–(1.2). Moreover, we assume that g satisfies the following conditions

$$\begin{cases} g(\cdot,y) & \text{is nonincreasing for all } y\in R,\\ g(x,\cdot) & \text{is nondecreasing for all } x\in R.\end{cases} \tag{2.8}$$

The following is the existence theorem of solution for problem (1.1)–(1.2).

Theorem 2.1. *Assume that $\widehat{u},\widetilde{u}\in C([0,T])$ are coupled lower and upper solutions of problem (1.1)–(1.2), such that (1.7) holds, $f\in C([0,T]\times R)$, and satisfies (2.5) and (2.7), g also satisfies condition (2.8). Then for problem (1.1)–(1.2), there exists one solution in the sector $\langle\widehat{u},\widetilde{u}\rangle$.*

Proof. In order to complete this proof, it is sufficient to prove that there exist coupled quasisolutions w, v of problem (1.1)–(1.2) and $v=w$. For this purpose, we have to find two solutions w and v for the following initial value problem

$$\begin{cases} D^{\alpha}v=f(t,v), & t\in(0,T],\\ v(0)=\rho,\end{cases} \tag{2.9}$$

where $\rho\in[\widehat{u}(0),\widetilde{u}(0)]$ is a constant, and then, prove that w and v are truly coupled quasisolutions of problem (1.1)–(1.2), and that $v=w$; that is, firstly, we will prove unique existence of solution $u\in C([0,T])$ of (2.9); second, we will prove that u satisfies $g(u(0),u(T))=0$. This procedure consists of six steps. □

Step 1. Constructing sequences $\{v^{(k)}\}$.

We see that problem (2.9) is equivalent to the following problem

$$\begin{cases} D^{\alpha}v+dv=dv+f(t,v), & t\in(0,T],\\ v(0)=\rho,\end{cases} \tag{2.9$'$}$$

where d is a given constant. So, in order to apply the method of upper and lower solutions and its associated monotone iterative to consider the existence of solution for problem (2.9), we firstly introduce the concepts of upper and lower solutions for initial value problem (2.9).

Definition. A function $\phi\in C([0,T])$ is called an upper solution of problem (2.9), if it satisfies

$$\begin{cases} D^{\alpha}\phi(t)\geq f(t,\phi), & t\in(0,T],\\ \phi(0)\geq\rho.\end{cases} \tag{2.10}$$

Similarly, function $\varphi\in C([0,T])$ is called a lower solution of problem (2.9), if it satisfies all reversed inequalities in (2.10).

From the above definition and Definitions 1.2, 1.3 and (1.7), we can easily obtain that $\widehat{u}$ is a lower solution, $\widetilde{u}$ is an upper solution of problem (2.9).

By adding $\underline{d}v$ ($\underline{d}$ is the function in (2.5)) on both sides of the differential equation of problem (2.9), and choosing a suitable initial iteration $v^{(0)}$, we construct a sequence $\{v^{(k)}\}$, $k=1,2,\ldots$, from the following iteration process

$$\begin{cases} D^{\alpha}v^{(k)}+\underline{d}v^{(k)}=\underline{d}v^{(k-1)}+f(t,v^{(k-1)}), & t\in(0,T],\\ v^{(k)}(0)=\rho.\end{cases} \tag{2.11}$$

Since for each k the right-hand side of (2.11) is known, Lemma 1.2 implies that the sequence $\{v^{(k)}\}$ is well defined. Of particular interest is the sequence obtained from (2.11) with an upper solution or lower solution of problem (2.9) as the initial iteration. From the above deductions, we know that $\widehat{u}$ is the lower solution, $\widetilde{u}$ is the upper solution of problem (2.9). Denote the sequence with the initial iteration $v^{(0)} = \widetilde{u}$ by $\{\overline{v}^{(k)}\}$ and the sequence with $v^{(0)} = \widehat{u}$ by $\{\underline{v}^{(k)}\}$.

Step 2. Monotone property of the two sequences.

The sequences $\{\overline{v}^{(k)}\}$, $\{\underline{v}^{(k)}\}$ constructed by (2.11) process the monotone property

$$\widehat{u} \le \underline{v}^{(k)} \le \underline{v}^{(k+1)} \le \overline{v}^{(k+1)} \le \overline{v}^{(k)} \le \widetilde{u}, \quad t \in [0, T], \tag{2.12}$$

for every $k = 1, 2, \ldots$.

