Abstract

The current paper is concerned with constructing multibump type solutions for a class of quasilinear Schrödinger type equations including the Modified Nonlinear Schrödinger Equations. Our results extend the existence results on multibump type solutions in Coti Zelati and Rabinowitz (1992) [17] to the quasilinear case. Our work provides a theoretic framework for dealing with quasilinear problems, which lack both smoothness and compactness, by using more refined variational techniques such as gluing techniques, Morse theory, Lyapunov–Schmidt reduction, etc.

Keywords: Multibump solutions; Quasilinear elliptic equations; Gluing method

1. Introduction

This paper is concerned with constructing multibump type solutions for quasilinear Schrödinger equations in the entire space. Multibump type solutions for semilinear elliptic PDEs with periodic potentials was first obtained by Coti Zelati and Rabinowitz [17,16] by using a gluing method which was used initially for Hamiltonian ODEs in [37,38] by Séré (see also [16] and the survey monographs of Rabinowitz [34,35] for more references therein). To start the gluing procedure one firstly uses a variational method to find a family of ground state solutions, which need to have certain non-degeneracy property and constitute the basic ‘one-bump’ solutions. Then
further variational arguments are used to construct multibump solutions, i.e. solutions near sums of sufficiently separated translates of the basic solutions. The goal of the current paper is to establish the phenomenon of multibump type solutions for quasilinear equations for which the variational formulation lacks both smoothness and compactness. More precisely, in this paper, we consider a class of quasilinear elliptic equations of Schrödinger type whose weak variational formulation is to look for $u \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that
\begin{equation}
\int_{\mathbb{R}^N} \sum_{i,j=1}^N b_{ij}(x, u) D_iu D_j \varphi dx + \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i,j=1}^N D_s b_{ij}(x, u) D_iu D_j u \varphi dx + \int_{\mathbb{R}^N} V(x) u \varphi dx - \int_{\mathbb{R}^N} f(x, u) \varphi dx = 0, \quad \forall \varphi \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \tag{1.1}
\end{equation}

Here and in the following $D_s b_{ij}(x, s) = \frac{\partial b_{ij}}{\partial s}(x, s)$ and $D^2_s b_{ij}(x, s) = \frac{\partial^2 b_{ij}}{\partial s^2}(x, s)$. For $b_{ij}(x, s) = (1 + 2s^2)\delta_{ij}$ Eq. (1.1) is reduced to the well known so-called Modified Nonlinear Schrödinger Equation
\begin{equation}
-\Delta u + V(x) u - (\Delta |u|^2) u = \lambda |u|^{q-2} u, \quad \text{in } \mathbb{R}^N \tag{1.2}
\end{equation}

where $\lambda > 0$, and $V = V(x)$, $x \in \mathbb{R}^N$ is a given potential, $4 \leq q < 22^* = \frac{4N}{N-2}$ with $2^* = \frac{2N}{N-2}$ for $N \geq 3$ and $2^* = \infty$ for $N = 1, 2$. Eq. (1.2) corresponds to $b_{ij}(s) = (1 + 2s^2)\delta_{ij}$ in (1.1). This type of equations arise from the study of steady states and standing wave solutions of time-dependent nonlinear Schrödinger equations, and are derived as models in various branches of mathematical physics such as in [8,9,6,10,19,7,20,21,32,36]. Mathematical analysis on ground states and bound state solutions of (1.2) have been studied by various variational arguments such as Nehari manifold, constrained minimizations and change of variables in recent years [33,26,29,2,30,13]. To our knowledge multibump type solutions have not been considered for this type of quasilinear Schrödinger equations. The difficulty lies in its lack of smoothness of the variational formulations due to the quasilinear nature and the lack of compactness due to a new compactness threshold. In particular a new critical exponent was revealed and due to the quasilinear presence the critical exponent is $22^* = \frac{4N}{N-2}$ which is twice the classical Sobolev exponent $\frac{2N}{N-2}$ (e.g., [26, 29]). This causes difficulty in treating the variational problem in $H^1(\mathbb{R}^N)$. The goal of the paper is to provide a theoretic framework for dealing with multibump type solutions of quasilinear equations. Our results extend the work by Coti Zelati and Rabinowitz in [17] to a class of general quasilinear equations. Our framework provides a foundation and basic techniques for working on similar problems for quasilinear equations. More precisely, we shall study (1.1), the more general setting, modeled on conditions based the simple model (1.2). Let us make the following assumptions.

(B) Let $b_{ij}: \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ for $1 \leq i, j \leq N$ satisfy

(b0) $b_{ij}(x, s)$ is of class $C^1$, and $D^2_s b_{ij}(x, s)$ is of class $C$.

(b1) For all $x \in \mathbb{R}^N$, $s \in \mathbb{R}$ and $1 \leq i, j \leq N$,

$$b_{ij}(x, s) = b_{ji}(x, s).$$
There exists a constant $c > 0$ such that for all $x \in \mathbb{R}^N$, $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$

$$
c(1 + s^2)|\xi|^2 \leq \sum_{i,j=1}^{N} b_{ij}(x,s)\xi_i\xi_j \leq c^{-1}(1 + s^2)|\xi|^2.
$$

There exist constants $0 < \beta < 2$, $4 < p < \frac{4N}{N-2}$ such that for all $x \in \mathbb{R}^N$, $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$

$$
(\beta - 2) \sum_{i,j=1}^{N} b_{ij}(x,s)\xi_i\xi_j \leq \sum_{i,j=1}^{N} sDsb_{ij}(x,s)\xi_i\xi_j \leq (p - 2 - \beta) \sum_{i,j=1}^{N} b_{ij}(x,s)\xi_i\xi_j.
$$

Let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfy

$(f_0)$ $f$ and $D_s f(x,s)$ are continuous.

$(f_1)$ $\lim_{s \to 0} \frac{f(x,s)}{s} = 0$ uniformly in $x$.

$(f_2)$ There exist $C > 0$, $p \leq q < \frac{4N}{N-2}$ such that for $x \in \mathbb{R}^N$ and $s \in \mathbb{R}$

$$
|f(x,s)| \leq C(|s| + |s|^{q-1}).
$$

$(f_3)$ For all $x \in \mathbb{R}^N$, $0 \neq s \in \mathbb{R}$, $\frac{1}{p} sf(x,s) \geq F(x,s) > 0$, where $F(x,s) = \int_0^s f(x,t)dt$.

$(V)$ $V \in L^\infty(\mathbb{R}^N)$, there is $c > 0$ such that $c \leq V(x) \leq c^{-1}$ for a.e. $x \in \mathbb{R}^N$.

$(T)$ $b_{ij}(x,s)$, $f(x,s)$ and $V(x)$ are $1$-periodic in $x_1, x_2, \ldots, x_N$.

Remark 1.1. We point out that these conditions are readily satisfied by Eq. (1.2) where $b_{ij}(s) = (1 + 2s^2)\delta_{ij}$, $f(s) = \lambda |u|^{q-2}u$ and $V$ is a periodic function satisfying $(V)$, $\lambda > 0$ and $4 < q < \frac{4N}{N-2}$.

Let $Y = \{u \mid u \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} u^2|Du|^2 dx < +\infty\}$. Formally the problem has a variational structure. Due to $(f_2)$ the functional $J : Y \to \mathbb{R}$ is well defined

$$
J(u) = \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i,j=1}^{N} b_{ij}(x,u)D_iuD_j u dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2 - \int_{\mathbb{R}^N} F(x,u)dx. \quad (1.3)
$$

For a function $\varphi \in C_0^\infty(\mathbb{R}^N)$, the derivative of $J$ in the direction $\varphi$ at $u \in Y$ exists, denoted by

$$
\langle DJ(u), \varphi \rangle = \lim_{t \to 0^+} \frac{1}{t}(J(u + t\varphi) - J(u))
$$

$$
= \int_{\mathbb{R}^N} \sum_{i,j=1}^{N} b_{ij}(x,u)D_iuD_j \varphi dx + \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i,j=1}^{N} D_s b_{ij}(x,u)D_iuD_j u \varphi dx
$$

$$
+ \int_{\mathbb{R}^N} V(x)u \varphi dx - \int_{\mathbb{R}^N} f(x,u)\varphi dx. \quad (1.4)
$$
Therefore that \( u \in Y \) is a solution of the quasilinear elliptic equation (1.1) is equivalent to that \( u \) is a critical point of the functional \( J \), that is

\[
\langle DJ(u), \varphi \rangle = 0 \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N).
\]

Now we equip \( Y \) with the following metric

\[
d_Y(u,v) = \|u - v\|_{H^1(\mathbb{R}^N)} + \|Du^2 - Dv^2\|_{L^2(\mathbb{R}^N)}.
\]

Then \( Y \) is a complete metric space and \( J \) is a continuous functional on \( Y \). In fact if \( \{u_n\} \subset Y \) is a Cauchy sequence, \( d_Y(u_n, u_m) \to 0 \) as \( n, m \to \infty \), then \( u_n \to u \) in \( H^1(\mathbb{R}^N) \), \( Du_n^2 \to v \) in \( L^2(\mathbb{R}^N) \). It is easy to verify that \( v = Du^2 \), hence \( u_n \to u \) in \( Y \). The critical point theory for continuous functionals has been well developed in recent years (e.g., [3,4,11,24]), which have been used for equations like (1.1) with \( b_{ij}(x,u) \) being uniformly bounded. Our results allow polynomial growth of the quasilinear term. Also to obtain multibump solutions we need to deal with both the smoothness issue and the compactness issue. We will use cut-off techniques to reduce the variational setting to one for a continuous functional defined on \( H^1(\mathbb{R}^N) \). Then critical point theory for continuous functionals can be adapted to our construction of multibump solutions. To outline our ideas let us first recall the \( G \)-differentiability (see [24] for details).

**Definition 1.1.** Let \( X \) be a Banach space and \( f \) be a continuous functional defined on \( X \). Let \( E \) be a dense subspace of \( X \). We say that \( f \) is \( G \)-differentiable with respect to \( E \) if

(1) for all \( u \in X \) and all \( \varphi \in E \), the derivative of \( f \) in the direction \( \varphi \) at \( u \) exists, denoted by \( \langle Df(u), \varphi \rangle \),

\[
\langle Df(u), \varphi \rangle = \lim_{t \to 0^+} \frac{1}{t} (f(u + t\varphi) - f(u)).
\]

(2) the map \((u, \varphi) \mapsto \langle Df(u), \varphi \rangle \) satisfies

(i) \( \langle Df(u), \varphi \rangle \) is linear in \( \varphi \),

(ii) \( \langle Df(u), \varphi \rangle \) is continuous in \( u \), that is \( \langle Df(u_n), \varphi \rangle \to \langle Df(u), \varphi \rangle \) as \( u_n \to u \) in \( X \).

**Definition 1.2.** The slope of a \( G \)-differentiable functional \( f \) at \( u \) denoted by \( |Df(u)| \), is an extended number in \([0, \infty]\):

\[
|Df(u)| = \sup\{|Df(u), \varphi| \mid \varphi \in E, \|\varphi\| = 1\}.
\]

where \( \|\cdot\| \) denotes the norm in \( X \). A point \( u \in X \) is said to be a critical point of \( f \) at the level \( c \) if \( |Df(u)| = 0 \) and \( f(u) = c \).

**Definition 1.3** (The Concrete Palais–Smale condition at level \( c \) ((CPS)\( c \) in short)). A \( G \)-differentiable functional \( f \) satisfies \( (CPS)_c \) if any sequence \( \{u_n\} \subset X \), satisfying \( |Df(u_n)| \to 0 \), \( f(u_n) \to c \), possesses a convergent subsequence.

In such a setting, we have the first and the second deformation lemmas, various types of min-max theorems, such as the mountain pass lemma, the saddle point theorem, etc. In this paper,
we will utilize the critical point theory for $G$-differentiable functionals. To do it, for $M > 1$ we introduce a modified functional $I_M$:

$$I_M(u) = \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij}(x,u)D_iuD_ju\,dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2\,dx - \int_{\mathbb{R}^N} G(x,u)\,dx$$

(1.5)

where $a_{ij}(x,s)$, $G(x,s)$ are truncation functions satisfying $a_{ij}(x,s) = b_{ij}(x,s)$, $G(x,s) = F(x,s)$ for $|s| \leq M$ and $D_s a_{ij}(x,s) = D_s G(x,s) \sim M^2 |s|^q/2 - 2$ for large $s$ with $M$ being a parameter. (See Section 2 for the precise definitions of these functions.)

For simplicity we omit the index $M$ in $I_M$ if there is no risk of confusion. The functional $I$ is well defined on the Hilbert space $H^1(\mathbb{R}^N)$ and $I$ is continuous. For a function $\phi \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, the $G$-derivative of $I$ in the direction $\phi$ at $u \in H^1(\mathbb{R}^N)$ exists, denoted by $\langle DI(u), \phi \rangle$:

$$\langle DI(u), \phi \rangle = \lim_{t \to 0^+} \frac{1}{t} (I(u + t\phi) - I(u)) = \int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij}(x,u)D_iuD_j\phi\,dx$$

$$+ \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i,j=1}^N D_s a_{ij}(x,u)D_iuD_ju\phi\,dx + \int_{\mathbb{R}^N} V(x)u\phi\,dx - \int_{\mathbb{R}^N} g(x,u)\phi\,dx.$$

If $u \in H^1(\mathbb{R}^N)$ is a critical point of $I$, i.e., $\langle DI(u), \phi \rangle = 0$ for all $\phi \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then $u$ satisfies the following quasilinear equation:

$$\int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij}(x,u)D_iuD_j\phi\,dx + \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i,j=1}^N D_s a_{ij}(x,u)D_iuD_ju\phi\,dx$$

$$+ \int_{\mathbb{R}^N} V(x)u\phi\,dx - \int_{\mathbb{R}^N} g(x,u)\phi\,dx = 0, \quad \forall \phi \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N).$$

(1.6)

It is clear that a solution $u$ of (1.6) is also one of (1.1) provided $\|u\|_{L^\infty} \leq M$.

Under the conditions we have imposed it is easy to see that the functional $I$ has a mountain-pass geometry and a mountain pass value $C_M$ can be defined. It is clear that $I$ possesses the translation invariance under the $\mathbb{Z}^N$ action, i.e., $\forall u \in H^1(\mathbb{R}^N)$, $\forall k \in \mathbb{Z}^N$, let $k * u = u(x - k)$, then $I(k * u) = I(u)$. Consequently, if $u_0$ is a critical point of $I$, so is $k * u$. Here are the main results of our paper.

**Theorem A.** For $M > 1$, let $C_M$ be the Mountain pass value of $I = I_M$:

$$C_M = \inf_{g \in \Gamma_M} \sup_{s \in [0,1]} I(g(s)),$$

$$\Gamma_M = \{ g \in C([0,1], H^1(\mathbb{R}^N)) \mid g(0) = 0, I(g(1)) < 0 \}.$$

Suppose that (B), (F), (V), (T) and the following (Z*) hold:
There exists $\epsilon > 0$ such that $I$ has no critical values in $(C_M, C_M + \epsilon]$ and the set $K^{C_M} / \mathbb{Z}^N$ is finite, where

$$K^a = \{ u \mid u \in H^1(\mathbb{R}^N), |DI(u)| = 0 \text{ and } I(u) \leq a \}.$$ 

Then the functional $I$ has a critical point $u_0$ at the level $C_M$ and the $L^\infty$-bound of $u_0$ is independent of the parameter $M$, thus for $M$ large enough $u_0$ is a critical point of the functional $J$, too. Moreover the critical group $C_1(I, u_0)$ is nontrivial.

It is intuitive that for $|k|$ large, $u_0 + k \ast u_0$ is an approximate solution. Two bump solutions are solutions close to approximate solutions of the form $u_0 + k \ast u_0$. Our next result is concerned with the existence of multibump solutions.

**Theorem B.** Let $u_0$ be an isolated critical point of $I$ having a nontrivial critical group. Then for any $\delta > 0$, there is $K>0$ such that for any $k \in \mathbb{Z}^N$, $|k| \geq K$ there is a critical point $u$ of $I$ satisfying

$$\|u - (u_0 + k \ast u_0)\|_{H^1(\mathbb{R}^N)} + \|u - (u_0 + k \ast u_0)\|_{L^\infty(\mathbb{R}^N)} \leq \delta.$$ 

**Remark 1.2.** Since $\lim_{|k| \to \infty} \|u_0 + k \ast u_0\|_{L^\infty(\mathbb{R}^N)} = \|u_0\|_{L^\infty}$, it is easy to show that $u$ is a critical point of $J$, too.

As a corollary we have the following.

**Theorem C.** Under the assumptions (B), (F), (V), (T) the functional $J$ has infinitely many critical points, which are different up to translations.

**Remark 1.3.** (1). We do not know whether $u_0$ in Theorem A is a critical point for functional $J$ at a mountain pass level. It would be interesting to look at this issue. See section for discussions on the relation between mountain pass values for $I$ and $J$.

(2). The arguments to prove two bump solutions of Theorem B can be extended to cover $m$ bump solutions for any integer $m > 1$.

**Remark 1.4.** Our results extend the classical work of Coti Zelati and Rabinowitz [17] for semilinear equations to the case of quasilinear problems with the new subcritical exponent. After the original work in [15,16,37,38] for Hamiltonian systems, multibump type solutions was initially studied in [17] for semilinear elliptic equations and have been further studied by many authors for semilinear elliptic equations for further results such as [1,2,5,22,27,31] and references therein. Different type of methods have also been explored for studying multibump phenomena such as methods involving using degree arguments in [1], using variant of implicit functions theorem in [31], using critical groups in [5], etc. Our method here is more related to that of using critical groups as in [5]. Due to the new critical exponent for the quasilinear problems we need to first make truncations to get modified problems for which the energy functionals are non-smooth. We work on the critical groups of these non-smooth functionals here.

This paper is organized as follows. In Section 2 we set up the modified functional $I$ and derive some estimates on the critical points of $I$. In Section 3 we make analysis of (CPS) sequences and
establish a deformation lemma. Section 4 is devoted to establishing the existence of a mountain pass type solution to the modified functional. In Section 5 we study in details on the local behavior of the modified functional near a critical point with nontrivial critical groups. In particular we establish a variant of the Shifting Theorem. Using the information from preceding sections we construct multibump type solutions in Section 6.

Throughout the paper, constants $C$ and $C_i$ are used in various places to denote constants independent of the sequences in the arguments.

2. The modified equations and their solutions

In this section, we firstly give the precise definition of the modified functional $I$. Then we study the properties of the critical points of $I$ (therefore the solutions of the modified Eq. (1.6)). In particular, we show that the $L^\infty$-bound of a solution $u$ depends on $I(u)$ only. Of course, $I$ depends on $M > 1$ so we have a family of modified functionals.

Let $M > 1$ be a parameter. We define firstly the function $a_{ij}(x, s) = b_{ij}(x, b(s))$, where $b : \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying

(b) $b(s) = s$, for $|s| \leq M - 1$, $b(-s) = -b(s)$; $b'(s) = 0$ for $s \geq M$ and $b'(s)$ is decreasing in $[M - 1, M]$.

Let $\tilde{F}(x, s) = \varphi(s)F(x, s) + C_0(1 - \varphi(s))|s|^q$, where $\varphi \in C_0^\infty(R, [0, 1])$ satisfies that $\varphi(-s) = \varphi(s)$, $\varphi(s) = 1$ for $|s| \leq M$, $\varphi(s) = 0$ for $|s| \geq 2M$ and $|\varphi'(s)| \leq \frac{2}{M}$ and $s\varphi'(s) \leq 0$, and $C_0$ is a large positive constant. Define $G(x, s) = \tilde{F}(x, m(s))$, where $m$ is a smooth function satisfying

(m) $m(s) = s$, for $|s| \leq M$, $m(-s) = -m(s)$; $\frac{m'(s)}{m(s)} = \frac{1}{2s-M}$ for $s \geq M + 1$ and $\frac{1}{2s-M} \leq \frac{m'(s)}{m(s)} \leq \frac{1}{M}$ for $M \leq s \leq M + 1$.

Define the functional $I = I_M : H^1(\mathbb{R}^N) \to \mathbb{R}$ as follows:

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij}(x, u)D_i u D_j u \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2 \, dx - \int_{\mathbb{R}^N} G(x, u) \, dx. \quad (2.1)$$

Lemma 2.1.

(1) $a_{ij}(x, s) = b_{ij}(x, s)$, $G(x, s) = F(x, s)$ for $|s| \leq M - 1$.

(2) $a_{ij}(x, s)$ $(1 \leq i, j \leq N)$ satisfies for all $x \in \mathbb{R}^N$, $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$

$$\sum_{i,j=1}^N a_{ij}(x, s)\xi_i \xi_j \leq \sum_{i,j=1}^N sD_i a_{ij}(x, s)\xi_i \xi_j \leq (p - 2 - \beta)\sum_{i,j=1}^N a_{ij}(x, s)\xi_i \xi_j. \quad (a_3)$$

(3) Let $\tilde{F}(x, s) = D_s \tilde{F}(x, s)$, then $\frac{1}{p}s\tilde{F}(s) \geq \tilde{F}(x, s)$.

(4) There exists $C > 0$ independent of $M > 1$ such that $|g(x, s)| \leq C(1 + |s|^{q-1})$ for $|s| \leq M$, $|g(x, s)| \leq C(1 + M^\frac{q}{2}|s|^{q-1})$, for $|s| \geq M$.

Proof. (1) is clear.

(2) For $s > 0$, $b(s) = \int_0^s b'(\tau) \, d\tau \geq sb'(s)$, and $1 \geq \frac{s b'(s)}{b(s)} \geq 0$. The same is true for $s < 0$. 
If $\sum_{i,j=1}^{N} s D_s a_{ij}(x,s) \xi_i \xi_j \geq 0$, then

$$\sum_{i,j=1}^{N} s D_s a_{ij}(x,s) \xi_i \xi_j \geq 0 \geq (-2 + \beta) \sum_{i,j=1}^{N} a_{ij}(x,s) \xi_i \xi_j.$$ 

If $\sum_{i,j=1}^{N} s D_s a_{ij}(x,s) \xi_i \xi_j \leq 0$, then

$$0 \geq \sum_{i,j=1}^{N} s D_s a_{ij}(x,s) \xi_i \xi_j = s b'(s) \sum_{i,j=1}^{N} b(s) D_s b_{ij}(x,b(s)) \xi_i \xi_j$$

$$\geq \sum_{i,j=1}^{N} b(s) D_s b_{ij}(x,b(s)) \xi_i \xi_j$$

$$\geq (-2 + \beta) \sum_{i,j=1}^{N} b_{ij}(x,b(s)) \xi_i \xi_j$$

$$= (-2 + \beta) \sum_{i,j=1}^{N} a_{ij}(x,s) \xi_i \xi_j,$$ 

since $\beta < 2$ and $\sum_{i,j=1}^{N} a_{ij}(x,s) \xi_i \xi_j \geq 0$. This proves the left-hand side of the inequality in $(a_3)$. In the same way, we can prove the right-hand side of the inequality in $(a_3)$.

(3) $\tilde{f}(x,s) = D_s \tilde{F}(x,s) = \varphi(s) f(x,s) + C_0 q(1 - \varphi(s)) |s|^{q-2} s + \varphi'(s)(F(x,s) - C_0 |s|^q)$, and

$$\frac{1}{p} s \tilde{f}(x,s) - \tilde{F}(x,s) = \left(\frac{1}{p} s f(x,s) - F(x,s)\right) \varphi(s) + C_0 \left(\frac{q}{p} - 1\right) |s|^q (1 - \varphi(s))$$

$$- \varphi'(s) s \frac{1}{p} (F(x,s) - C_0 |s|^q).$$

Taking $C_0$ large enough we have $F(x,s) - C_0 |s|^q \leq 0$ for $|s| \geq M$. Hence $\frac{1}{p} s \tilde{f}(x,s) - \tilde{F}(x,s) \geq 0$.

(4) Since $|\varphi'(s)| \leq \frac{2}{M}, |\varphi'(s)(F(x,s) - C_0 |s|^q)| \leq C |s|^{q-1} |s|^q$ for $M \leq |s| \leq 2M$. Hence

$$|\tilde{f}(x,s)| \leq C (1 + |s|^{q-1}),$$

$$g(x,s) = D_s G(x,s) = \tilde{f}(x,m(s)) m'(s),$$

$$|g(x,s)| \leq C (1 + |m(s)|^{q-1}) |m'(s)|.$$ 

We have $m(s) = s$ for $|s| \leq M$. Using the properties of $m$ for some $C > 0$ independent of $M > 1$, $|m(s)| \leq C M^\frac{q}{2} |s|^{\frac{q}{2}-1}$ for $|s| \geq M$. Hence $|g(x,s)| \leq C (1 + |s|^{q-1})$, for $|s| \leq M$ and $|g(x,s)| \leq (1 + C M^\frac{q}{2} |s|^{\frac{q}{2}-1})$ for $|s| \geq M$. □
Lemma 2.2. Let \( u \) be a critical point of \( I \), then

\[
\int_{\mathbb{R}^N} u^2 \, dx + \int_{\mathbb{R}^N} \left(1 + \theta_M^2(u)\right) |Du|^2 \, dx \leq CI(u),
\]

where \( C \) is a constant independent of \( M \) and \( \theta_M(u) \) is a truncation of \( u \):

\[
\theta_M(u) = \begin{cases} 
    u & \text{if } |u| \leq M, \\
    M & \text{if } u \geq M, \\
    -M & \text{if } u \leq -M.
\end{cases}
\]

Proof. Notice that \( D_s a_{ij}(x, u) = 0 \) where \( |u| \geq M \). Let

\[
\phi = \begin{cases} 
    u & \text{if } |u| \leq M, \\
    2u - M & \text{if } u \geq M, \\
    2u + M & \text{if } u \leq -M.
\end{cases}
\]

Let \( k > M \) and

\[
\phi_k = \begin{cases} 
    2k - M & \text{if } u \geq k, \\
    \phi & \text{if } |u| \leq k, \\
    -2k + M & \text{if } u \leq -k.
\end{cases}
\]

We choose \( \phi_k \) as the test function in (1.6). Let \( k \to +\infty \), we have

\[
I(u) = \lim_{k \to \infty} \left( I(u) - \frac{1}{p} \langle I'(u), \phi_k \rangle \right)
\]

\[
= \int_{|u| \leq M} \sum_{i,j=1}^{N} \left[ \left( \frac{1}{2} - \frac{1}{p} \right) a_{ij}(x, u) - \frac{1}{2p} D_s a_{ij}(x, u) u \right] D_i u D_j u \, dx
\]

\[
+ \left( \frac{1}{2} - \frac{2}{p} \right) \int_{|u| \geq M} \sum_{i,j=1}^{N} a_{ij}(x, u) D_i u D_j u \, dx
\]

\[
+ \int_{\mathbb{R}^N} V(x) \left( \frac{1}{2} u^2 - \frac{1}{p} u \phi \right) \, dx + \int_{\mathbb{R}^N} \left( \frac{1}{p} g(x, u) \phi - G(x, u) \right) \, dx.
\]

By the condition on \( m \),

\[
\frac{1}{p} g(x, u) \phi = \frac{1}{p} \tilde{f}(x, m(u)) m'(u) \phi
\]

\[
\geq \frac{1}{p} \tilde{f}(x, m(u)) m(u)
\]

\[
\geq \tilde{F}(x, m(u)) = G(x, u).
\]
Meanwhile, $\frac{1}{2}u^2 - \frac{1}{p}u\varphi \geq (\frac{1}{2} - \frac{2}{p})u^2$. Hence by the condition \((a_3)\), we have for some \(C\) independent of \(M\) that

$$I(u) \geq C \int \sum_{i,j=1}^{N} a_{ij}(x,u) D_i u D_j u \, dx + C \int u^2 \, dx$$

$$\geq C \int (1 + b^2(u)) |Du|^2 \, dx + C \int u^2 \, dx$$

$$\geq C \int (1 + \theta_M^2(u)) |Du|^2 \, dx + C \int u^2 \, dx. \quad \square$$

**Remark 2.3.** Checking the proof of the above result and noting that $\|\varphi_k\| \leq 2\|u\|$ in the proof, we see we have proved that for $u \in H^1(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} u^2 \, dx + \int_{\mathbb{R}^N} (1 + \theta_M^2(u)) |Du|^2 \, dx \leq C (I(u) + |D I(u)|\|u\|). \quad (2.2)$$

Using the approach of Moser, we can estimate the $L^\infty$-bound of a critical point of $I$. Here we give some details to show that the estimate is controlled by the value of the functional, and is independent of the parameter $M$. Hence for $M$ large enough, we obtain critical points of the original functional $J$, or solutions of our problem \((1.1)\).

**Lemma 2.4.** Let $u$ be a critical point of $I$. Then $u \in W^{1,\infty}(\mathbb{R}^N)$ and $\|u\|_{W^{1,\infty}(\mathbb{R}^N)} \leq C$, where the constant $C$ depends on $I(u)$ only and is independent of $M$.

**Proof.** Choose $0 < \epsilon < c$ where $c$ is from the condition for $V(x)$. Then by Lemma 2.1 there is $C = C(\epsilon) > 0$ (independent of $M$) such that

$$|g(x,s)| \leq \epsilon |s| + C |s|^{q-1}, \quad \text{for } |s| \leq M,$$

$$|g(x,s)| \leq \epsilon |s| + CM^\frac{1}{2} |s|^{\frac{q}{2}-1}, \quad \text{for } |s| \geq M. \quad (2.3)$$

Inductively, we assume $u \in L^{r+\frac{q}{2}-1}(\mathbb{R}^N)$ for some $r > 1$. Since $u \in H^1(\mathbb{R}^N)$ we may start with $r_0 = 2^* + 1 - \frac{q}{2} > 1$. Let

$$\varphi = \begin{cases} |u|^{2(r-1)}u & \text{if } |u| \leq M, \\ M^{r-1}u^r & \text{if } u \geq M, \\ -M^{r-1}|u|^r & \text{if } u \leq -M \end{cases}$$

and for $k > M$

$$\varphi_k = \begin{cases} \varphi & \text{if } |u| \leq k, \\ M^{r-1}k^r & \text{if } u \geq k, \\ -M^{r-1}k^r & \text{if } u \leq -k. \end{cases}$$
Taking \( \varphi_k \) as a test function in (1.6) and letting \( k \to \infty \), we claim that \( \varphi \) satisfies (1.6). This can be done first because \( \int g(x,u)\varphi_k \to \int g(x,u)\varphi \) and \( \int Vu_\varphi_k \to \int Vu\varphi \) as \( k \to \infty \). Noting \( D_s a_{ij}(x,u) = 0 \) for \( |u| \geq M \), the term involving \( D_s a_{ij}(x,u) \) stays as constant for \( k \geq M \). The terms involving \( a_{ij}(x,u) \) are \( \int_{|u| \leq k} rM^{r-1}|u|^{r-1}a_{ij}(x,u)D_iuD_j u \, dx \) and \( \int_{|u| \geq k} rM^{r-1}|u|^{r-1}a_{ij}(x,u)D_iuD_j u \, dx \). The first of these stays constant and the second converges as \( k \to \infty \) to \( \int_{|u| \leq k} rM^{r-1}|u|^{r-1}a_{ij}(x,u)D_iuD_j u \, dx \) by Lebesgue convergence theorem.

Now we estimate the terms in Eq. (1.6) with \( \varphi \). We have

\[
\int_{\mathbb{R}^N} g(x,u)\varphi \, dx \leq \epsilon \int_{\mathbb{R}^N} u\varphi \, dx + C \int_{\mathbb{R}^N} (u\theta_M(u))^{\frac{q}{2}} |u|^{-1} |\varphi| \, dx
\]

\[
= \epsilon \int_{\mathbb{R}^N} u\varphi \, dx + C \int_{\mathbb{R}^N} |u\theta_M(u)|^{r+\frac{q}{2}-1} \, dx,
\]

\[
\int_{\mathbb{R}^N} V(x)u\varphi \, dx \geq c \int_{\mathbb{R}^N} u\varphi \, dx > 0.
\]

By \((a_3)\) and Sobolev inequality,

\[
\int_{\mathbb{R}^N} \left( \sum_{i,j=1}^{N} a_{ij}(x,u)D_iuD_j \varphi + \frac{1}{2} D_s a_{ij}(x,u)D_iuD_j u \varphi \right) \, dx
\]

\[
= \int_{|u| \leq M} \sum_{i,j=1}^{N} ((2r-1)a_{ij}(x,u) + \frac{1}{2} D_s a_{ij}(x,u)u) |u|^{2r-2} D_iuD_j u \, dx
\]

\[
+ \int_{|u| \geq M} \sum_{i,j=1}^{N} ra_{ij}(x,u)M^{r-1}|u|^{r-1} D_iuD_j u \, dx
\]

\[
\geq \left( 2r - 2 + \frac{\beta}{2} \right) c \int_{|u| \leq M} |u|^{r-2} |Du|^2 \, dx + \int_{|u| \geq M} M^{r+1}|u|^{r-1} |Du|^2 \, dx
\]

\[
\geq \frac{C_0}{r+1} \int_{\mathbb{R}^N} |D(u\theta_M(u))|^{\frac{r+1}{2}} \, dx
\]

\[
\geq \frac{C_0}{r+1} \left( \int_{\mathbb{R}^N} |(u\theta_M(u))|^{(r+1)\frac{N}{N-2}} \, dx \right)^{\frac{N-2}{N}}
\]

where \( C_0 = \frac{\beta c}{2} \). Hence

\[
\frac{C_0}{r+1} \left( \int_{\mathbb{R}^N} |(u\theta_M(u))|^{(r+1)\frac{N}{N-2}} \, dx \right)^{\frac{N-2}{N}} + c \int_{\mathbb{R}^N} u\varphi \, dx \leq \epsilon \int_{\mathbb{R}^N} u\varphi \, dx + C \int_{\mathbb{R}^N} |u\theta_M(u)|^{r+1+\frac{2}{N-2}} \, dx.
\]
Let \( s = \frac{4N}{4N - (N - 2)(q - 4)} \), then \( 1 < s < \frac{N}{N - 2} \), \( \frac{q - 4}{2} \frac{s}{s - 1} = \frac{2N}{N - 2} \),

\[
\int_{\mathbb{R}^N} |u_{\theta M}(u)|^{r \frac{q - 4}{2} + \frac{q - 4}{2} s} \, dx \leq \left( \int_{\mathbb{R}^N} |u_{\theta M}(u)|^{(r + 1)s} \, dx \right)^{\frac{1}{s}} \left( \int_{\mathbb{R}^N} |u_{\theta M}(u)|^{\frac{2N}{N - 2}} \, dx \right)^{\frac{r - 1}{s}}
\]

\[
\leq C_1 \left( \int_{\mathbb{R}^N} |u_{\theta M}(u)|^{(r + 1)s} \, dx \right)^{\frac{1}{s}}
\]

where the constant \( C_1 \) depends on \( \| D(u_{\theta M}(u)) \|_{L^2} \), which is controlled by \( I(u) \).

Therefore, we have

\[
\left( \int_{\mathbb{R}^N} |u_{\theta M}(u)|^{(r + 1)\frac{N}{N - 2}} \, dx \right)^{\frac{N - 2}{(r + 1)N}} \leq \left( C_2 (r + 1) \right)^{\frac{1}{r + 1}} \left( \int_{\mathbb{R}^N} |u_{\theta M}(u)|^{(r + 1)s} \, dx \right)^{\frac{1}{s}}
\]

where \( C_2 = \frac{C_1}{c} \). With \( r_0 = 2^* + 1 - \frac{q}{2} > 1 \) we have \( (r_0 + 1)s = \frac{2N}{N - 2} \). Inductively we choose \( (r_k + 1)s = (r_{k - 1} + 1)\frac{N}{N - 2} = (r_{k - 1} + 1)s d, d = \frac{N}{s(N - 2)} > 1 \). Then

\[
\left( \int_{\mathbb{R}^N} |u_{\theta M}(u)|^{(r_k + 1)s} \, dx \right)^{\frac{1}{(r_k + 1)s}} \leq \left( C_2 (r_{k - 1} + 1) \right)^{\frac{1}{r_k - 1 + 1}} \left( \int_{\mathbb{R}^N} |u_{\theta M}(u)|^{(r_{k - 1} + 1)s} \, dx \right)^{\frac{1}{s}}
\]

\[
\leq \exp \left\{ \sum_{i=1}^{k} \frac{\ln[C_2 (r_{i - 1} + 1)]}{r_{i - 1} + 1} \right\} \left( \int_{\mathbb{R}^N} |u_{\theta M}(u)|^{(r_0 + 1)s} \, dx \right)^{\frac{1}{(r_0 + 1)s}}
\]

\[
= \exp \left\{ \sum_{i=1}^{k} \frac{\ln[C_2 d^{i - 1}(r_0 + 1)]}{d^{i - 1}(r_0 + 1)} \right\} \left( \int_{\mathbb{R}^N} |u_{\theta M}(u)|^{\frac{2N}{N - 2}} \, dx \right)^{\frac{N - 2}{2N}}
\]

\[
\leq C_3 \left\| D(u_{\theta M}(u)) \right\|_{L^2(\mathbb{R}^N)}
\]

Let \( k \to +\infty \),

\[
\left\| u_{\theta M}(u) \right\|_{L^\infty(\mathbb{R}^N)} \leq C_3 \left\| D(u_{\theta M}(u)) \right\|_{L^2(\mathbb{R}^N)}
\]

which implies that

\[
\left\| u \right\|_{L^\infty(\mathbb{R}^N)} \leq C_4
\]

where \( C_4 \) depends only on \( I(u) \) and is independent of \( M \). By regularity theory, we get similar estimate for \( \| Du \|_{L^\infty(\mathbb{R}^N)} \). \( \Box \)

**Lemma 2.5.** Let \( u \) be a critical point of \( I \), then \( u(x) \to 0, D(u(x)) \to 0 \), exponentially as \( |x| \to \infty \).

With slight modifications Lemma 5.10 from [30] applies. We omit it here.
Lemma 2.6. There exists $\alpha > 0$ such that $\int_{\mathbb{R}^N} \theta_M^2(u)|Du|^2 \, dx \geq \alpha$ and $I(u) \geq \alpha$ for all non-zero critical points of $I$. Also, given $K > 0$ there is $\alpha' > 0$ such that for all non-zero critical points of $I$ with $I(u) \leq K$ it holds that $\|u\|_{L^2(\mathbb{R}^N)} \geq \alpha'$.

Proof. Let $u$ be a nonzero critical point. Then we have

$$c \int_{\mathbb{R}^N} \left( u^2 + \frac{\beta}{2} \theta_M^2(u)|Du|^2 \right) \, dx$$

$$\leq \int_{\mathbb{R}^N} \sum_{i,j=1}^N \left( a_{ij}(x,u) + \frac{1}{2} u D_s a_{ij}(x,u) \right) D_i u D_j u \, dx + \int_{\mathbb{R}^N} V(x) u^2 \, dx$$

$$= \int_{\mathbb{R}^N} g(x,u) u \, dx$$

$$\leq \int_{\mathbb{R}^N} \epsilon u^2 \, dx + C_\epsilon \int_{\mathbb{R}^N} \left( \theta_M(u) u \right)^q \, dx$$

$$\leq \epsilon \int_{\mathbb{R}^N} u^2 \, dx + C_\epsilon \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \theta_M^2(u)|Du|^2 \, dx \right)^{\frac{q}{2}}$$

Hence there is $\alpha > 0$ such that $\int_{\mathbb{R}^N} \theta_M^2(u)|Du|^2 \, dx \geq \alpha$ for all non-zero critical points.