In fact, let $r = \overline{v}^{(0)} - \overline{v}^{(1)}$. By (2.10), (2.11) and (1.7), and $\overline{v}^{(0)} = \widetilde{u}$, we have

$$\begin{aligned} D^\alpha r + \underline{d}r &= (D^\alpha \widetilde{u} + \underline{d}\widetilde{u}) - (\underline{d}\overline{v}^{(0)} + f(t, \overline{v}^{(0)})) \\ &= D^\alpha \widetilde{u} - f(t, \widetilde{u}) \ge 0, \quad t \in (0, T], \\ r(0) &= \widetilde{u}(0) - \rho \ge 0. \end{aligned}$$

In view of Lemma 2.1, $r \ge 0$ for $t \in [0, T]$, which leads to $\overline{v}^{(1)} \le \overline{v}^{(0)} = \widetilde{u}$. A similar argument using the property of a lower solution of problem (2.9) gives $\underline{v}^{(1)} \ge \underline{v}^{(0)} = \widehat{u}$. In fact, let $r = \underline{v}^{(1)} - \underline{v}^{(0)}$. By (2.10), (2.11) and (1.7), and $\underline{v}^{(0)} = \widehat{u}$, we have

$$\begin{aligned} D^\alpha r + \underline{d}r &= \underline{d}\widehat{u} + f(t, \widehat{u}) - D^\alpha \widehat{u} - \underline{d}\widehat{u} \\ &= f(t, \widehat{u}) - D^\alpha \widehat{u} \ge 0, \quad t \in (0, T], \\ r(0) &= \rho - \widehat{u}(0) \ge 0. \end{aligned}$$

In view of Lemma 2.1, $r \ge 0$ for $t \in [0, T]$, which leads to $\widehat{u} = \underline{v}^{(0)} \le \underline{v}^{(1)}$. Let $r^{(1)} = \overline{v}^{(1)} - \underline{v}^{(1)}$. By (2.5), (2.6) and (1.7), we have

$$\begin{aligned} D^\alpha r^{(1)} + \underline{d}r^{(1)} &= \underline{d}\overline{v}^{(0)} + f(t, \overline{v}^{(0)}) - (\underline{d}\underline{v}^{(0)} + f(t, \underline{v}^{(0)})) \\ &= \underline{d}(\widetilde{u} - \widehat{u}) + f(t, \widetilde{u}) - f(t, \widehat{u}) \\ &\ge 0, \quad t \in (0, T] \qquad , \\ r^{(1)}(0) &= \overline{v}^{(1)}(0) - \underline{v}^{(1)}(0) = 0. \end{aligned}$$

Again, in view of Lemma 2.1, $r^{(1)} \ge 0$ for $t \in [0, T]$, the above conclusion shows that

$$\widehat{u} = \underline{v}^{(0)} \le \underline{v}^{(1)} \le \overline{v}^{(1)} \le \overline{v}^{(0)} = \widetilde{u}.$$

Assume, by induction that

$$\widehat{u} \le \underline{v}^{(k-1)} \le \underline{v}^{(k)} \le \overline{v}^{(k)} \le \overline{v}^{(k-1)} \le \widetilde{u}, \quad t \in [0, T]. \tag{2.13}$$

Then by (2.11), (2.5), (2.6) and (2.13), the function $r^{(k)} = \overline{v}^{(k)} - \overline{v}^{(k+1)}$ satisfies the relations

$$\begin{aligned} D^\alpha r^{(k)} + \underline{d}r^{(k)} &= \underline{d}\overline{v}^{(k-1)} + f(t, \overline{v}^{(k-1)}) - (\underline{d}\overline{v}^{(k)} + f(t, \overline{v}^{(k)})) \\ &\ge 0, \quad t \in (0, T], \\ r^{(k)}(0) &= \overline{v}^{(k)}(0) - \overline{v}^{(k+1)}(0) = 0. \end{aligned}$$

In view of Lemma 2.1, $r^{(k)} \ge 0$, that is $\overline{v}^{(k+1)} \le \overline{v}^{(k)}$ for $t \in [0, T]$. Similar reasoning gives $\underline{v}^{(k)} \le \underline{v}^{(k+1)}$ and $\underline{v}^{(k+1)} \le \overline{v}^{(k+1)}$. Hence, the monotone property (2.12) follows from the principle of induction.