The estimate for $I(u) \geq \alpha$ follows from Lemma 2.2.

We show the last assertion by contradiction. Suppose that $(u_n)$ is a sequence of critical points such that $I(u_n) \leq K$ and $\|u_n\|_{L^2(\mathbb{R}^N)} \to 0$ as $n \to \infty$. By Lemma 2.2, $\|u_n\|$ is bounded and by Sobolev inequality $\|u_n\|_{L^{2^*}(\mathbb{R}^N)}$ remains bounded. By interpolation $\|u_n\|_{L^r(\mathbb{R}^N)} \to 0$ for $2 \leq r < 2^*$. We also have

$$C_1 \int_{\mathbb{R}^N} \left( u_n^2 + \theta_M^2(u_n)|Du_n|^2 \right) \, dx \leq \int_{\mathbb{R}^N} g(x,u_n) u_n \, dx$$

$$\leq \int_{\mathbb{R}^N} \epsilon u_n^2 \, dx + C_\epsilon \int_{\mathbb{R}^N} \left( \theta_M(u_n) u_n \right)^q \, dx \to 0$$

a contradiction with the conclusion from the first part of this lemma. \qed

3. Deformation lemma

In this section, we analyze the behavior of $(CPS)$ sequences and prove a version of the second deformation lemma. Using this deformation lemma we establish the existence of a critical point of mountain pass type.
We recall that a sequence \( \{u_n\} \) in \( H^1(\mathbb{R}^N) \) is a Concrete Palais–Smale sequence (\( (CPS) \) in short), if \( I(u_n) \to c \) and \( |DI(u_n)| \to 0 \). By Remark 2.3 a \( (CPS) \) sequence is bounded in \( H^1(\mathbb{R}^N) \). We have the following lemma about the weak limit of a \( (CPS) \) sequence.

**Lemma 3.1.** Let \( \{u_n\} \subset H^1(\mathbb{R}^N) \) be a \( (CPS) \) sequence. Assume \( \{u_n\} \) weakly converges to \( u \) in \( H^1(\mathbb{R}^N) \). Then \( u \) is a solution of (1.6) and \( u_n \to u \) in \( H^1_{\text{loc}}(\mathbb{R}^N) \).

**Proof.** Basically, the same device from [12, 25, 28] is used. Passing to a subsequence, we assume that \( u_n \to u \) in \( L^p_{\text{loc}}(\mathbb{R}^N) \), \( 2 \leq p < \frac{2N}{N-2} \), \( u_n \to u \), a.e. in \( \mathbb{R}^N \). We have

\[
\int \sum_{i,j=1}^N a_{ij}(x,u_n)D_i u_n D_j \varphi dx + \frac{1}{2} \int \sum_{i,j=1}^N D_s a_{ij}(x,u_n)D_i u_n D_j u_n \varphi dx + \int V(x)u_n \varphi dx - \int g(x,u_n)\varphi dx = \langle w_n, \varphi \rangle, \quad \forall \varphi \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)
\]  

(3.1)

where \( w_n \in H^{-1}(\mathbb{R}^N) \) and \( \| w_n \|_{H^{-1}(\mathbb{R}^N)} \to 0 \) as \( n \to \infty \). Let \( \psi \geq 0, \psi \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \). Take the test function \( \varphi = \psi \exp\{-K\theta_M(u_n)\} \), where \( K > 0 \) is a constant to be chosen later. We have

\[
\int \sum_{i,j=1}^N a_{ij}(x,u_n)D_i u_n D_j \psi \exp\{-K\theta_M(u_n)\} dx
\]

\[
+ \int \sum_{i,j=1}^N \left( -K a_{ij}(x,u_n) + \frac{1}{2} D_s a_{ij}(x,u_n) \right) D_i \theta_M(u_n) D_j \theta_M(u_n) \psi \exp\{-K\theta_M(u_n)\} dx
\]

\[
+ \int V(x)u_n \psi \exp\{-K\theta_M(u_n)\} dx - \int g(x,u_n)\psi \exp\{-K\theta_M(u_n)\} dx = o(1).
\]  

(3.2)

In the above, we have used the fact that \( D_s a_{ij}(x,u_n) = 0, D_i \theta_M(u_n) = 0 \), if \( |u_n| \geq M \). Take \( K \) large enough, so that

\[
\sum_{i,j=1}^N \left( -K a_{ij}(x,s) + \frac{1}{2} D_s a_{ij}(x,s) \right) \xi_i \xi_j \leq 0, \quad \text{for } x \in \mathbb{R}, \ s \in \mathbb{R}, \ \xi \in \mathbb{R}^N.
\]

Taking the limit in (3.2) and by Fatou’s Lemma, we have
\[ \int_{\mathbb{R}^N} \sum_{i,j=1}^{N} a_{ij}(x,u)D_i u D_j \psi \exp \{ -K \theta_M(u) \} \, dx \\
+ \int_{\mathbb{R}^N} \sum_{i,j=1}^{N} \left( -K a_{ij}(x,u) + \frac{1}{2} D_s a_{ij}(x,u) \right) D_i \theta_M(u) D_j \theta_M(u) \psi \exp \{ -K \theta_M(u) \} \, dx \\
+ \int_{\mathbb{R}^N} V(x)u \psi \exp \{ -K \theta_M(u) \} \, dx - \int_{\mathbb{R}^N} g(x,u) \psi \exp \{ -K \theta_M(u) \} \, dx \\
\geq 0, \quad \forall \psi \geq 0, \; \psi \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \quad (3.3) \]

For \( \varphi \geq 0, \varphi \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \), we can take \( \psi = \varphi \exp \{ K \theta_M(u) \} \) in (3.3) and obtain
\[
\int_{\mathbb{R}^N} \sum_{i,j=1}^{N} a_{ij}(x,u)D_i u D_j \varphi \, dx + \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i,j=1}^{N} D_s a_{ij}(x,u)D_i u D_j \varphi \, dx \\
+ \int_{\mathbb{R}^N} V(x)u \varphi \, dx - \int_{\mathbb{R}^N} g(x,u) \varphi \, dx \\
\geq 0, \quad \forall \varphi \geq 0, \; \varphi \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \quad (3.4) \]

If we take the test function \( \varphi = \psi \exp \{ K \theta_M(u) \} \) in (3.1), then we can obtain the opposite inequality of (3.4). Hence we have:
\[
\int_{\mathbb{R}^N} \sum_{i,j=1}^{N} a_{ij}(x,u)D_i u D_j \varphi \, dx + \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i,j=1}^{N} D_s a_{ij}(x,u)D_i u D_j \varphi \, dx \\
+ \int_{\mathbb{R}^N} V(x)u \varphi \, dx - \int_{\mathbb{R}^N} g(x,u) \varphi \, dx \\
= 0, \quad \forall \varphi \geq 0, \; \varphi \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \quad (3.5) \]

It is clear that (3.5) holds for all \( \varphi \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \), if we apply (3.5) to \( \varphi_+ - \varphi_- \). Let \( \chi \) be a \( C_0^\infty(\mathbb{R}^N) \) function. We have
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \sum_{i,j=1}^{N} \left( a_{ij}(x,u_n) + \frac{1}{2} D_s a_{ij}(x,u_n)u_n \right) D_i u_n D_j u_n \chi \, dx \\
= \lim_{n \to \infty} \left\{ [D I(u_n), u_n \chi] - \int_{\mathbb{R}^N} V(x)u_n^2 \chi \, dx + \int_{\mathbb{R}^N} g(x,u_n)u_n \chi \, dx \\
- \int_{\mathbb{R}^N} \sum_{i,j=1}^{N} a_{ij}(x,u_n)D_i u_n u_n D_j \chi \, dx \right\} 
\]
\[
\begin{align*}
&= - \int_{ \mathbb{R}^N } V(x) u^2 \chi \, dx + \int_{ \mathbb{R}^N } g(x,u)u \chi \, dx - \int_{ \mathbb{R}^N } \sum_{ i,j=1 }^{ N } a_{ij}(x,u) D_i u D_j \chi \, dx \\
&= \int_{ \mathbb{R}^N } \left( \sum_{ i,j=1 }^{ N } \left( a_{ij}(x,u) + \frac{1}{2} D_s a_{ij}(x,u) u \right) D_i u D_j \chi \right) \, dx.
\end{align*}
\]
(3.6)

Hence
\[
c \lim_{ n \to \infty } \int_{ \mathbb{R}^N } |D u_n - D u|^2 \chi \, dx \\
\leq \lim_{ n \to \infty } \int_{ \mathbb{R}^N } \sum_{ i,j=1 }^{ N } \left( a_{ij}(x,u_n) + \frac{1}{2} D_s a_{ij}(x,u_n) u_n \right) D_i (u_n - u) D_j (u_n - u) \chi \, dx \\
= \lim_{ n \to \infty } \int_{ \mathbb{R}^N } \sum_{ i,j=1 }^{ N } \left( a_{ij}(x,u_n) + \frac{1}{2} D_s a_{ij}(x,u_n) u_n \right) (D_i u_n D_j u_n - 2D_i u_n D_j u_n + D_i u_n D_j u_n) \chi \, dx \\
= 0.
\]

That is \( D u_n \to D u \) in \( L^2_{\text{loc}}(\mathbb{R}^N) \), and consequently \( u_n \to u \) in \( H^1_{\text{loc}}(\mathbb{R}^N) \). \( \square \)

It is clear that \( I \) possesses a \( \mathbb{Z}^N \) invariance. For \( u \in H^1(\mathbb{R}^N), k \in \mathbb{Z}^N \), let
\[(k \ast u)(x) = u(x - k)\].

Then \( I(k \ast u) = I(u) \). Consequently, if \( u \) is a critical point, so is \( k \ast u \). So the (CPS) condition does not hold as the (CPS) sequence \( \{k_n \ast u\}, |k_n| \to \infty \) has no convergent subsequence, unless \( u \equiv 0 \). However, the following conclusion can be drawn from a (CPS) sequence, see [16,17] for the cases of semilinear differential equations.

**Lemma 3.2.** Let \( \{u_n\} \subset H^1(\mathbb{R}^N) \) be such that \( I(u_n) \leq b \) and \( |D I(u_n)| \to 0 \). Then there exists an \( l \in \mathbb{N} \) bounded by a constant \( l(b) \) depending only on \( b \), nontrivial critical points \( v_1, v_2, \ldots, v_l \) of \( I \) and corresponding \( \{k_n^i\} \subset \mathbb{Z}^N, 1 \leq i \leq l \), such that, passing to a subsequence,
\[
\left\| u_n - \sum_{ i=1 }^{ l } k_n^i \ast v_i \right\| \to 0, \tag{3.7}
\]
\[
I(u_n) - \sum_{ i=1 }^{ l } I(v_i) \to 0, \tag{3.8}
\]
\[
|k_n^i - k_n^j| \to \infty, \quad \text{as } n \to \infty \text{ for } 1 \leq i \neq j \leq l. \tag{3.9}
\]

**Proof.** By Remark 2.3, \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^N) \). If \( u_n \to 0 \) in \( H^1(\mathbb{R}^N) \), we are done. Otherwise we apply the concentration-compactness principle [23] to the sequence \( \{u_n\} \). There are two cases, passing to a subsequence:
(i) (Vanishing) \( \lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} u_n^2 \, dx = 0 \) for all \( R > 0 \).

(ii) (Non-vanishing) There exist \( \gamma > 0 \), \( R < +\infty \) and \( \{y_n\} \subset \mathbb{R}^N \) such that

\[
\lim_{n \to \infty} \int_{B_R(y_n)} u_n^2 \, dx \geq \gamma > 0.
\]

We show that Vanishing cannot occur. Otherwise, for any given \( \epsilon > 0 \), there exists \( N_0 \) such that

\[
\lim_{n \to \infty} \int_{B_1(y)} u_n^2 \, dx \leq \epsilon, \quad n \geq N_0.
\]

Let \( \chi \) be a cut-off function such that \( \chi = 1 \) in \( B_1 \), \( \chi = 0 \) outside \( B_2 \) and \( |D\chi| \leq 2 \). By Hölder inequality and Sobolev embedding theorem, we have

\[
\int_{B_1(y)} |u_n|^{2N+4/N} \, dx \leq \left( \int_{B_1(y)} u_n^2 \, dx \right)^{2/N} \left( \int_{B_1(y)} |u_n|^{2N/N} \, dx \right)^{N-2/N} \leq C \left( \int_{B_1(y)} u_n^2 \, dx \right)^{2/N} \int_{B_1(y)} |D(u_n\chi)|^2 \, dx \leq C \epsilon^{2/N} \int_{B_2(y)} (|Du_n|^2 + u_n^2) \, dx.
\]

A covering consideration gives

\[
\int_{\mathbb{R}^N} |u_n|^{2N+4/N} \, dx \leq C \epsilon^{2/N} \int_{\mathbb{R}^N} (|Du_n|^2 + u_n^2) \, dx \leq C \epsilon^{2/N}
\]

which implies that \( \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{2N+4/N} \, dx = 0 \). Since both \( \|u_n\|_{L^2} \) and \( \|u_n\|_{L^{2N/(N-2)}} \) are bounded, we have \( \|u_n\|_{L^r} \to 0 \) for \( 2 < r < \frac{2N}{N-2} \). For \( \epsilon > 0 \), there is \( C_\epsilon > 0 \) such that

\[
|g(x, s)| \leq \epsilon |s| + C_\epsilon |t|^\frac{q}{2}-1,
\]

and

\[
\int_{\mathbb{R}^N} |g(x, u_n)| \, dx \leq \epsilon \int_{\mathbb{R}^N} u_n^2 \, dx + C_\epsilon \int_{\mathbb{R}^N} |u_n|^\frac{q}{2} \, dx.
\]

We have

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} g(x, u_n) \, dx = 0.
\]
Since \( \{u_n\} \) is a \((\text{CPS})\) sequence we have
\[
\begin{align*}
c \int_{\mathbb{R}^N} (|Du_n|^2 + u_n^2) \, dx &\leq \int_{\mathbb{R}^N} \left( \sum_{ij=1}^{N} a_{ij}(x, u_n) + \frac{1}{2} D_s a_{ij}(x, u_n)u_n \right) D_i u_n D_j u_n \, dx \\
&\quad + \int_{\mathbb{R}^N} V(x) u_n^2 \, dx \\
&= \int_{\mathbb{R}^N} g(x, u_n) u_n \, dx + \{DI(u_n), u_n\}.
\end{align*}
\]

Hence \( u_n \to 0 \) in \( H^1(\mathbb{R}^N) \).

Note that the above Vanishing result can be localized. Namely, let \( \Omega \) be an open subset of \( \mathbb{R}^N \), bounded or unbounded, and \( \Omega_0 \) be an open subset of \( \Omega \) such that \( \bar{\Omega}_0 \subset \Omega \). Suppose that
\[
\lim_{n \to \infty} \sup_{y \in \Omega} \int_{B_R(y)} u_n^2 \, dx = 0,
\]
then \( \lim_{n \to \infty} \int_{\Omega_0} (u_n^2 + |Du_n|^2) \, dx = 0. \)

Now suppose that Non-vanishing occurs. There exists \( \nu > 0 \), \( R < \infty \) and \( \{k_n^1\} \subset \mathbb{R}^N \) such that
\[
\lim_{n \to \infty} \int_{B_R(k_n^1)} u_n^2 \, dx \geq \nu,
\]
without loss of generality, we assume that \( \{k_n^1\} \subset \mathbb{Z}^N \). By Lemma 3.1, there is a critical point \( v_1 \) of \( I \), passing to a subsequence, \( u_n(\cdot + k_n^1) \) converges to a solution \( v_1 \) of (1.6) weakly in \( H^1(\mathbb{R}^N) \) and strongly in \( H^1_{\text{loc}}(\mathbb{R}^N) \). Let \( \tilde{u}_n = u_n - k_n^1 * v_1 \). We claim that
\[
\lim_{n \to \infty} \|u_n\|^2 = \lim_{n \to \infty} \|\tilde{u}_n\|^2 + \|v_1\|^2, \quad (3.10)
\]
\[
\lim_{n \to \infty} I(u_n) = \lim_{n \to \infty} I(\tilde{u}_n) + I(v_1). \quad (3.11)
\]
To get these we first show
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \sum_{ij=1}^{N} a_{ij}(x, u_n) D_i u_n D_j u_n \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} \sum_{ij=1}^{N} a_{ij}(x, \tilde{u}_n) D_i \tilde{u}_n D_j \tilde{u}_n \, dx \\
+ \int_{\mathbb{R}^N} \sum_{ij=1}^{N} a_{ij}(x, v_1) D_i v_1 D_j v_1 \, dx. \quad (3.12)
\]
Since \( v_1 \in H^1(\mathbb{R}^N) \), \( |v_1(x)|, |Dv_1(x)| \to 0 \) as \( |x| \to \infty \) and \( \|\tilde{u}_n\|_{H^1(B_R(k_n^1))} = \|u_n - k_n^1 * v_1\|_{H^1(B_R(0))} \to 0 \) as \( n \to \infty \) for all \( R > 0 \), we have
\[
\int \sum_{ij=1}^{N} a_{ij}(x, u_n) D_i u_n D_j u_n \, dx \\
= \int \sum_{ij=1}^{N} a_{ij}(x, k_n^1 v_1) D_i (k_n^1 v_1) D_j (k_n^1 v_1) \, dx + o_n(1) \\
= \int \sum_{ij=1}^{N} a_{ij}(x, v_1) D_i v_1 D_j v_1 \, dx + o_n(1) \\
= \int \sum_{ij=1}^{N} a_{ij}(x, v_1) D_i v_1 D_j v_1 \, dx + o_R(1) + o_n(1)
\]

(3.13)

where \( o_R(1) \rightarrow 0 \) as \( R \rightarrow \infty \) and \( o_n(1) \rightarrow 0 \) as \( n \rightarrow \infty \). We have also

\[
\int \sum_{ij=1}^{N} a_{ij}(x, u_n) D_i u_n D_j u_n \, dx \\
= \int \sum_{ij=1}^{N} a_{ij}(x, \tilde{u}_n + k_n^1 v_1) D_i (\tilde{u}_n + k_n^1 v_1) D_j (\tilde{u}_n + k_n^1 v_1) \, dx \\
= \int \sum_{ij=1}^{N} a_{ij}(x, \tilde{u}_n) D_i \tilde{u}_n D_j \tilde{u}_n \, dx + o_R(1) \\
= \int \sum_{ij=1}^{N} a_{ij}(x, \tilde{u}_n) D_i \tilde{u}_n D_j \tilde{u}_n \, dx + o_R(1) + o_n(1).
\]

(3.14)

Thus (3.12) follows from (3.13) and (3.14). In a similar way, we can prove that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |D u_n|^2 \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} |D \tilde{u}_n|^2 \, dx + \int_{\mathbb{R}^N} |D v_1|^2 \, dx,
\]

(3.15)

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \theta_M^2(u_n)|D u_n|^2 \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} \theta_M^2(\tilde{u}_n)|D \tilde{u}_n|^2 \, dx + \int_{\mathbb{R}^N} \theta_M^2(v_1)|D v_1|^2 \, dx
\]

and

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} H(x, u_n) \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} H(x, \tilde{u}_n) \, dx + \lim_{n \to \infty} \int_{\mathbb{R}^N} H(x, v_1) \, dx
\]

(3.16)
where $H(x, s): \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is measurable in $x$ for all $s \in \mathbb{R}$, and is $C^1$ in $s$ for a.e. $x \in \mathbb{R}^N$ and satisfies for some constant $K > 0$, $2 < r < \frac{2N}{N-2}$

$$|D_s H(x, s)| \leq K(|s| + |s|^{r-1}), \quad H(x, s) \leq K(|s|^2 + |s|^r).$$

In particular (3.10), (3.11) hold.

Now we apply the concentration compactness principle to the sequence $\{\tilde{u}_n\}$. If Vanishing occurs for the sequence $\{\tilde{u}_n\}$, then $\int_{\mathbb{R}^N} |\tilde{u}_n|^p \, dx \to 0$ for $2 < p < \frac{2N}{N-2}$. Given $\epsilon > 0$, talking $R$ large enough, we have

$$\int_{\mathbb{R}^N \setminus B_R(k_1^n)} |u_n|^p \, dx \leq C \left( \int_{\mathbb{R}^N} |\tilde{u}_n|^p \, dx + \int_{\mathbb{R}^N \setminus B_R(0)} |v|^p \, dx \right) \leq C \epsilon$$

(3.17)

which in turn implies that $\|u_n\|_{H^1(\mathbb{R}^N \setminus B_R(k_1^n))}$ is small and $\tilde{u}_n$ converges to zero in $H^1(\mathbb{R}^N)$.

If Non-vanishing occurs for the sequence $\{\tilde{u}_n\}$, then there exists $\nu > 0$, $R < \infty$ and a sequence $\{k_2^n\} \subset \mathbb{Z}^N$ such that

$$\lim_{n \to \infty} \int_{B_R(k_2^n)} |u_n - k_2^n \ast v_1|^2 \, dx \geq \nu.$$ 

(3.18)

Note that for any sequence $\{y_n\}$ such that $|y_n - k_1^n|_1$ is bounded, say $|y_n - k_1^n| \leq r$, we have

$$\lim_{n \to \infty} \int_{B_R(y_n)} |u_n - k_1^n \ast v_1|^2 \, dx \leq \lim_{n \to \infty} \int_{B_{R+r}(k_1^n)} |u_n - k_1^n \ast v_1|^2 \, dx = 0.$$

Hence $|k_2^n - k_1^n| \to \infty$ as $n \to \infty$. Suppose $u_n(x - k_2^n) \rightharpoonup v_2$ in $H^1(\mathbb{R}^N)$. By Lemma 3.1, $v_2$ is a solution of (1.6) and $u_n \rightharpoonup v_2$ in $H^1_{loc}(\mathbb{R}^N)$. We have the following

$$\lim_{n \to \infty} \|u_n\|^2 = \lim_{n \to \infty} \|u_n - k_2^n \ast v_1 - k_1^n \ast v_2\|^2 + \|v_1\|^2 + \|v_2\|^2,$$

$$\lim_{n \to \infty} I(u_n) = \lim_{n \to \infty} I(u_n - k_1^n \ast v_1 - k_2^n \ast v_2) + I(v_1) + I(v_2).$$

In fact, let us prove for example that

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^2 \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n - k_2^n \ast v_1 - k_1^n \ast v_2|^2 \, dx + \int_{\mathbb{R}^N} v_1^2 \, dx + \int_{\mathbb{R}^N} v_2^2 \, dx.$$

Divide $\int_{\mathbb{R}^N} |u_n|^2 \, dx$ to three parts:

$$\int_{\mathbb{R}^N} |u_n|^2 \, dx = \int_{B_R(k_1^n)} |u_n|^2 \, dx + \int_{B_R(k_2^n)} |u_n|^2 \, dx + \int_{\mathbb{R}^N \setminus (B_R(k_1^n) \cup B_R(k_2^n))} |u_n|^2 \, dx.$$
Then for $i = 1, 2$, we have
\[
\int_{B_R(k^i_n)} |u_n|^2 \, dx = \int_{B_R(0)} |u_n(x + k^i_n)|^2 \, dx = \int_{B_R(0)} v_i^2 \, dx + o_n(1) = \int_{R^N} v_i^2 \, dx + o_R(1) + o_n(1),
\]
and
\[
\int_{\mathbb{R}^N \setminus (B_R(k^1_n) \cup B_R(k^2_n))} |u_n|^2 \, dx = \int_{\mathbb{R}^N \setminus (B_R(k^1_n) \cup B_R(k^2_n))} |u_n - k^1_n * v_1 - k^2_n * v_2|^2 \, dx + o_R(1) = \int_{\mathbb{R}^N} |u_n - k^1_n * v_1 - k^2_n * v_2|^2 \, dx + o_R(1) + o_n(1),
\]

since $u_n - k^1_n * v_1 \rightarrow 0$, $k^2_n * v_2 \rightarrow 0$ in $H^1(B_R(k^1_n))$ and $u_n - k^2_n * v_2 \rightarrow 0$, $k^1_n * v_1 \rightarrow 0$ in $H^1(B_R(k^2_n))$.

Continuing the above arguments, we obtain nonzero critical points $v_1, v_2, \ldots, v_l$ and sequences $\{k^1_n\}, \{k^2_n\} \ldots \{k^n_l\} \subset \mathbb{Z}^N$. By Lemma 2.6, there is $\alpha > 0$ such that $\|v\|_{L^2(\mathbb{R}^N)} \geq \alpha$ for all nonzero critical point, the above procedure must stop after a finite number of steps, say $l$ steps. That is, for the sequence $\{u_n - \sum_{i=1}^l k^i_n * v_i\}$ vanishing happens, and hence $\|u_n - \sum_{i=1}^l k^i_n * v_i\| \rightarrow 0$. The lemma is proved. \hfill \Box

Now we are ready to prove the deformation lemma. The classical deformation lemmas have been extended to continuous functionals, for example, the following form was proved in [24]:

**Proposition.** Let $f$ be a $G$-differentiable functional defined on a Banach space $X$ and satisfy the (CPS) condition. Let $f^b = \{u \in X \mid f(x) \leq b\}$, $K = \{u \in X \mid |Df(u)| = 0\}$. Suppose that $f$ has no critical points in the set $f^b \setminus f^a$ and $K_a = \{u \in X \mid f(u) = a, \ |Df(u)| = 0\}$ consists of isolated critical points. Then $f^a$ is a strong deformation retract of $f^b$.

In the present case, the functional $I$ does not satisfy the (CPS) condition. However, the (CPS) condition can be replaced by the analysis of the (CPS) sequence to prove the global existence of the pseudo gradient flow. We follow the idea in [12,24]. We will use the following notations

\[
I^b = \{u \mid u \in H^1(\mathbb{R}^N), \ I(u) \leq b\},
\]

\[
I^b_a = \{u \mid u \in H^1(\mathbb{R}^N), \ a \leq I(u) \leq b\},
\]

\[
K^a = \{u \mid u \in H^1(\mathbb{R}^N), \ u \neq 0, \ |Df(u)| = 0, \ I(u) \leq a\},
\]

\[
K_a = \{u \mid u \in H^1(\mathbb{R}^N), \ |Df(u)| = 0, \ I(u) = a\}.
\]

We need the following assumptions:

\((Z^*)\) $I$ has no critical points in $I^b \setminus I^a$ and the set $K^a / \mathbb{Z}^N$ is finite.

Choose one representative in each class of $K^a / \mathbb{Z}^N$ to form a finite set. For simplicity of notation we just use $K^a / \mathbb{Z}^N$ to denote this finite set.
Theorem 3.3. Assume (B), (F), (V), (T) and (Z*). Then $I^a$ is a strong deformation retract of $I^b$.

Proof of Theorem 3.3. We define

$$D_n = \left\{ w = \sum_{i=1}^{l} k_i^* v_i \mid l \leq l(b), \; v_1, v_2, \ldots, v_l \in K^a/Z^N, \; k_1, k_2, \ldots, k_l \in Z^N, \; \|k_i - k_j\| \geq n, \; \text{for} \; i \neq j \right\},$$

and

$$A_n = \left\{ u \mid u \in I^b_a, \; \|u - w\| < \frac{1}{n} \; \text{for some} \; w \in D_n \right\},$$

where $l = l(b)$ is the constant in Lemma 3.2.

Claim 1. For each $n$, $\alpha_n := \frac{1}{2} \inf\{|DI(u)|u \in I^b_a \setminus A_n\} > 0$.

Proof. We prove this by an indirect argument. Suppose that $\{u_m\} \subset I^b_a \setminus A_n$ and $|DI(u_m)| \to 0$ as $m \to \infty$. By Lemma 3.2, there exist $l \leq l(b), v_1, v_2, \ldots, v_l \in K^a/Z^N$ and $\{k^1_m\}, \{k^2_m\}, \ldots, \{k^l_m\} \subset Z^N$ such that as $m \to \infty$

$$\left\|u_m - \sum_{i=1}^{l} k^i_m \ast v_i\right\| \to 0,$$

$$|k^i_m - k^j_m| \to \infty \; \text{for} \; 1 \leq i < j \leq l.$$

In particular, $|k^i_m - k^j_m| \geq n$ for $1 \leq i < j \leq l$ and $\|u_m - \sum_{i=1}^{l} k^i_m \ast v_i\| < \frac{1}{n}$ as $m$ large enough, hence $u_m \in A_n$, a contradiction.

Claim 2. There exist $n_1 \in N$, $\delta > 0$ such that $\|w - \tilde{w}\| \geq \delta$ for $w, \tilde{w} \in D_n, w \neq \tilde{w}$ and $n \geq n_1$.

Proof. If the conclusion in Claim 2 is not true, there exist two sequences $\{w_n\}, \{\tilde{w}_n\}$ such that $w_n \neq \tilde{w}_n, w_n, \tilde{w}_n \in D_n$ and $\|w_n - \tilde{w}_n\| \to 0$ as $n \to \infty$. Passing to a subsequence, we have $l, \tilde{l} \leq l(b), v_1, v_2, \ldots, v_l; \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_{\tilde{l}} \in K^a/Z^N$ and $\{k^1_n\}, \{k^2_n\}, \ldots, \{k^l_n\}, \{\tilde{k}^1_{\tilde{n}}\}, \{\tilde{k}^2_{\tilde{n}}\}, \ldots, \{\tilde{k}^\tilde{l}_{\tilde{n}}\} \subset Z^N$ such that

$$w_n = \sum_{i=1}^{l} k^i_n \ast v_i, \quad \tilde{w}_n = \sum_{i=1}^{\tilde{l}} \tilde{k}^i_n \ast \tilde{v}_i,$$

$$|k^i_n - k^j_n| \to \infty, \; \text{for} \; 1 \leq i < j \leq l; \; \text{as} \; n \to \infty,$$

$$|\tilde{k}^i_n - \tilde{k}^j_n| \to \infty, \; \text{for} \; 1 \leq i < j \leq \tilde{l}; \; \text{as} \; n \to \infty.$$

If for all $i = 1, 2, \ldots, \tilde{l}$, $|k^i_n - k^i_n| \to \infty$, as $n \to \infty$, then
\[
\|v_l\| \leq \lim_{n \to \infty} \left\| v_l + \sum_{i=1}^{l-1} (k^l_n - \tilde{k}^l_n) * v_i - \sum_{i=1}^{\tilde{l}} (\tilde{k}^l_n - k^l_n) * \tilde{v}_i \right\|
\]

\[
= \lim_{n \to \infty} \left\| \sum_{i=1}^{l} k^l_n * v_i - \sum_{i=1}^{\tilde{l}} \tilde{k}^l_n * \tilde{v}_i \right\|
\]

\[
= 0
\]

which is a contradiction. If for some \( i \), say \( i = \tilde{i} \), \( \{k^l_n - \tilde{k}^l_n\} \) is bounded, then \( \{k^l_n - \tilde{k}^l_n\} \) is unbounded for \( 1 \leq i \leq l - 1 \) and up to a subsequence \( k^l_n - \tilde{k}^l_n = k^l \) is a constant. We have

\[
\|v_l - k^l \ast \tilde{v}_l\| \leq \lim_{n \to \infty} \left\| v_l + \sum_{i=1}^{l-1} (k^l_n - \tilde{k}^l_n) * v_i - k^l \ast \tilde{v}_l - \sum_{i=1}^{\tilde{l}-1} (\tilde{k}^l_n - k^l_n) * \tilde{v}_i \right\|
\]

\[
= \lim_{n \to \infty} \left\| \sum_{i=1}^{l} k^l_n * v_i - \sum_{i=1}^{\tilde{l}} \tilde{k}^l_n * \tilde{v}_i \right\|
\]

\[
= 0.
\]

Since \( v_l, \tilde{v}_l \in K^a / \mathbb{Z}^N \), we have \( k^l = 0 \), \( v_l = \tilde{v}_l \), and

\[
\lim_{n \to \infty} \left\| \sum_{i=1}^{l-1} k^l_n \ast v_i - \sum_{i=1}^{\tilde{l}-1} \tilde{k}^l_n \ast \tilde{v}_i \right\| = 0.
\]

By induction we conclude that \( l = \tilde{l} \) and \( w_n = \tilde{w}_n \).

**Claim 3.** Let \( B_n = \{ u \in I_a^b, \| u - w \| \leq \frac{1}{n} \text{ for some } w \in D_n \} \). Then there exists \( n_2 \in \mathbb{N} \) such that \( \tilde{A}_n \subset B_n \subset A_{n-1} \) for \( n \geq n_2 \).

**Proof.** Let \( u \in \tilde{A}_n, n \geq n_1 \). There exist \( \{u_m\} \subset A_n \) and \( \{w_m\} \subset D_n \subset D_{n_1} \) such that \( u_m \to u \) and \( \|w_m - u_m\| < \frac{1}{n_1} \). For \( m, m' \) large enough, we have

\[
\|w_m - w_{m'}\| \leq \frac{2}{n} + \|u_m - u_{m'}\| \leq \frac{3}{n}.
\]

Let \( \delta \) be the constant in Claim 2. Choose \( n_2 > \max(n_1, \frac{3}{\delta}) \). Then for \( n \geq n_2 \), we have

\[
\|w_m - w_{m'}\| \leq \frac{3}{n} < \delta.
\]

By Claim 2, \( w_m = w_{m'} = w \in D_n \) and \( \|w - u_m\| < \frac{1}{n} \). Let \( m \to \infty \), we have \( \|w - u\| \leq \frac{1}{n} \) and \( u \in B_n \). The inclusion \( B_n \subset A_{n-1} \) is obvious.

**Claim 4.** \( \bigcap_{n=1}^{\infty} A_n = K_a \).

**Proof.** It is clear that \( K_a \subset \bigcap_{n=1}^{\infty} D_n \subset \bigcap_{n=1}^{\infty} A_n \). Suppose that \( u \in \bigcap_{n=1}^{\infty} A_n \). Then there exists \( w_n \in D_n \) and \( \|u - w_n\| \leq \frac{1}{n}, n = 1, 2, \ldots \). Passing to a subsequence, we can assume that
\[ w_n = \sum_{i=1}^l k_n^i \ast v_i \], where \( l \leq l(b) \), \( v_1, v_2, \ldots, v_l \in K^a/\mathbb{Z}^N \), and \( \{k_1^i\}, \{k_2^i\}, \ldots, \{k_l^i\} \) satisfying \( |k_n^i - k_n^j| \to \infty \) as \( n \to \infty \) for \( 1 \leq i < j \leq l \). If \( \{k_n^i\} \) is unbounded for all \( i \), then

\[ \|u\| \leq \lim_{n \to \infty} \|u - w_n\| = 0. \]

If one of \( \{k_n^i\} \), say \( \{k_1^1\} \), is bounded, then up to a subsequence \( k_1^1 = k^1 \) and

\[ \|u - k^1 \ast v_1\| \leq \lim_{n \to \infty} \|u - w_n\| = 0. \]

The claim is proved.

Define

\[ g(u) = \begin{cases} 
\frac{\alpha_{n+1}}{d(u, A_{n+1})} \alpha_n, & u \in I^b_a \setminus A_n, \\
\frac{\alpha_{n+1}}{d(u, A_{n+1}) + d(u, I^b_a \setminus A_n)} \alpha_n, & u \in A_n \setminus A_{n+1}, n \geq n_2.
\end{cases} \]

By Claims 3, 4, \( g \) is well defined and locally Lipschitz continuous on \( I^b_a \setminus K_a \). By the definition of \( \alpha_n \) from Claim 2

\[ |DI(u)| \geq 2g(u), \quad \text{for } u \in I^b_a \setminus K_a. \]

We construct a pseudo-gradient vector field \( v \) on \( I^b_a \setminus K_a \) as follows. For \( u \in I^b_a \setminus K_a \), we choose \( \phi = \phi(u) \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) and an open neighborhood \( O(u) \) of \( u \) such that \( \|\phi\| = 1 \) and

\[ \langle DI(v), \phi \rangle \geq g(v), \quad \forall v \in O(u). \]

There is a locally finite partition of unity \( \{\eta_i, i \in \Lambda\} \) with \( \text{supp} \eta_i \subset O(u_i) \) for some \( u_i \in I^b_a \setminus K_a \) and

\[ \sum_{i \in \Lambda} \eta_i(u) = 1, \quad \forall u \in I^b_a \setminus K_a. \]

Define \( V(u) = -\sum_{i \in \Lambda} \eta_i(u) \phi(u_i)/g(u), u \in I^b_a \setminus K_a \). Then

\[ \begin{align*}
\|V(u)\| & \leq \frac{1}{g(u)}, \\
\langle DI(u), V(u) \rangle & \geq 1, \quad \forall u \in I^b_a \setminus K_a.
\end{align*} \]

Consider the flow \( \xi \) on \( I^b_a \setminus K_a \)

\[ \begin{cases} 
\frac{d}{ds} \xi(s, u) = -V(\xi(s, u)), \\
\xi(0, u) = u \in I^b_a \setminus K_a.
\end{cases} \]

The local existence and continuous dependence on the initial date of the flow \( \xi \) were proved in [24].
Claim 5. Let \( \tilde{s} = \tilde{s}(u) \) be the maximal existing time of \( \xi \). Then \( \lim_{s \to \tilde{s}} \xi(s) = v \) exists. Moreover \( I(v) = a \).

Proof. We have \( \tilde{s} \leq b - a \) due to the fact
\[
\frac{d}{ds} I(\xi(s)) = \left( DI(\xi(s)), \frac{d}{ds} \xi(s) \right) = -\left( DI(\xi(s)), V(\xi(s)) \right) \leq -1.
\]
The above equality is clear, if \( I \) is a \( C^1 \) functional. In [24] it was proved to be true for \( G \)-differentiable functional.

Now two cases may occur.

Case I. There exist \( n_0 \geq n_2 \) and \( s_0 < \tilde{s} \) such that for all \( s > s_0, \xi(s) \notin A_{n_0} \).

Case II. There exists a sequence \( s_n \to \tilde{s} \) such that \( \xi(s_n) \in A_n \).

Case I. By Lemma 2 we have
\[
\left\| \frac{d}{ds} \xi(s) \right\| = \left\| V(\xi(s)) \right\| \leq \frac{1}{g(\xi(s))} \leq \frac{1}{\alpha_{n_0+1}}, \quad s > s_0
\]
hence \( \lim_{s \to \tilde{s}} \xi(s) \) exists.

Case II. We have a sequence \( s_n \) such that \( s_n \) increases and converges to \( \tilde{s} \) and \( u_n := \xi(s_n) \in A_n \).