Step 3. The two sequences constructed by (2.11) have pointwise limits and satisfy some relations, that is,

$$\lim_{k\to\infty} \overline{v}^{(k)}(t) = v(t), \qquad \lim_{k\to\infty} \underline{v}^{(k)}(t) = w(t) \tag{2.14}$$

exist and satisfy the relation

$$\widehat{u} \le \underline{v}^{(k)} \le \underline{v}^{(k+1)} \le w \le v \le \overline{v}^{(k+1)} \le \overline{v}^{(k)} \le \widetilde{u}, \quad t \in [0, T], \tag{2.15}$$

for every $k = 1, 2, \ldots$.

In fact, by (2.12), we see that the upper sequence $\{\overline{v}^{(k)}\}$ is monotone nonincreasing and is bounded from below and that the lower sequence $\{\underline{v}^{(k)}\}$ is monotone nondecreasing and is bounded from above. Therefore, the pointwise limits exist and these limits are denoted by v and w as in (2.14). Moreover, by (2.12), the limits v, w satisfy (2.15).

Step 4. To prove that v and w are solutions of initial value problem (2.9).

Let $v^{(k)}$ be either $\overline{v}^{(k)}$ or $\underline{v}^{(k)}$ and let

$$(F(v^{(k)}))(t) = \underline{d}v^{(k)}(t) + f(t, v^{(k)}(t)). \tag{2.16}$$

By the integral representation (1.11) for the linear initial value problem, the solution $v^{(k)}$ of problem (2.11) may be expressed as

$$v^{(k)}(t) = \rho E_{\alpha,1}(-\underline{d}t^{\alpha}) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\underline{d}(t-s)s^{\alpha})(F(v^{(k)}))(s)ds. \tag{2.17}$$

By the assumption of function f, the function F is continuous and is monotone nondecreasing in v, the monotone convergence of $v^{(k)}$ to v (see (2.14)) implies that $(F(v^{(k-1)}))(t)$ converges to $(F(v))(t)$. Let $k \to \infty$ in (2.17) and apply the dominated convergence theorem, v satisfies the integral equation

$$v(t) = \rho E_{\alpha,1}(-\underline{d}t^{\alpha}) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\underline{d}(t-s)^{\alpha})(F(v))(s)ds. \tag{2.18}$$

That is, $v(t)$ is the integral representation of the solution of problem (2.9). And that, by Remark 1.1, we can easily obtain that $v \in C([0,T])$, this proves that the upper sequence $\{\overline{v}^{(k)}\}$ converges monotonically from above to a solution v of problem (2.9); the lower sequence $\{\underline{v}^{(k)}\}$ converges monotonically from below to a solution w of problem (2.9), and satisfies the relation $v(t) \ge w(t), t \in [0,T]$.

Step 5. To prove $g(v(0), w(T)) = 0 = g(w(0), v(T))$.

In fact, from (2.15), (2.8) and (1.6), we have

$$\begin{aligned}
&g(v(0), w(T)) \le g(\widehat{u}(0), w(T)) \le g(\widehat{u}(0), \widetilde{u}(T)) \le 0,\\
&g(v(0), w(T)) \ge g(\widetilde{u}(0), w(T)) \ge g(\widetilde{u}(0), \widehat{u}(T)) \ge 0,\\
&g(w(0), v(T)) \le g(\widehat{u}(0), v(T)) \le g(\widehat{u}(0), \widetilde{u}(T)) \le 0,\\
&g(w(0), v(T)) \ge g(\widetilde{u}(0), v(T)) \ge g(\widetilde{u}(0), \widehat{u}(T)) \ge 0,
\end{aligned}$$

which imply that $g(v(0), w(T)) = 0 = g(w(0), v(T))$. Thus, w and v are coupled quasisolutions of problem (1.1)–(1.2), by definition (1.4).

Steps 6. Verifying $v(t) = w(t), t \in [0,T]$.

It is sufficient to prove that $v(t) \le w(t), t \in [0,T]$, by $v(t) \ge w(t), t \in [0,T]$ obtained in Step 4. In fact, by (1.1) and (2.7), the function $r = w - v$ satisfies the relations

$$\begin{cases} D^{\alpha}r = -(f(t,v) - f(t,w)) \ge \overline{d}r, & t \in (0,T],\\ r(0) = 0. \end{cases}$$

Then, Lemma 2.1 implies that $r(t) \ge 0, t \in [0,T]$, this proves $w \ge v$, therefore, we obtain that $v = w$ is unique solution of problem (2.9) in function sector $\langle \widehat{u}, \widetilde{u}\rangle$, that is, $v = w$ is one solution of problem (1.1)–(1.2). Thus, we complete this proof.

Corollary 2.1. *Assume that $\widehat{u}, \widetilde{u} \in C([0,T])$ are coupled lower and upper solutions of problem (1.1)–(1.2), such that (1.7) holds. g also satisfies condition (2.8). And $f(t,u) = j(t,u) + \lambda u$, $\lambda < \frac{\Gamma(1+\alpha)}{T^{\alpha}}$, j satisfies the Lipschitz condition*

$$|j(t,u_1) - j(t,u_2)| \le K|u_1 - u_2|, \quad u_1, u_2 \in \langle \widehat{u}, \widetilde{u}\rangle \tag{2.19}$$

where K is the Lipschitz constant satisfying $0 < K < \frac{\Gamma(1+\alpha)}{T^{\alpha}} - \lambda$, then for problem (1.1)–(1.2) there exists one solution in the sector $\langle \widehat{u}, \widetilde{u}\rangle$.

Proof. From (2.19), we obtain that

$$(\lambda - K)(u_1 - u_2) \le f(t,u_1) - f(t,u_2) \le (K+\lambda)(u_1 - u_2), \quad \widehat{u} \le u_2 \le u_1 \le \widetilde{u},$$

that is, (2.5) and (2.6) hold with $\underline{d} = K - \lambda$, $\overline{d} = K + \lambda$, then Theorem 2.1 implies that for problem (1.1)–(1.2) there exists one unique solution in the sector $\langle \widehat{u}, \widetilde{u}\rangle$. □

Corollary 2.2. *Let $\widetilde{u} \in C([0,T])$ be a nonnegative coupled upper solution of problem (1.1)–(1.2). g also satisfies condition (2.8). And $f(t,u) = j(t,u) + \lambda u$, $\lambda < \frac{\Gamma(1+\alpha)}{T^{\alpha}}$, j satisfies (2.19), and that f, j, g satisfy the relations*

$$f(t,0) = j(t,0) \ge 0, \tag{2.20}$$

$$g(0, \widetilde{u}(T)) \le 0 \le g(\widetilde{u}(0), 0). \tag{2.21}$$

Then for problem (1.1)–(1.2) there exists one unique solution u with $0 \le u \le \widetilde{u}$.

Proof. By (2.20), (2.21) and Definitions 1.2, 1.3, we know that $u(t) = 0$ is a coupled lower solution of problem (1.1)–(1.2), then Corollary 2.1 implies that this result holds.

Corollary 2.3. *Assume that* $f(t,u) = j(t,u) + \lambda u$, $\lambda < \frac{\Gamma(1+\alpha)}{T^\alpha}$, *and conditions* (2.19), (2.20) *hold.* g *also satisfies condition* (2.8). *Moreover, there exists a positive constant* ρ *such that*

$$f(t,\rho) = j(t,\rho) + \lambda\rho \le 0, \quad g(0,\rho) \le 0 \le g(\rho,0). \tag{2.22}$$

Then for problem (1.1)–(1.2) *there exists one unique solution* u *with* $0 \le u \le \rho$.

Proof. From (2.22) and Definitions 1.2, 1.3, we know that $u(t) = \rho$ is a nonnegative coupled upper solution of problem (1.1)–(1.2), then Corollary 2.2 implies that this result holds. □

Remark 2.1. In a similar way, we can deal with the existence results of solutions for problem (1.1)–(1.2) with more general nonlinear boundary conditions

$$g(u(t_0), u(t_1), \ldots, u(t_r)) = 0,$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_r = T$, under some conditions.

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