By the definition of \( A_n \), there exists \( w_n \in D_n \) and \( \|w_n - u_n\| \leq \frac{1}{n} \). Passing to a subsequence, we can assume that \( w_n = \sum_{i=1}^l k_n^i \ast v_i \), i.e.,
\[
\left\| u_n - \sum_{i=1}^l k_n^i \ast v_i \right\| \to 0, \quad \text{as} \quad n \to \infty,
\]
where \( l \leq l(b), v_1, v_2, \ldots, v_l \in K^a / \mathbb{Z}^N, \{k_n^1\}, \{k_n^2\}, \ldots, \{k_n^l\} \subset \mathbb{Z}^N \) and \( |k_n^j - k_n^j| \to \infty \) for \( 1 \leq i < j \leq N \) as \( n \to \infty \).

We show that by passing to a subsequence, \( u_n \to v \in K \). Three subcases may occur:

(1) \( l = 1 \) and \( \{k_n^1\} \) is bounded;
(2) \( l = 1 \) and \( \{k_n^1\} \) is unbounded;
(3) \( l \geq 2 \).

In the case (1), passing to a subsequence, we have \( k_n^1 = k, \|u_n - k \ast v_1\| \to 0 \), and \( v = k \ast v_1 \in K \). In cases (2) and (3), \( w_n \neq w_m \). By Claim 2, \( \|w_n - w_m\| \geq \delta \) for \( n, m \geq n_1 \), hence \( \|u_n - u_m\| \geq \frac{1}{2} \delta \).

By Lemma 2.6, there is \( \alpha > 0 \) such that \( \|u\| \geq \alpha \) for all nonzero critical points of \( I \). Let \( r \leq \frac{1}{2} \alpha \). We claim that there exist \( \mu > 0, n_0 \in N \) such that
\[
g(\xi(s)) \geq \mu, \quad \text{for} \quad \xi(s) \in B_r(u_n) \setminus B_{\frac{1}{8} r}(u_n), \quad n \geq n_0.
\]

Otherwise, there is a sequence \( \{\tilde{s}_n\} \) such that for \( \tilde{u}_n = \xi(\tilde{s}_n) \)
\[
\frac{r}{8} \leq \|u_n - \tilde{u}_n\| \leq r, \quad g(\tilde{u}_n) \to 0 \quad \text{as} \quad n \to \infty.
\]

\[
\frac{r}{8} \leq \|u_n - \tilde{u}_n\| \leq r, \quad g(\tilde{u}_n) \to 0 \quad \text{as} \quad n \to \infty.
\]

\[
\frac{r}{8} \leq \|u_n - \tilde{u}_n\| \leq r, \quad g(\tilde{u}_n) \to 0 \quad \text{as} \quad n \to \infty.
\]
By the definition of $g$, if $g(u) < \alpha_n + 1$, then $u \in A_n$. Passing to a subsequence we can assume that there exists $\{\tilde{v}_i\}_1^{\tilde{I}} \subset K \setminus \{0\}$, $h_n^i \in \mathbb{Z}^N$, $1 \leq i \leq \tilde{I}$ such that $|h_n^i - h_n^j| \to \infty$ as $n \to \infty$, $i \neq j$:

$$
\left\| \tilde{u}_n - \sum_{i=1}^{\tilde{I}} h_n^i \ast \tilde{v}_i \right\| \to 0, \quad \text{as } n \to \infty. \quad (3.22)
$$

By (3.19), (3.21) and (3.22), we have $\epsilon_n \to 0$ such that

$$
\frac{r}{8} - \epsilon_n \leq \left\| \sum_{i=1}^{\tilde{I}} k_n^i \ast v_i - \sum_{i=1}^{\tilde{I}} h_n^i \ast \tilde{v}_i \right\| \leq r + \epsilon_n.
$$

If $|h_n^i - k_n^i| \to \infty$ for all $i$, then $\|v_1\| \leq r$, which is impossible. By rearrangement, passing to a subsequence, $k_n^i - h_n^1 = k^1$ is constant, and $\|v_1 - k^1 \ast \tilde{v}_1\| \leq r$. Since $v_1$ is isolated, we have $v_1 = k^1 \ast \tilde{v}_1$ provided $r < r_0$, where $r_0$ satisfies

$$
K \cap B_{r_0}(v_i) = \{v_i\}, \quad i = 1, 2, \ldots, l.
$$

Repeating this process, we have that $l = \tilde{I}$ and

$$
\left\| \sum_{i=1}^{\tilde{I}} k_n^i \ast v_i - \sum_{i=1}^{\tilde{I}} h_n^i \ast \tilde{v}_i \right\| = 0
$$

which is a contradiction. Thus (3.20) is proved.

Recall that we have $\|u_n - u_m\| \geq \frac{\delta}{2} > 0$, with $u_n = \xi(s_n)$. If we take $r$ less than $\frac{1}{6} \delta$ in (3.20), then we can find two sequences $\{s_n^1\}$, $\{s_n^2\}$ such that

$$
s_n^1 < s_n^2 < s_n^1, \quad \left\| \xi(s_n') - u_n \right\| = \frac{r}{8}, \quad \left\| \xi(s_n') - u_n \right\| = r, \quad \frac{r}{8} < \left\| \xi(s) - u_n \right\| < r, \quad \text{if } s_n^1 < s < s_n^2.
$$

Then we have

$$
\frac{7}{8} r \leq \left\| \xi(s_n') - \xi(s_n'') \right\| \leq \int_{s_n'}^{s_n''} \left\| \frac{d}{ds} \xi(s) \right\| ds \leq \int_{s_n'}^{s_n''} \frac{1}{g(\xi(s))} ds
$$

$$
\leq \frac{1}{\mu} (s_n'' - s_n') \to 0, \quad \text{as } n \to \infty.
$$

This contradiction shows that in (3.19) $l = 1$, $\{k_n^1\}$ is bounded and $u_n \to v = k \ast v_1 \in K$. 

Since \( c = \lim_{s \to s^*} I(\xi(s)) \geq a \) and \( K \cap (I^b \setminus I^a) = \emptyset \), it follows that \( v \in K_a = \{ u \in K \mid I(u) = a \} \). It remains to show \( \lim_{s \to s^*} \xi(s) = v \). Similar to the above proof, we conclude that there exists \( \bar{\mu} = \bar{\mu}(r) > 0 \) such that \( g(u) \geq \bar{\mu}, \) for \( u \in B_{r}(v) \setminus B_{\bar{r}}(v) \), for all \( 0 < r < \bar{r}, \bar{r} = \inf \{ \| \bar{v} - v \| \mid \bar{v} \neq v, \bar{v} \in K \} \). If \( \xi(s) \) does not converge to \( v \), we find \( 0 < r < \bar{r} \) and two sequences \( \{ s'_n \}, \{ s''_n \} \), such that \( s'_n < s''_n < s, s'_n \to s, s''_n \to s \) and

\[
\| \xi(s'_n) - v \| = \frac{r}{8}, \quad \| \xi(s''_n) - v \| = r,
\]

\[
\frac{r}{8} < \| \xi(s) - v \| < r, \quad \text{if} \ s'_n < s < s''_n.
\]

Again, this is impossible and Claim 5 is proved.

Finally we finish the proof of the Deformation Theorem. Let \( \xi(s,u) \) be the flow \( \xi(s) \) with initial data \( u, 0 < s < s_a \), where \( s_a \) is defined by \( I(\xi(s_a - 0, u)) = a \). We define

\[
\eta(t,u) = \begin{cases} 
\xi(ts_a,u), & (t,u) \in [0,1] \times (I^b \setminus I^a), \\
\lim_{t \to 1-} \xi(ts_a,u), & (t,u) \in [1] \times (I^b \setminus I^a), \\
u, & (t,u) \in [0,1] \times I^a.
\end{cases}
\]

This is the deformation retract we need. The continuity of \( \eta \) is verified in the same way as in [12,24], we omit it. Theorem 3.3 is proved. \( \square \)

4. Solutions of mountain pass type

In this section we prove Theorem A. We define the mountain pass value \( C_J \) by

\[
C_J = \inf_{g \in \Gamma_J} \sup_{s \in [0,1]} J(g(s)) \tag{4.1}
\]

where \( \Gamma_J = \{ g \in C([0,1], Y) \mid g(0) = \theta, J(g(1)) < 0 \} \).

We need to define similar values for the functional \( I \). To emphasize the dependence of \( I \) on the parameter \( M \), we denote \( I \) by \( I_M \). We define

\[
C_M = \inf_{g \in \Gamma_M} \sup_{s \in [0,1]} I_M(g(s)) \tag{4.2}
\]

where \( \Gamma_M = \{ g \in C([0,1], H^1(\mathbb{R}^N)) \mid g(0) = \theta, I_M(g(1)) < 0 \} \).

It is easy to show that \( \lim_{M \to \infty} I_M(u) = J(u) \), for \( u \in Y \). Similarly for \( g \in \Gamma_J \)

\[
\lim_{M \to \infty} \sup_{s \in [0,1]} I_M(g(s)) = \sup_{s \in [0,1]} J(g(s)).
\]

Therefore we have

\[
\limsup_{M \to \infty} C_M \leq C_J. \tag{4.3}
\]
By (4.3) and Lemma 2.4, we choose $M_0$ such that for $M \geq M_0$

(i) $C_M \leq C_J + 1,$

(ii) if $|\nabla I_M(u)| = 0,$ $I_M(u) \leq C_J + 1,$ then $\|u\|_{L^\infty} \leq M.$ (4.4)

In the following we always assume $M \gg M_0,$ thus if $C_M$ is a critical value of $I_M,$ it is also one of $J.$

**Theorem 4.1.** Assume (B), (F), (V), (T). Assume that $I_M$ satisfies $(Z^*)$ in $(C_M, C_M + \epsilon)$ for some $\epsilon > 0.$ Then $C_M$ is a critical value of $I_M$ (hence of $J$). And there exists a critical point $u_0$ of $I_M$ such that $I_M(u) = C_M$ and $C_1(u_0, I_M) \neq 0.$

**Proof.** By $(b_2), (f_1), (f_2)$ and the definition of $I_M,$ there exists $\rho_0 > 0$ such that

$$I_M(u) \geq c \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \, dx$$

provided $\|u\|_{H^1(\mathbb{R}^N)} \leq \rho_0.$ Let $g \in \Gamma_M.$ Then $\sup_{0 \leq s \leq 1} I_M(g(s)) \geq c\rho_0^2.$ Therefore

$$C_M = \inf_{g \in I_M} \sup_{s \in [0, 1]} I_M(g(s)) \geq c\rho_0^2.$$ (4.6)

Since $I_M$ satisfies the condition $Z^*$ in $(C_M, C_M + \epsilon), I_M^{C_M}$ is a strong deformation retract of $I_M^{C_M + \epsilon}.$ Moreover the family $\Gamma_M$ is invariant under this deformation, and $C_M$ is a critical value of $I_M.$ By our choice of $M,$ it is also a critical value of $J.$

Essentially, the proof of the existence of critical point $u_0$ with $I_M(u_0) = C_M,$ $C_1(u_0, I_M) \neq 0$ is the same as in [12], we reproduce it here for completeness.

By the deformation retract, there is a curve $g \in \Gamma_M$ such that

$$\sup_{s \in [0, 1]} I_M(g(s)) \leq C_M.$$ 

On one hand, by the definition of $C_M, g([0, 1]) \cap K_{C_M} \neq \emptyset,$ otherwise by a further deformation of the curve $g,$ we would have a new curve $\tilde{g} \in \Gamma_M$ such that $\sup_{s \in [0, 1]} I_M(\tilde{g}(s)) < C_M.$ On the other hand, since $I_M$ satisfies the condition $Z^*$ in $(C_M, C_M + \epsilon), g([0, 1]) \cap K_{C_M}$ consists of a finite number of points, say $v_1, v_2, \ldots, v_l.$ After removing some circles emanating from $v_1, v_2, \ldots, v_l,$ we may assume that the curve $g$ passes $v_1, v_2, \ldots, v_k,$ $k \leq l,$ successively. That is, there are $0 < s_1 < s_2 < \cdots < s_k < 1$ such that $g(s_i) = v_i, 1 \leq i \leq k$ and $I(g(s)) < \epsilon$ if $s \notin \{s_1, s_2, \ldots, s_k\}.$ Choose $t_0 = 0, t_i \in (s_i, s_{i+1}), i = 1, 2, \ldots, k - 1$ and $t_k = 1.$ Define

$$C_i = \inf_{\gamma \in \Gamma_i} \sup_{s \in [0, 1]} I_M(\gamma(s)),$$

$$\Gamma_i = \{\gamma \in C([0, 1], H^1(\mathbb{R}^N)) \mid \gamma(0) = g(t_{i-1}), \gamma(1) = g(t_i)\}, \quad 1 \leq i \leq k.$$ 

Then there is an $i_0, 1 \leq i_0 \leq k,$ with $C_{i_0} = C_M.$ Let $U$ be an open neighborhood of $v_{i_0}$ such that $v_{i_0}$ is the unique critical point of $I_M$ in $U.$
We consider a piece of $g$: $g_0 = g|_{[s', s'']}$, where $t_{i_0 - 1} < s' < s_i < s'' < t_{i_0}$, such that $g([s', s'']) \subset U$. Then $g(s')$ and $g(s'')$ are connected in $(I_M)^{CM} \cap U$ by the piece of $g_0$, but cannot be connected in $(I_M)^{CM} \cap U \setminus \{v_{i_0}\}$. Otherwise by moving the curve connecting $g(s')$ and $g(s'')$ in $(I_M)^{CM} \cap U \setminus \{v_{i_0}\}$ a little along with the flow generated by the pseudo vector field we obtain a curve connecting $g(s')$ and $g(s'')$ and entirely contained in $I^{-1}_M(\infty, CM)$. Consequently, we obtain a curve connecting $g(t_{i_0 - 1})$ and $g(t_{i_0})$ and entirely contained in $I^{-1}_M(\infty, CM)$, which is a contradiction with the fact $C_{i_0} = CM$. Therefore we have

$$C_1(I_M, v_{i_0}) = H_1((I_M)^{CM} \cap U, (I_M)^{CM} \cap U \setminus \{v_{i_0}\}) \neq 0. \quad \Box$$

**Proof of Theorem A.** Since $J$ satisfies the condition $Z^*$ in $(C_J, C_J + \epsilon)$ for some $\epsilon > 0$, by (4.4), $I_M$ has no critical points in $(C_J, C_J + \epsilon)$, hence $C_M \leq C_J$ and $I_M$ satisfies the condition $Z^*$ in $(C_M, C_M + \epsilon')$ for some $\epsilon' > 0$. Now Theorem A follows from Theorem 4.1 and the choice of $M$ and (4.4). \(\square\)

5. Local property of the modified functional near a critical point

In this section, we study the local behavior of the functional $I_M$ near a critical point. As the statements and arguments are independent of $M$ we drop the subscript $M$ for simplicity. We reduce $I$ to a functional defined on a finite dimensional space and prove a shift theorem for the critical groups of an isolated critical point.

Let $u_0$ be a critical point of $I$. We know that $u_0 \in W^{1, \infty}(\mathbb{R}^N)$ and $u_0(x) \to 0$, $Du_0(x) \to 0$ as $|x| \to \infty$ by Lemmas 2.4 and 2.5. Even if the functional $I$ is only $G$-differentiable, we may still define the Hessian operator at the critical point $u_0$ as follows:

$$\begin{align*}
(L \psi, \varphi) &= \int_{\mathbb{R}^N} \sum_{i=1}^{N} a_{ij}(x, u_0) D_i \psi D_j \varphi \, dx + \int_{\mathbb{R}^N} V(x) \psi \varphi \, dx \\
&\quad + \int_{\mathbb{R}^N} \sum_{i=1}^{N} D_z a_{ij}(x, u_0) D_i u_0 D_j (\psi \varphi) \, dx \\
&\quad + \int_{\mathbb{R}^N} \left( \frac{1}{2} \sum_{i=1}^{N} D_z^2 a_{ij}(x, u_0) D_i u_0 D_j u_0 - D_z g(x, u_0) \right) \psi \varphi \, dx,
\end{align*}
$$

(5.1)

for $\psi, \varphi \in H^1(\mathbb{R}^N)$. By the boundedness of $u_0$ and $Du_0$, $(L \psi, \varphi)$ is well defined. Let

$$(L \psi, \varphi) = (L_0 \psi, \varphi) + (K \psi, \varphi),$$

(5.2)

where

$$(L_0 \psi, \varphi) = \int_{\mathbb{R}^N} \sum_{i=1}^{N} a_{ij}(x, u_0) D_i \psi D_j \varphi \, dx + \int_{\mathbb{R}^N} V(x) \psi \varphi \, dx,$$

and
\[(K\psi, \varphi) = \int_{\mathbb{R}^N} \sum_{i,j=1}^{N} D_s a_{ij}(x, u_0) D_i u_0 D_j (\psi \varphi) \, dx \]
\[+ \int_{\mathbb{R}^N} \left( \frac{1}{2} \sum_{i,j=1}^{N} D_s^2 a_{ij}(x, u_0) D_i u_0 D_j u_0 - D_s g(x, u_0) \right) \psi \varphi \, dx. \]

Then \(L_0, K\) are bounded linear operators from \(H^1(\mathbb{R}^N)\) to itself. Moreover \(L_0\) is positively definite, hence invertible, and \(K\) is compact. To show the compactness of \(K\), we need only to prove that for a bounded sequence \(\{\psi_n\} \subset H^1(\mathbb{R}^N)\), passing to a subsequence, \((K \psi_n, \varphi)\) converges uniformly in \(\varphi \in B\), where \(B\) is the unit ball of \(H^1(\mathbb{R}^N)\). Let \(\psi_n \rightharpoonup \psi\) in \(H^1(\mathbb{R}^N)\),

\[(K \psi_n, \varphi) = \int_{\mathbb{R}^N} \sum_{i,j=1}^{N} D_s a_{ij}(x, u_0) D_i u_0 D_j \psi_n \varphi \, dx + \int_{\mathbb{R}^N} D_s a_{ij}(x, u_0) D_i u_0 \psi_n D_j \varphi \, dx \]
\[+ \int_{\mathbb{R}^N} \left( \frac{1}{2} \sum_{i,j=1}^{N} D_s^2 a_{ij}(x, u_0) D_i u_0 D_j u_0 - D_s g(x, u_0) \right) \psi_n \varphi \, dx. \tag{5.3} \]

We consider the first term in (5.3)

\[\int_{\mathbb{R}^N} \sum_{i,j=1}^{N} D_s a_{ij}(x, u_0) D_i u_0 D_j \psi_n \varphi \, dx = \int_{B_R} \sum_{i,j=1}^{N} D_s a_{ij}(x, u_0) D_i u_0 D_j \psi_n \varphi \, dx \]
\[+ \int_{\mathbb{R}^N \setminus B_R} \sum_{i,j=1}^{N} D_s a_{ij}(x, u_0) D_i u_0 D_j \psi_n \varphi \, dx \]

where \(R\) is a large number. The integral over \(\mathbb{R}^N \setminus B_R\) is small uniformly in \(n\) and \(\varphi \in B\) by the decay of \(D u_0\). The integral over \(B_R\) converges to 
\[\int_{B_R} \sum_{i,j=1}^{N} D_s a_{ij}(x, u_0) D_i u_0 D_j \psi \varphi \, dx \]
uniformly in \(\varphi \in B\), since the ball \(B\) is compact in \(L^2(B_R)\). The other terms in (5.3) can be verified in a similar way. Therefore \((K \psi_n, \varphi)\) converges to \((K \psi, \varphi)\) as \(n \to \infty\) uniformly in \(\varphi \in B\), and \(K\) is compact.

Let \(J : H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)\) be defined by

\[(J \psi, \varphi) = \int_{\mathbb{R}^N} \psi \varphi \, dx, \quad \text{for } \psi, \varphi \in H^1(\mathbb{R}^N). \]

Consider the spectrum of \(L\) with respect to \(J\), that is for \(\lambda \in \mathbb{R}\), we consider \(L - \lambda J\). We say that \(\lambda\) is an eigenvalue, if there is an eigenfunction \(v \neq 0, v \in H^1(\mathbb{R}^N)\), such that

\[(Lv, \varphi) = \lambda (Jv, \varphi), \quad \text{for all } \varphi \in H^1(\mathbb{R}^N). \]

That is,
\[
\int \sum_{ij=1}^{N} a_{ij}(x, u_0) D_i v D_j \varphi \, dx + \int V(x) v \varphi \, dx + \int D_s a_{ij}(x, u_0) D_i u_0 D_j (v \varphi) \, dx \\
+ \int \left( \frac{1}{2} \sum_{ij=1}^{N} D_s^2 a_{ij}(x, u_0) D_i u_0 D_j u_0 - D_s g(x, u_0) \right) v \varphi \, dx = \lambda \int v \varphi \, dx, \quad (5.4)
\]
for all \( \varphi \in H^1(\mathbb{R}^N) \). By assumption (V) there exists \( \lambda_0 > 0 \) such that for \( \lambda < \lambda_0 \), \( L_0 - \lambda J \) is positively definite. Then for such a \( \lambda \) either the operator \( L - \lambda J = (L_0 - \lambda J) + K \) is invertible, or \( \lambda \) is an eigenvalue of finite multiplicity. In particular any nonpositive eigenvalue of \( L \) is of finite multiplicity. In a similar and easier way as in Lemma 2.4, we can prove that the eigenfunction \( v \), corresponding to a nonpositive eigenvalue, belongs to \( W^{1, \infty}(\mathbb{R}^N) \) and \( v(x) \to 0, \ Dv(x) \to 0 \) as \( |x| \to +\infty \).

Let \( V \) be the finite dimensional subspace of \( H^1(\mathbb{R}^N) \) spanned by eigenfunctions of \( L \) corresponding to nonpositive eigenvalues. Decompose \( H^1(\mathbb{R}^N) \) as a direct sum \( H^1(\mathbb{R}^N) = V \oplus W \), where \( V \) and \( W \) are orthogonal in the sense that

\[
(Lv, w) = 0, \quad (Jv, w) = 0, \quad \text{for } v \in V, \ w \in W. \quad (5.5)
\]
We see that when restricted on \( W \), \( L \) is positively definite

\[
(Lw, w) \geq \gamma \|w\|^2
\]
for some \( \gamma > 0 \). If \( I \) were a \( C^2 \) functional, there would be an expansion as

\[
I(u_0 + v + w) = I(u_0) + \frac{1}{2} (Lv, v) + \frac{1}{2} (Lw, w) + o(\|v\|^2 + \|w\|^2).
\]
Such an expansion does not hold here as our functional is merely continuous. However we have the following lemma.

**Lemma 5.1.** Let \( u_0 \) be a critical point of \( I \), \( v \in V, \ w \in W, \ u = u_0 + v + w \). Then as \( \|v\| + \|w\| \to 0 \)

\[
I(u) = I(u_0) + \frac{1}{2} (Lv, v) + \frac{1}{2} (\tilde{L}(w)w, w) + o(\|v\|^2 + \|w\|^2), \quad (5.6)
\]
where the operator \( \tilde{L}(w) \) is defined as

\[
(\tilde{L}(w)\psi, \varphi) = (L\psi, \varphi) + \int \sum_{ij=1}^{N} a_{ij}(x, u_0 + w) - a_{ij}(x, u_0)) D_i \psi D_j \varphi \, dx. \quad (5.7)
\]

**Proof.** We have
\[ I(u) = I(u_0 + w + v) \]
\[ = I(u_0) + \langle DI(u_0), v + w \rangle + \frac{1}{2}(Lv, v) \]
\[ + \frac{1}{2}(Lw, w) + \frac{1}{2}(L(w)w, w) + R, \]

where
\[
R = \int_{\mathbb{R}^N} \sum_{ij=1}^N h_{ij}^{(2)} D_i u_0 D_j u_0 \, dx + \int_{\mathbb{R}^N} \sum_{ij=1}^N h_{ij}^{(1)} D_i u_0 D_j (v + w) \, dx
\]
\[ + \int_{\mathbb{R}^N} \sum_{ij=1}^N h_{ij}^{(0)} (D_i v D_j w + \frac{1}{2} D_i v D_j v) \, dx \]
\[ + \frac{1}{2} \int_{\mathbb{R}^N} \sum_{ij=1}^N \tilde{h}_{ij}^{(0)} D_i w D_j w \, dx + \int_{\mathbb{R}^N} H \, dx, \tag{5.8} \]

and
\[
h_{ij}^{(2)} = a_{ij}(x, u_0 + v + w) - a_{ij}(x, u_0) - D_s a_{ij}(x, u_0)(v + w) - \frac{1}{2} D^2_s a_{ij}(x, u_0)(v + w)^2,
\]
\[
h_{ij}^{(1)} = a_{ij}(x, u_0 + v + w) - a_{ij}(x, u_0) - D_s a_{ij}(x, u_0)(v + w),
\]
\[
h_{ij}^{(0)} = a_{ij}(x, u_0 + v + w) - a_{ij}(x, u_0),
\]
\[
\tilde{h}_{ij}^{(0)} = a_{ij}(x, u_0 + v + w) - a_{ij}(x, u_0 + w),
\]
\[
H = G(x, u_0 + v + w) - G(x, u_0) - g(x, u_0)(v + w) - \frac{1}{2} D_s g(x, u_0)(v + w)^2.
\]

We estimate the remainder term by term. For example
\[
h_{ij}^{(2)} = a_{ij}(x, u_0 + v + w) - a_{ij}(x, u_0) - D_s a_{ij}(x, u_0)(v + w) - \frac{1}{2} D^2_s a_{ij}(x, u_0)(v + w)^2
\]
\[ = \int_0^1 (1 - s)(D^2_s a_{ij}(x, u_0 + s(v + w)) - D^2_s a_{ij}(x, u_0)) \, ds(v + w)^2
\]
\[ := k(x)|v + w|^2. \]

Taking \( R \) large we have
\[
\int_{\mathbb{R}^N} h_{ij}^{(2)} D_i u_0 D_j u_0 \, dx = \int_{B_R} h_{ij}^{(2)} D_i u_0 D_j u_0 \, dx + \int_{\mathbb{R}^N \setminus B_R} h_{ij}^{(2)} D_i u_0 D_j u_0 \, dx,
\]
and

\[
\left| \int_{\mathbb{R}^N \setminus B_R} h_{ij}^{(2)} D_i u_0 D_j u_0 \, dx \right| \leq o(R(1)) \int_{\mathbb{R}^N \setminus B_R} |h_{ij}^{(2)}| \, dx \leq o(R(1)) \| v + w \|_{L^2(\mathbb{R}^N)}^2,
\]

since \( |Du_0(x)| \to 0 \) as \( |x| \to \infty \),

\[
\left| \int_{B_R} h_{ij}^{(2)} D_i u_0 D_j u_0 \, dx \right| \leq C \int_{B_R} |h_{ij}^{(2)}| \, dx \leq C \left( \int_{B_R} |k(x)|_{N \over 2}^N \right)^{N \over 2N} \left( \int_{B_R} |v + w|_{N \over 2}^{2N} \right) = o(1)(\| v \| + \| w \|)^2.
\]

Here we used \( \int_{B_R} |k(x)|_{N \over 2}^N \, dx \to 0 \) as \( \| v \| + \| w \| \to 0 \), which follows from Lebesgue dominated convergence theorem. So \( \int_{\mathbb{R}^N} h_{ij}^{(2)} D_i u_0 D_j u_0 \, dx = o(1)(\| v \| + \| w \|)^2 \).

The other terms of \( R \) can be estimated in a similarly way. Finally we have \( \langle DI(u_0), v + w \rangle = 0 \). In fact, \( u_0 \) is a critical point of \( I \), \( \langle DI(u_0), \phi \rangle = 0 \) for \( \phi \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \).

Now we consider the quadratic form \( \langle \tilde{L}(w) \psi, \psi \rangle \).

**Lemma 5.2.** There exist \( \gamma_0, \mu > 0 \) such that if \( \| w \|_{H^1(\mathbb{R}^N)} \leq \gamma_0 \) then

\[
\langle \tilde{L}(w) \psi, \psi \rangle \geq \mu \| \psi \|_{H^1(\mathbb{R}^N)}^2, \quad \text{for } \psi \in W.
\]

**Proof.** We have

\[
\langle \tilde{L}(w) \psi, \psi \rangle \geq c_1 \| D\psi \|_{L^2(\mathbb{R}^N)}^2 - c_2 \| \psi \|_{L^2(\mathbb{R}^N)}^2,
\]

for some \( c_1, c_2 > 0 \). If we can prove that

\[
\langle \tilde{L}(w) \psi, \psi \rangle \geq \mu_0 \| \psi \|_{L^2(\mathbb{R}^N)}^2, \quad \psi \in W \quad (5.9)
\]

for some \( \mu_0 > 0 \), then taking \( c_3 > 0 \) satisfying \( c_3 \mu_0 - c_2 > 0 \), we have

\[
(c_3 + 1) \langle \tilde{L}(w) \psi, \psi \rangle \geq c_1 \| D\psi \|_{L^2(\mathbb{R}^N)}^2 + (c_3 \mu_0 - c_2) \| \psi \|_{L^2(\mathbb{R}^N)}^2 \geq \mu \| \psi \|_{H^1(\mathbb{R}^N)}^2
\]

for \( \mu = \min(c_1, c_3 \mu_0 - c_2) \).

We prove (5.9) by an indirect argument. If (5.9) is not true, there exist sequences \( \{w_n\} \) and \( \{\psi_n\} \) such that \( \| w_n \|_{H^1(\mathbb{R}^N)} \to 0 \), \( \{\psi_n\} \subset W \), \( \| \psi_n \|_{L^2(\mathbb{R}^N)} = 1 \) and

\[
\lim_{n \to \infty} \langle \tilde{L}(w_n) \psi_n, \psi_n \rangle = 0. \quad (5.10)
\]
Passing to subsequence, we may assume that \( w_n \to 0 \), a.e., \( \psi_n \to \psi \) in \( H^1(\mathbb{R}^N) \), \( \psi_n \to \psi \) in \( L^p_{\text{loc}}(\mathbb{R}^N), 2 \leq p < \frac{2N}{N-2} \). Let \( R > 0 \). By (5.3) and the lower semicontinuity, passing to a subsequence,

\[
\lim_{n \to \infty} \left\{ \int_{B_R} \sum_{i,j=1}^N a_{ij}(x,u_0 + w_n)D_i\psi_n D_j\psi_n \, dx + \int_{B_R} V(x)\psi_n^2 \, dx + \int_{B_R} \sum_{i,j=1}^N a_{ij}(x,u_0)D_iu_0 D_j\psi_n \right. \\
+ \int_{B_R} \left( \frac{1}{2} \sum_{i,j=1}^N D_x^2 a_{ij}(x,u_0)D_iu_0 D_ju_0 - D_s g(x,u_0) \right) \psi_n^2 \, dx \right. \\
\geq \int_{B_R} \sum_{i,j=1}^N a_{ij}(x,u_0)D_i\psi D_j\psi \, dx + \int_{B_R} V(x)\psi^2 \, dx + \int_{B_R} \sum_{i,j=1}^N a_{ij}(x,u_0)D_iu_0 D_j\psi^2 \\
+ \int_{B_R} \left( \frac{1}{2} \sum_{i,j=1}^N D_x^2 a_{ij}(x,u_0)D_iu_0 D_ju_0 - D_s g(x,u_0) \right) \psi^2 \, dx \\
= (\mathcal{L}\psi,\psi) + o_R(1) \geq \gamma \|\psi\|_{L^2(\mathbb{R}^N)}^2 + o_R(1) \\
\geq \gamma \lim_{n \to \infty} \int_{B_R} \psi_n^2 \, dx + o_R(1). \tag{5.11}
\]

Since \( u_0(x), Du_0(x) \to 0 \) as \( |x| \to \infty \) and \( D_s g(x,0) = 0 \), for some \( \mu_0 \),

\[
\lim_{n \to \infty} \left\{ \int_{\mathbb{R}^N \setminus B_R} \sum_{i,j=1}^N a_{ij}(x,u_0 + w_n)D_i\psi_n D_j\psi_n \, dx + \int_{\mathbb{R}^N \setminus B_R} V(x)\psi_n^2 \, dx \\
+ \int_{\mathbb{R}^N \setminus B_R} \sum_{i,j=1}^N a_{ij}(x,u_0)D_iu_0 D_j\psi_n \right. \\
+ \int_{\mathbb{R}^N \setminus B_R} \left( \frac{1}{2} \sum_{i,j=1}^N D_x^2 a_{ij}(x,u_0)D_iu_0 D_ju_0 - D_s g(x,u_0) \right) \psi_n^2 \, dx \right. \\
\geq \lim_{n \to \infty} \left\{ 2\mu_0 \int_{\mathbb{R}^N \setminus B_R} (|D\psi_n|^2 + \psi_n^2) \, dx + o_R(1) \int_{\mathbb{R}^N \setminus B_R} (|\psi_n| |Du_0| + \psi_n^2) \, dx \right\} \\
\geq \mu_0 \lim_{n \to \infty} \int_{\mathbb{R}^N \setminus B_R} \psi_n^2 \, dx. \tag{5.12}
\]

It follows from (5.10), (5.11) and (5.12) that
\[ 0 = \lim_{n \to \infty} \left( \tilde{L}(w_n) \psi_n, \psi_n \right) \]
\[ \geq \lim_{n \to \infty} \gamma \int_{B_R} \psi_n^2 \, dx + o_R(1) + \lim_{n \to \infty} \mu_0 \int_{\mathbb{R}^N \setminus B_R} \psi_n^2 \, dx \]
\[ \geq m \lim_{n \to \infty} \int_{\mathbb{R}^N} \psi_n^2 \, dx + o_R(1) \]
\[ = m + o_R(1), \]
where \( m = \min\{\gamma, \mu_0\} \). This is a contradiction. \( \Box \)

Given \( v \in V \), we consider the functional \( I(u_0 + v + w) \), \( w \in W \) and look for \( w \in W \) such that the partial derivative with respect to \( w \) of \( I \) at \( u = u_0 + v + w \) is zero, that is,
\[ \langle DI(u), z \rangle = 0 \quad \text{for} \quad z \in W \cap L^\infty(\mathbb{R}^N), \]

or
\[ \int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij}(x,u) D_i u D_j z \, dx + \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i,j=1}^N D_i a_{ij}(x,u) D_i u D_j z \, dx + \int_{\mathbb{R}^N} V(x) u z \, dx \]
\[ - \int_{\mathbb{R}^N} g(x,u) z \, dx = 0, \quad \text{for} \quad z \in W \cap L^\infty(\mathbb{R}^N). \] (5.13)

We have the following theorem

**Theorem 5.3.** Let \( B_\delta = \{ u \in H^1(\mathbb{R}^N) \mid \|u\| \leq \delta \} \). There exist \( \delta_1, \delta_2 > 0 \) such that for \( v \in V \cap B_{\delta_1} \) the following statements hold.

(a) There exists a unique \( w = \phi(v) \in W \cap B_{\delta_2} \) satisfying (5.13). Moreover \( w \in W^{1,\infty}(\mathbb{R}^N) \).

(b) The mapping \( \phi : V \cap B_{\delta_1} \mapsto W \cap B_{\delta_2} \) is Lipschitz continuous both in \( L^\infty(\mathbb{R}^N) \) and in \( H^1(\mathbb{R}^N) \), for some \( c > 0 \) and \( v_1, v_2 \in V \cap B_{\delta_1} \)
\[ \| \phi(v_2) - \phi(v_1) \|_{L^\infty(\mathbb{R}^N)} + \| \phi(v_2) - \phi(v_1) \|_{H^1(\mathbb{R}^N)} \leq c \| v_2 - v_1 \|_{H^1(\mathbb{R}^N)}. \] (5.14)

(c) Moreover, \( h(v) = I(u_0 + v + \phi(v)), v \in V \cap B_{\delta_1} \), is a \( C^1 \) functional and
\[ \langle Dh(v), \varphi \rangle = \langle DI(u_0 + v + \phi(v)), \varphi \rangle, \quad \forall \varphi \in V. \] (5.15)

In particular, \( v \) is a critical point of \( h \) if and only if \( u = u_0 + v + \phi(v) \) is a critical point of \( I \).

Theorem 5.3 resembles the classical Lyapunov–Schmidt reduction for smooth functionals in studying bifurcation problems via the implicit function theorem. Here we used the variational method. See [25] for applications of such a method to bifurcation problems. The proof of this theorem is lengthy and will be broken into several lemmas. The existence of solution of (5.13) is proved in Lemma 5.4, while the uniqueness and the continuity (5.14) is proved in Lemma 5.5.
Lemma 5.4. Given \( v \in V \cap B_{\delta_1} \), let \( g_v(w) = I(u_0 + v + w), w \in W \cap B_{\delta_2} \). Then there exist \( \delta_1, \delta_2 > 0 \) such that \( g_v \) assumes its minimum at a point \( w \in W, \|w\| < \delta_2 \). Moreover \( u = u_0 + v + w \) solves (5.13), and \( u \in W^{1,\infty}(\mathbb{R}^N) \) and for some \( C > 0 \)

\[
\|u\|_{W^{1,\infty}(\mathbb{R}^N)} \leq C \|u\|_{H^1(\mathbb{R}^N)}.
\] (5.16)

**Proof.** By Lemma 5.1, Lemma 5.2, there exist \( c_1, c_2, \epsilon, \delta_1, \delta_2 > 0 \) such that

\[
g_v(w) = I(u_0 + v + w) \geq I(u_0) - c_1 \|v\|_{H^1(\mathbb{R}^N)}^2 + c_2 \|w\|_{H^1(\mathbb{R}^N)}^2,
\]

\[
g_v(0) \leq I(u_0) + \epsilon \|v\|_{H^1(\mathbb{R}^N)}^2, \quad v \in V \cap B_{\delta_1}, \quad w \in W \cap B_{\delta_2}.
\] (5.17)

We require \( c_2 \delta_2^2 - c_1 \delta_1^2 > \epsilon \delta_1^2 \) so that \( g_v(w) > g_v(0) \) for \( w \in W \cap \partial B_{\delta_2} \). (5.18)

We show that the functional \( g_v(w) \) is lower semicontinuous provided \( \delta_2 \) is small enough. Hence by (5.18) \( g_v \) assumes its minimum at a point \( w \in W \cap \text{int}B_{\delta_2}, \) for which \( u = u_0 + v + w \) solves (5.13).

Let \( \{w_n\} \subset W \cap B_{\delta_2} \) be a minimizing sequence:

\[
\lim_{n \to \infty} g_v(w_n) = \inf\{g_v(w) \mid w \in W \cap B_{\delta_2}\}.
\]

Passing to a subsequence \( w_n \rightarrow w \) in \( H^1(\mathbb{R}^N), w_n \rightarrow w \) in \( L^p_{\text{loc}}(\mathbb{R}^N), 2 \leq p < \frac{2N}{N-2}, w_n \rightarrow w \) a.e. in \( \mathbb{R}^N \). Let \( u_n = u_0 + v + w_n \). Choosing \( R > 0 \) and passing to a subsequence we have

\[
\lim_{n \to \infty} \frac{1}{2} \int_{B_R} \sum_{i,j=1}^N a_{ij}(x, u_n)D_i u_n D_j u_n \, dx + \frac{1}{2} \int_{B_{2R}} V(x) u_n^2 \, dx - \int_{B_R} G(x, u_n) \, dx
\]

\[
\geq \frac{1}{2} \int_{B_R} \sum_{i,j=1}^N a_{ij}(x, u)D_i u D_j u \, dx + \frac{1}{2} \int_{B_{2R}} V(x) u^2 \, dx - \int_{B_R} G(x, u) \, dx
\]

\[
= g_v(u) + o_R(1).
\] (5.19)

Since \( u_0(x) \rightarrow 0, Du_0(x) \rightarrow 0, v(x) \rightarrow 0, Dv(x) \rightarrow 0 \) as \(|x| \rightarrow \infty\) uniformly in \( v \in V \cap B_{\delta_1}, \) for \( R \) large enough,

\[
\frac{1}{2} \int_{\mathbb{R}^N \setminus B_R} \sum_{i,j=1}^N a_{ij}(x, u_n)D_i u_n D_j u_n \, dx + \frac{1}{2} \int_{B_N \setminus B_R} V(x) u_n^2 \, dx - \int_{B_R} G(x, u_n) \, dx
\]

\[
\geq \frac{1}{2} \int_{\mathbb{R}^N \setminus B_R} \sum_{i,j=1}^N a_{ij}(x, u_n)D_i w_n D_j w_n \, dx + \frac{1}{2} \int_{B_R} V(x) w_n^2 \, dx
\]
\[ - \int_{\mathbb{R}^N \setminus B_{2R}} G(x, w_n) \, dx + o_R(1) \]
\[ \geq c \int_{\mathbb{R}^N \setminus B_R} (|Dw_n|^2 + w_n^2) \, dx - \int_{\mathbb{R}^N \setminus B_{2R}} (\epsilon w_n^2 + C_\epsilon |w_n|^p) \, dx + o_R(1). \]  

(5.20)

Take a cut-off function \( \chi \) such that \( \chi = 0 \) in \( B_R, \chi = 1 \) in \( \mathbb{R}^N \setminus B_{2R}, |\nabla \chi| \leq 1 \). Then by Hölder inequality and Sobolev imbedding theorem

\[ \int_{B_{N} \setminus B_{2R}} |w_n|^p \, dx \leq \int_{\mathbb{R}^N} |w_n \chi|^p \, dx \]
\[ \leq \left( \int_{\mathbb{R}^N} |w_n \chi|^2 \, dx \right)^{\frac{p}{2}} \left( \int_{\mathbb{R}^N} |w_n \chi|^{2^*} \, dx \right)^{\frac{p-2}{2^*-2}} \]
\[ \leq C \left( \int_{\mathbb{R}^N} |w_n \chi|^2 + |D(w_n)\chi|^2 \, dx \right)^{\frac{p}{2}} \]
\[ \leq C \left( \int_{\mathbb{R}^N \setminus B_R} |w_n|^2 + |D(w_n)|^2 \, dx \right)^{\frac{p}{2}}. \]

Thus the right-hand side of (5.20) is not less than the following

\[ c_3 \int_{\mathbb{R}^N \setminus B_{2R}} (|Dw_n|^2 + w_n^2) \, dx - c_4 \left( \int_{\mathbb{R}^N \setminus B_{2R}} (|Dw_n|^2 + w_n^2) \, dx \right)^{\frac{p}{2}} + o_R(1) \geq o_R(1) \]

provided \( c_3 \delta_2^2 - c_4 \delta_2^p > 0 \).

Let \( R \to \infty \), then \( \epsilon \to 0 \), it follows from this and (5.19), (5.20) that

\[ \inf \{ g_v(w) \mid w \in W \cap B_{\delta_2} \} = \lim_{n \to \infty} g_v(w_n) \geq g_v(w). \]

By (5.18), \( \|w\| < \delta_2 \), hence \( u = u_0 + v + w \) solves (5.13).

Now let \( u \) be a solution of (5.13), \( u = u_0 + v + w, v \in V \cap B_{\delta_2}, w \in W \cap B_{\delta_2} \). Let \( f \in V: f = \sum_{k=1}^d \phi_k(DI(u), \phi_k) \), where \( \{\phi_1, \ldots, \phi_d\} \) is a base of \( V \), then \( \|f\|_{L^\infty(\mathbb{R}^N)} \leq c\|u\|_{H^1(\mathbb{R}^N)} \), for \( \|u\|_{H^1(\mathbb{R}^N)} \) small. We have

\[ \int_{\mathbb{R}^N} \left[ \sum_{i,j=1}^N a_{ij}(x,u)D_iuD_j\varphi \, dx + \frac{1}{2} \sum_{i,j=1}^N D_s a_{ij}(x,u)D_iuD_ju \varphi \, dx \right] 
+ \int_{\mathbb{R}^N} V(x)u \varphi \, dx - \int_{\mathbb{R}^N} g(x,u)\varphi \, dx = \int_{\mathbb{R}^N} f \varphi \, dx, \forall \varphi \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \]  

(5.21)
As in Lemma 2.4, by Moser’s iteration, we have
\[ \|u\|_{L^\infty(\mathbb{R}^N)} \leq C(\|u\|_{H^1(\mathbb{R}^N)}) \|f\|_{L^\infty(\mathbb{R}^N)} \leq C\|u\|_{H^1(\mathbb{R}^N)} \]
and also \( \|Du\|_{L^\infty(\mathbb{R}^N)} \leq C\|u\|_{H^1(\mathbb{R}^N)} \) by regularity theory for elliptic equations (e.g., [18]).

**Lemma 5.5.** Let \( u_k = u_0 + v_k + w_k \) solve (5.13) with \( v_k \in V \cap B_{\delta_1}, w_k \in W \cap B_{\delta_2}, k = 1, 2 \). Then for some \( C > 0 \)
\[ \|w_2 - w_1\|_{L^\infty(\mathbb{R}^N)} + \|w_2 - w_1\|_{H^1(\mathbb{R}^N)} \leq C\|v_2 - v_1\|_{H^1(\mathbb{R}^N)}. \]

**Proof.** The proof of this lemma is known, e.g., [28]. For completeness we reproduce it here. Let \( f_l \in V, f_l = \sum_{k=1}^d \psi_k(DI(u_l), \psi_k) \), \( l = 1, 2 \). We have \( \|f_2 - f_1\|_{L^\infty(\mathbb{R}^N)} \leq C\|u_2 - u_1\|_{H^1(\mathbb{R}^N)} \) where \( u_k \) satisfies the following equation:
\[
\int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij}(x, u_k) D_i u_k D_j \psi dx + \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i,j=1}^N D_s a_{ij}(x, u_k) D_i u_k D_j u_k \psi dx \\
+ \int_{\mathbb{R}^N} V(x) u_k \psi dx - \int_{\mathbb{R}^N} g(x, u_k) \psi dx = \int_{\mathbb{R}^N} f_k \psi dx, \quad \forall \psi \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \]

The difference \( u_2 - u_1 \) satisfies
\[
\int_{\mathbb{R}^N} \sum_{i,j=1}^N A_{ij}(x) D_i (u_2 - u_1) D_j \psi dx + \int_{\mathbb{R}^N} \sum_{j=1}^N B_j(x) D_j ((u_2 - u_1)\psi) dx \\
+ \int_{\mathbb{R}^N} D(x) (u_2 - u_1) \psi dx = \int_{\mathbb{R}^N} (f_2 - f_1) \psi dx, \quad \forall \psi \in H^1(\mathbb{R}^N), \tag{5.22}
\]
where
\[
A_{ij}(x) = \int_1^2 a_{ij}(x, u_t) dt,
\]
\[
B_j(x) = \int_1^2 D_s a_{ij}(x, u_t) D_i u_t dt,
\]
\[
D(x) = \int_1^2 \left\{ \frac{1}{2} D^2_s a_{ij}(x, u_t) D_i u_t D_j u_t + v(x) - D_s g(x, u_t) \right\} dt,
\]
and \( u_t = (2-t)u_2 + (t-1)u_1, t \in (1, 2) \). By regularity theorem
\[ \|u_2 - u_1\|_{L^\infty(\mathbb{R}^N)} \leq C \left( \|u_2 - u_1\|_{H^1(\mathbb{R}^N)} + \|f_2 - f_1\|_{L^\infty(\mathbb{R}^N)} \right) \leq C \|u_2 - u_1\|_{H^1(\mathbb{R}^N)}, \]  

(5.23)

where \( C = C(\|A_{ij}\|_{L^\infty(\mathbb{R}^N)}, \|B_j\|_{L^\infty(\mathbb{R}^N)}, \|D\|_{L^\infty(\mathbb{R}^N)}) \). It follows from (5.23) that

\[ \|w_2 - w_1\|_{L^\infty(\mathbb{R}^N)} \leq C \left( \|w_2 - w_1\|_{H^1(\mathbb{R}^N)} + \|v_2 - v_1\|_{H^1(\mathbb{R}^N)} \right). \]  

(5.24)

If \( v = 0 \), then \( w = 0 \), and \( u = u_0 \) solves (5.13). It follows from (5.24) that if \( u = u_0 + v + w \) solves (5.13) then

\[ \|w\|_{L^\infty(\mathbb{R}^N)} \leq C \left( \|w\|_{H^1(\mathbb{R}^N)} + \|v\|_{H^1(\mathbb{R}^N)} \right). \]  

(5.25)

We next estimate \( \|w_2 - w_1\|_{H^1(\mathbb{R}^N)} \). By (5.1) and (5.22)

\[
\gamma \|w_2 - w_1\|_{H^1(\mathbb{R}^N)}^2 \leq \left( L(w_2 - w_1), w_2 - w_1 \right) = \left( L(u_2 - u_1), w_2 - w_1 \right)
\]

\[
= \int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij}(x,u_0)D_i(u_2 - u_1)D_j(w_2 - w_1) \, dx
\]

\[
+ \int_{\mathbb{R}^N} \sum_{j=1}^N b_j(x)D_j\left\{(u_2 - u_1)(w_2 - w_1)\right\} \, dx
\]

\[
+ \int_{\mathbb{R}^N} \sum_{j=1}^N d(x)\left\{(u_2 - u_1)(w_2 - w_1)\right\} \, dx
\]

\[
= \int_{\mathbb{R}^N} \sum_{i,j=1}^N (a_{ij}(x,u_0) - A_{ij}(x))D_i(u_2 - u_1)D_j(w_2 - w_1) \, dx
\]

\[
+ \int_{\mathbb{R}^N} \sum_{j=1}^N (b_j(x) - B_j(x))D_j\left\{(u_2 - u_1)(w_2 - w_1)\right\} \, dx
\]

\[
\times \int_{\mathbb{R}^N} \sum_{j=1}^N (d(x) - D(x))\left\{(u_2 - u_1)(w_2 - w_1)\right\} \, dx
\]

\[ =: I + II + III \]

where \( b_j(x) = D_s a_{ij}(x,u_0)D_i u_0, d(x) = \frac{1}{2} D_s^2 a_{ij}(x,u_0)D_i u_0 D_j u_0 + V(x) - D_s g(x, u_0) \). We have

\[
|I| \leq C \sum_{i,j=1}^N \sup_{x \in \mathbb{R}^N} \left| a_{ij}(x,u_0) - A_{ij}(x) \right| \cdot \|u_2 - u_1\|_{H^1(\mathbb{R}^N)} \|w_2 - w_1\|_{H^1(\mathbb{R}^N)}
\]

\[
\leq C(\delta_1 + \delta_2) \|u_2 - u_1\|_{H^1(\mathbb{R}^N)} \|w_2 - w_1\|_{H^1(\mathbb{R}^N)},
\]
since
\[ |a_{ij}(x, u_0) - a_{ij}(x, u_0 + v + w)| \leq c \left( \|v\|_{L^\infty} + \|w\|_{L^\infty} \right) \leq c(\delta_1 + \delta_2). \]

Similarly
\[ |II| \leq C \sum_{j=1}^{N} \|b_j - B_j\|_{L^2(\mathbb{R}^N)} \left( \|u_2 - u_1\|_{H^1(\mathbb{R}^N)} \|w_2 - w_1\|_{L^\infty(\mathbb{R}^N)} + \|u_2 - u_1\|_{L^\infty(\mathbb{R}^N)} \|w_2 - w_1\|_{H^1(\mathbb{R}^N)} \right). \]

Similarly
\[ |III| \leq \frac{1}{t} \int_{\mathbb{R}^N} \sum_{i,j=1}^{N} |D_s^2 a_{ij}(x, u_0)D_i u_0 D_j u_0 - D_s^2 a_{ij}(x, u_t)D_i u_t D_j u_t| \, dx \times \|u_2 - u_1\|_{L^\infty(\mathbb{R}^N)} \|w_2 - w_1\|_{L^\infty(\mathbb{R}^N)} \]
\[ + \sup_{t \in [0, 1]} \sup_{x \in \mathbb{R}^N} \left| D_s g(x, u_0) - D_s g(x, u_t) \right| \|u_2 - u_1\|_{H^1(\mathbb{R}^N)} \|w_2 - w_1\|_{H^1(\mathbb{R}^N)} = C \epsilon(\delta_1, \delta_2)(\|u_2 - u_1\|_{L^\infty(\mathbb{R}^N)} \|w_2 - w_1\|_{L^\infty(\mathbb{R}^N)} + \|u_2 - u_1\|_{H^1(\mathbb{R}^N)} \|w_2 - w_1\|_{H^1(\mathbb{R}^N)}). \]

where $\epsilon(\delta_1, \delta_2) \to 0$ as $\delta_1, \delta_2 \to 0$. It follows from the above estimates that
\[ \|w_2 - w_1\|_{H^1(\mathbb{R}^N)} \leq C \epsilon(\delta_1, \delta_2)(\|v_2 - v_1\|_{H^1(\mathbb{R}^N)} + \|w_2 - w_1\|_{L^\infty(\mathbb{R}^N)}). \quad (5.26) \]

By (5.24) and (5.26),
\[ \|w_2 - w_1\|_{L^\infty(\mathbb{R}^N)} + \|w_2 - w_1\|_{H^1(\mathbb{R}^N)} \leq C \|v_2 - v_1\|_{H^1(\mathbb{R}^N)}. \]

**Proof of Theorem 5.3.** We need only to derive the derivative of the function $h(v) = I(u_0 + v + \phi(v))$. Let $\varphi \in V$. Then by Lemma 5.5
\[ \left\| \frac{1}{t}(\phi(v + t\varphi) - \phi(v)) \right\|_{H^1(\mathbb{R}^N)} + \left\| \frac{1}{t}(\phi(v + t\varphi) - \phi(v)) \right\|_{L^\infty(\mathbb{R}^N)} \leq c \|\varphi\|_{H^1(\mathbb{R}^N)}. \]

Notice that the functional $I$ is $C^1$ differential in $H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, we have
\[
\left| \frac{1}{t}(h(v + t\varphi) - h(v)) - \left\langle DI(u_0 + v + \phi(v)), \varphi \right\rangle \right| \\
= \left| \frac{1}{t}(I(u_0 + v + t\varphi + \phi(v + t\varphi)) - I(u_0 + v + \phi(v))) - \left\langle DI(u_0 + v + \phi(v)), \varphi \right\rangle \right| \\
= \left| \int_0^1 \left( DI(u_0 + v + \phi(v)) + s(t\varphi + \phi(v + t\varphi) - \phi(v)) \right) \right|
\]
Thus we have \( \lim_{s \to 0} \frac{1}{t}(h(v + t\varphi) - h(v)) = \langle DI(u_0 + v + \phi(v)), \varphi \rangle \), which is continuous in \( v \in V \). Therefore \( h \) is a \( C^1 \) function and \( (Dh(v), \varphi) = \langle DI(u_0 + v + \phi(v)), \varphi \rangle \), \( \forall \varphi \in V \). Since \( \langle DI(u_0 + v + \phi(v)), z \rangle = 0 \) for \( z \in W \cap L^\infty \), \( v \) is a critical point of \( h \) if and only if \( u = u_0 + v + \phi(v) \) is a critical point of \( I \). \( \square \)

Let \( g_v(w) = I(u_0 + v + w) \), \( v \in V \cap B_{\delta_1} \). Then \( g_v \) is a \( G \)-differential functional defined on \( W \cap B_{\delta_2} \), \( \langle Dg_v(w), z \rangle = \langle DI(u_0 + v + w), z \rangle \) for \( z \in W \cap L^\infty(\mathbb{R}^N) \).

**Lemma 5.6.** \( g_v \) satisfies the (CPS) condition uniformly for \( v \in V \cap B_{\delta_1} \), that is, if \( \{v_n\} \subset V \cap B_{\delta_1} \), \( \{w_n\} \subset W \cap B_{\delta_2} \), \( \langle Dg_{v_n}(w_n) \rangle \to 0 \) and \( g_{v_n}(w_n) \to c \) as \( n \to \infty \), then \( (v_n, w_n) \) possesses a convergent subsequence.

**Proof.** Let \( u_n = u_0 + v_n + w_n \). Then for \( z \in W \cap L^\infty(\mathbb{R}^N) \),

\[
\langle Dg_{v_n}(w_n), z \rangle = \langle DI(u_n), z \rangle = \int_{\mathbb{R}^N} \sum_{i,j=1}^n a_{ij}(x, u_n) D_i u_n D_j z \, dx + \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i,j=1}^n D_a a_{ij}(x, u_n) D_i u_n D_j u_n z \, dx \int_{\mathbb{R}^N} V(x) u_n z \, dx - \int_{\mathbb{R}^N} g(x, u_n) z \, dx
\]

\[
= \langle \tilde{w}_n, z \rangle, \quad z \in W \cap L^\infty(\mathbb{R}^N),
\]

where \( \tilde{w}_n \in W \), \( \|\tilde{w}_n\|_{H^1(\mathbb{R}^N)} \to 0 \). Let \( \{\varphi_1, \varphi_2, \ldots, \varphi_d\} \) be an orthogonal base of \( V \subset L^2(\mathbb{R}^N) \), and let \( f_n = \sum_{k=1}^d \varphi_k(DI(u_n), \phi_k) \). Assume \( f_n \to f \) in \( V \). Then

\[
\langle DI(u_n), \varphi \rangle - \langle f, \varphi \rangle = \langle \tilde{w}_n, \varphi \rangle, \quad \forall \varphi \in H^1(\mathbb{R}^N) \cap L^\infty(T^N)
\]

where \( \langle \tilde{w}_n, \varphi \rangle = \langle \tilde{w}_n, \varphi \rangle + \int_{\mathbb{R}^N} (f_n - f) \varphi, \forall \varphi \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \). Using \( \|\tilde{w}_n\|_{H^1(\mathbb{R}^N)} \to 0 \), we have that \( \{u_n\} \) is a CPS sequence of the functional

\[
I_f(u) = I(u) - \int_{\mathbb{R}^N} f u \, dx.
\]

By similar analysis for a CPS sequence of \( I \) as in Lemmas 3.1 and 3.2, we conclude that

(i) Assume \( u_n \to u \) in \( H^1(\mathbb{R}^N) \), then \( u \) is a critical point of \( I_f \).
(ii) There exist critical points $v_1, v_2, \ldots, v_l$ of $I$, sequences $\{k_n^i\} \subset \mathbb{Z}^N$, $1 \leq i \leq l$ such that, passing to a subsequence

$$
\|u_n - \left( u + \sum_{i=1}^l k_n^i * v_i \right) \|_{H^1(\mathbb{R}^N)} \to 0,
$$

$$
|k_n^i| \to \infty, \quad |k_n^i - k_n^j| \to \infty, \quad 1 \leq i < j \leq n,
$$

$$
\lim_{n \to \infty} I_f(u_n) = I_f(u) + \sum_{i=1}^l I(v_i),
$$

$$
\lim_{n \to \infty} \|u_n\|^2_{H^1(\mathbb{R}^N)} = \|u\|^2_{H^1(\mathbb{R}^N)} + \sum_{i=1}^l \|v_i\|^2_{H^1(\mathbb{R}^N)}.
$$

Consequently, we may assume $v_n \to v \in V$, $w_n \to w \in W$ in $H^1(\mathbb{R}^N)$. Then

$$
\|w_n - \left( w + \sum_{i=1}^l k_n^i * v_i \right) \|_{H^1(\mathbb{R}^N)} \to 0,
$$

$$
\lim_{n \to \infty} \|w_n\|^2_{H^1(\mathbb{R}^N)} = \|w\|^2_{H^1(\mathbb{R}^N)} + \sum_{i=1}^l \|v_i\|^2_{H^1(\mathbb{R}^N)}. \quad (5.27)
$$

Remember that, there is an $\alpha > 0$ such that any non-trivial critical point $v$ of $I$ satisfies $\|v\| \geq \alpha$. Let $\delta_2 < \alpha$, it follows from (5.27) that $l = 0$ in (5.27), $\lim_{n \to \infty} \|w_n\|^2_{H^1(\mathbb{R}^N)} = \|w\|^2_{H^1(\mathbb{R}^N)}$, hence $w_n \to w$ in $H^1(\mathbb{R}^N)$. \hfill \Box

**Theorem 5.7.** If $u_0$ is an isolated critical point of $I$, then $C_q(I, u_0) = C_q(h, 0)$ for all $q = 0, 1, 2, \ldots$, where $h(v) = I(u_0 + v + \phi(v))$.

This follows from the following lemma.

**Lemma 5.8.** Let $A = \{u = u_0 + v + w \mid I(u) \leq I(u_0), \|v\| \leq \delta_1, \|w\| \leq \delta_2\}$ and $B = \{v \mid I(u_0 + v + \phi(v)) \leq I(u_0), \|v\| \leq \delta_1\}$. Then $(B, B \setminus \{0\})$ is a deformation retract of $(A, A \setminus \{u_0\})$.

**Proof.** Choose $\delta_1 > 0$, $\delta_2 > 0$ as in Theorem 5.3. When we treat $v \in V \cap B_{\delta_1}$ as a parameter, the functional $g_v(w) = I(u_0 + v + w)$ has a unique critical point $\phi(v) \in W \cap B_{\delta_2}$, which is a minimum point. For such a $v$, as in the proof of Theorem 3.3, we can construct a pseudo gradient vector field $V_v(w)$ on $W \cap B_{\delta_2}$. By Lemma 5.6, (CPS) condition holds uniformly in $v$, hence we can do this in a way that $V_v(w)$ depends on $v$ continuously in $v \in V \cap B_{\delta_1}$. Then the pseudo gradient flow $\xi(s, v, w)$ is continuous in $(s, v, w)$. For fixed $v$ and $w$, by the (CPS) condition there is a unique time $\tilde{s} = \tilde{s}(v, w)$ such that $\tilde{s}(v, w)$ is continuous in $(v, w)$ and $\tilde{x}(\tilde{s}, v, w)$ reaches $\phi(v)$, the unique minimum point of $g_v$. Then $\eta(t, v, w) = \tilde{x}(t \tilde{s}(v, w), v, w)$ is the desired deformation. \hfill \Box
6. Multibump solutions

Let $u_0$ be a critical point of $I$ with the nontrivial $q$-th critical group $C_q(I, u_0)$. An example of such a critical point is the critical point of mountain pass type obtained in the previous sections, see Theorem A.

We are looking for two-bump critical points of $I$ near $u_0 + \kstar u_0$ for $k \in \mathbb{Z}^N$ with $|k|$ sufficiently large. Recall that $\kstar u$ is defined as $(\kstar u)(x) = u(x - k)$, the translation of $u$.

**Lemma 6.1.** $\lim_{|k| \to \infty} |DI(u_0 + \kstar u_0)| = 0$.

**Proof.** Let $\varphi \in H^1(\mathbb{R}^N)$

$$
\langle DI(u_0 + \kstar u_0), \varphi \rangle = \int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij}(x, u_0 + \kstar u_0) D_i u_0 D_j \varphi \, dx \\
+ \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i,j=1}^N D_i a_{ij}(x, u_0 + \kstar u_0) D_i u_0 D_j (u_0 + \kstar u_0) \varphi \, dx \\
+ \int_{\mathbb{R}^N} V(x)(u_0 + \kstar u_0) \varphi \, dx - \int_{\mathbb{R}^N} g(x, u_0 + \kstar u_0) \varphi \, dx.
$$

Let $R > 0$, $|k| > 3R$ and $\Omega = \mathbb{R}^N \setminus (B_R(0) \cup B_R(k))$, then

$$
\int_{B_R(0)} \sum_{i,j=1}^N a_{ij}(x_0, u_0 + \kstar u_0) D_i u_0 D_j \varphi \, dx \\
= \int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij}(x, u_0) D_i u_0 D_j \varphi \, dx - \int_{\mathbb{R}^N \setminus B_R(0)} \sum_{i,j=1}^N a_{ij}(x, u_0) D_i u_0 D_j \varphi \, dx \\
+ \int_{B_R(0)} \sum_{i,j=1}^N (a_{ij}(x, u_0 + \kstar u_0) - a_{ij}(x, u_0)) D_i u_0 D_j \varphi \, dx \\
+ \int_{B_R(0)} \sum_{i,j=1}^N a_{ij}(x, u_0 + \kstar u_0) D_i (k \star u_0) D_j \varphi \, dx \\
= \int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij}(x, u_0) D_i u_0 D_j \varphi \, dx + o_R(1)\|\varphi\|.
$$

In the above, we have used the exponential decay of $u_0$, i.e. for some $\alpha > 0$, $\|u_0\|_{H^1(\mathbb{R}^N \setminus B_R(0))} = o_R(1)$ and $\|u_0\|_{W^{1,\infty}(\mathbb{R}^N \setminus B_R(0))} = o_R(1)$. Also we have
\[ \int_{B_R(0)} \sum_{i,j=1}^{N} D_s a_{ij}(x, u + k \ast u_0) D_i(u_0 + k \ast u_0) D_j(u_0 + k \ast u_0) \varphi \, dx \]

\[ = \int_{\mathbb{R}^N} \sum_{i,j=1}^{N} D_s a_{ij}(x, u_0) D_i u_0 D_j u_0 \varphi \, dx - \int_{\mathbb{R}^N \setminus B_R(0)} \sum_{i,j=1}^{N} D_s a_{ij}(x, u_0) D_i u_0 D_j u_0 \varphi \, dx \]

\[ + \int_{B_R(0)} \sum_{i,j=1}^{N} (D_s a_{ij}(x, u_0 + k \ast u_0) - D_s a_{ij}(x, u_0)) D_i u_0 D_j u_0 \varphi \, dx \]

\[ + \int_{B_R(0)} \sum_{i,j=1}^{N} D_s a_{ij}(x, u_0 + k \ast u_0) (2 D_i u_0 D_j (k \ast u_0) + D_i (k \ast u_0) D_j (k \ast u_0)) \varphi \, dx \]

\[ = \int_{\mathbb{R}^N} \sum_{i,j=1}^{N} D_s a_{ij}(x, u_0) D_i u_0 D_j u_0 \varphi \, dx + o_R(1) \| \varphi \|, \]

\[ \int_{B_R(0)} V(x)(u_0 + k \ast u_0) \varphi \, dx \]

\[ = \int_{\mathbb{R}^N} V(x) u_0 \varphi \, dx - \int_{\mathbb{R}^N \setminus B_R(0)} V(x) u_0 \varphi \, dx + \int_{B_R(0)} V(x) k \ast u_0 \varphi \, dx \]

\[ = \int_{\mathbb{R}^N} V(x) u_0 \varphi \, dx + o_R(1) \| \varphi \|, \]

\[ \int_{B_R(0)} g(x, u_0 + k \ast u_0) \varphi \, dx \]

\[ = \int_{\mathbb{R}^N} g(x, u_0) \varphi \, dx - \int_{\mathbb{R}^N \setminus B_R(0)} g(x, u_0) \varphi \, dx + \int_{B_R(0)} (g(x, u_0 + k \ast u_0) - g(x, u_0)) \varphi \, dx \]

\[ = \int_{\mathbb{R}^N} g(x, u_0) \varphi \, dx + o_R(1) \| \varphi \|, \]

since \(|g(x, u_0)| \leq C|u_0|, |g(x, u_0 + k \ast u_0) - g(x, u_0)| \leq C|k \ast u_0|\). Altogether we have

\[ \int_{B_R(0)} \sum_{i,j=1}^{N} a_{ij}(x, u_0 + k \ast u_0) D_i(u_0 + k \ast u_0) D_j \varphi \, dx \]

\[ + \frac{1}{2} \int_{B_R(0)} \sum_{i,j=1}^{N} D_s a_{ij}(x, u_0 + k \ast u_0) D_i(u_0 + k \ast u_0) D_j(u_0 + k \ast u_0) \varphi \, dx \]
\[ + \int_{B_R(0)} V(x)(u_0 + k \ast u_0)\varphi \, dx - \int_{B_R(0)} g(x, u_0 + k \ast u_0)\varphi \, dx = \{D1(u_0), \varphi\} + o_R(1)\|\varphi\|. \]

Similarly
\[
\int_{B_R(k)} \sum_{i,j=1}^{N} a_{ij}(x, u_0 + k \ast u_0)D_i(u_0 + k \ast u_0)D_j \varphi \, dx
+ \frac{1}{2} \int_{B_R(k)} \sum_{i,j=1}^{N} D_s a_{ij}(x, u_0 + k \ast u_0)D_i(u_0 + k \ast u_0)D_j(u_0 + k \ast u_0)\varphi \, dx
+ \int_{B_R(k)} V(x)(u_0 + k \ast u_0)\varphi \, dx - \int_{B_R(k)} g(x, u_0 + k \ast u_0)\varphi \, dx
= \{D1(k \ast u_0), \varphi\} + o_R(1)\|\varphi\|
\]

and
\[
\int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(x, u_0 + k \ast u_0)D_i(u_0 + k \ast u_0)D_j \varphi \, dx
+ \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{N} D_s a_{ij}(x, u_0 + k \ast u_0)D_i(u_0 + k \ast u_0)D_j(u_0 + k \ast u_0)\varphi \, dx
+ \int_{\Omega} V(x)(u_0 + k \ast u_0)\varphi \, dx - \int_{\Omega} g(x, u_0 + k \ast u_0)\varphi \, dx
= o_R(1)\|\varphi\|.
\]

Finally
\[
\{D1(u_0 + k \ast u_0), \varphi\} = \{D1(u_0), \varphi\} + \{D1(k \ast u_0), \varphi\} + o_R(1)\|\varphi\|. \tag{6.1}
\]

Lemma 6.1 is proved. \(\square\)

**Remark 6.1.** Let \(u, v \in H^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)\) and have exponentially decay, that is, for some \(\alpha > 0\)
\[
\|u\|_{H^1(\mathbb{R}^N \setminus B_R(0))} = o_R(1), \quad \|v\|_{H^1(\mathbb{R}^N \setminus B_R(0))} = o_R(1).
\]

It is clear that our arguments yield
\[
\{D1(u + k \ast v), \varphi\} = \{D1(u), \varphi\} + \{D1(k \ast v), \varphi\} + o_R(1)\|\varphi\|.
\]
Now we study the behavior of $I$ near the approximate critical point $u_0 + k * u_0$. First we have the following Taylor’s expansion, which is a counterpart of Lemma 5.1.

**Lemma 6.2 (Taylor’s expansion).** Let $v, \tilde{v} \in V, w \in V^\bot \cap (k * V)^\bot$. Then

$$I(u_0 + k * u_0 + v + k * \tilde{v} + w) = I(u_0 + k * u_0) + \langle DI(u_0 + k * u_0), v + k * \tilde{v} + w \rangle$$

$$+ \frac{1}{2} \left( \mathcal{L}_k(v + k * \tilde{v}), v + k * \tilde{v} \right) + \left( \mathcal{L}_k(v + k * \tilde{v}), w \right) + \frac{1}{2} \left( \mathcal{L}_k(w), w \right)$$

$$+ o(1)\left( \|v\|_{H^1}^2 + \|\tilde{v}\|_{H^1}^2 + \|w\|_{H^1}^2 \right),$$

where

$$(\mathcal{L}_k \psi, \varphi) = D^2 I(u_0 + k * u_0)(\psi, \varphi)$$

$$= \int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij}(x, u_0 + k * u_0) D_i \psi D_j \varphi \, dx$$

$$+ \int_{\mathbb{R}^N} \sum_{i,j=1}^N D_i a_{ij}(x, u_0 + k * u_0) D_i (u_0 + k * u_0) D_j (\psi \varphi) \, dx$$

$$+ \int_{\mathbb{R}^N} \left\{ \frac{1}{2} \sum_{i,j=1}^N D^2 s a_{ij}(x, u_0 + k * u_0) D_i (u_0 + k * u_0) D_j (u_0 + k * u_0) \right. $$

$$+ V(x) - D_s g(x, u_0 + k * u_0) \right\} (\psi \varphi) \, dx,$$

and

$$(\tilde{\mathcal{L}}_k(w) \psi, \varphi) = (\mathcal{L}_k \psi, \varphi) + \int_{\mathbb{R}^N} \sum_{i,j=1}^N \left( a_{ij}(x, u_0 + k * u_0 + w) - a_{ij}(x, u_0 + k * u_0) \right) D_i \psi D_j \varphi \, dx.$$ 

**Proof.** Denote $U_k = u_0 + k * u_0$, $V_k = v + k * \tilde{v}$.

$$I(U_k + V_k + w) = I(U_k) + \langle DI(U_k), V_k + w \rangle + \frac{1}{2} \left( \mathcal{L}_k(V_k + w), V_k + w \right)$$

$$+ \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i,j=1}^N \left( a_{ij}(x, U_k + w) - a_{ij}(x, U_k) \right) D_i w D_j w \, dx$$

$$+ R_1 + R_2 + R_3 + R_4 + R_5,$$

where
\[ R_1 = \int_{\mathbb{R}^N} \frac{1}{2} \sum_{i,j=1}^{N} \left\{ a_{ij}(x, U_k + V_k + w) - a_{ij}(x, U_k) - D_s a_{ij}(x, U_k)(V_k + w) \right\} D_t U_k D_j U_k \, dx \\
= \int_{0}^{1} (1 - t) \int_{\mathbb{R}^N} \left\{ D_s a_{ij}(x, U_k + t(V_k + w)) - D_s a_{ij}(x, U_k) \right\}(V_k + w)^2 D_t U_k D_j U_k \, dx, \]

\[ R_2 = \int_{\mathbb{R}^N} \frac{1}{2} \sum_{i,j=1}^{N} \left\{ a_{ij}(x, U_k + V_k + w) - a_{ij}(x, U_k) - D_s a_{ij}(x, U_k)(V_k + w) \right\} D_t U_k D_j (V_k + w) \, dx \\
= \int_{0}^{1} dt \int_{\mathbb{R}^N} \left\{ D_t a_{ij}(x, U_k + t(V_k + w)) - D_t a_{ij}(x, U_k) \right\}(V_k + w) D_t U_k D_j (V_k + w) \, dx, \]

\[ R_3 = \int_{\mathbb{R}^N} \frac{1}{2} \sum_{i,j=1}^{N} \left( a_{ij}(x, U_k + V_k + w) - a_{ij}(x, U_k) \right)(2 D_t V_k D_j w + D_t V_k D_j V_k) \, dx, \]

\[ R_4 = \int_{\mathbb{R}^N} \frac{1}{2} \sum_{i,j=1}^{N} \left( a_{ij}(x, U_k + V_k + w) - a_{ij}(x, U_k) \right) D_t w D_j w \, dx, \]

\[ R_5 = \int_{\mathbb{R}^N} \left( G(x, U_k + V_k + w) - G(x, U_k) - g(x, U_k)(V_k + w) - \frac{1}{2} D_s g(x, U_k)(V_k + w)^2 \right) \, dx \\
= \int_{0}^{1} (1 - t) \int_{\mathbb{R}^N} \left( D_s g(x, U_k + t(V_k + w)) - D_s g(x, U_k) \right)(V_k + w)^2 \, dx. \]

As \( a_{ij}(x, s), D_s a_{ij}(x, s) \) and \( D_s^2 a_{ij}(x, s) \) are continuous and

\[
|D_s^2 a_{ij}(x, U_k + t(V_k + w)) - D_s^2 a_{ij}(x, U_k)| \leq C \epsilon |V_k + w|^\frac{4}{N-2}, \\
|D_s a_{ij}(x, U_k + t(V_k + w)) - D_s a_{ij}(x, U_k)| \leq C \epsilon |V_k + w|^\frac{2}{N-2}, \\
|a_{ij}(x, U_k + t(V_k + w)) - a_{ij}(x, U_k)| \leq C |V_k + w|,
\]
we have

\[
|R_1| \leq C \int_{\mathbb{R}^N} \left( \epsilon + C \epsilon |V_k + w|^\frac{4}{N-2} \right)(V_k + w)^2 \, dx \leq \epsilon \|V_k + w\|_{H^1(\mathbb{R}^N)}^2 + C \epsilon \|V_k + w\|_{H^1(\mathbb{R}^N)}^{\frac{2N}{N-2}},
\]
\[ |R_2| \leq C \int_{\mathbb{R}^N} \left( \epsilon + C_\epsilon |V_k + w|^{\frac{4}{N-2}} \right) |V_k + w| |D(V_k + w)| \, dx \]

\[ \leq C \epsilon \|V_k + w\|^2_{H^1_{\mathbb{R}^N}} + C_\epsilon \|V_k + w\|^\frac{2N-4}{N-2} \]

\[ |R_3| \leq C \int_{\mathbb{R}^N} |V_k + w|(2|DV_k||Dw| + |DV_k|^2) \, dx \]

\[ \leq C \|DV_k\|_{L^\infty}(\|V_k\|_{H^1_{\mathbb{R}^N}}^2 + \|w\|_{H^1_{\mathbb{R}^N}}^2) \]

\[ \leq C \|V_k\|_{H^1_{\mathbb{R}^N}}(\|V_k\|_{H^1_{\mathbb{R}^N}} + \|w\|_{H^1_{\mathbb{R}^N}}^2) \]

\[ |R_4| \leq \int_{\mathbb{R}^N} |V_k||Dw|^2 \, dx \leq \|V_k\|_{L^\infty_{\mathbb{R}^N}} \|w\|_{H^1_{\mathbb{R}^N}}^2 \leq C \|V_k\|_{H^1_{\mathbb{R}^N}} \|w\|_{H^1_{\mathbb{R}^N}}^2 \]

Similarly,

\[ \left| D_s g(x, U_k + t(V_k + w)) - D_s g(x, U_k) \right| \leq \epsilon + C_\epsilon |V_k + w|^{\frac{4}{N-2}} \]

and

\[ |R_5| \leq C \int_{\mathbb{R}^N} \left( \epsilon + C_\epsilon |V_k + w|^{\frac{4}{N-2}} \right) |V_k + w|^2 \, dx \leq C \epsilon \|V_k + w\|^2_{H^1_{\mathbb{R}^N}} + C_\epsilon \|V_k + w\|^{\frac{2N}{N-2}}_{H^1_{\mathbb{R}^N}}. \]

Altogether we have

\[ I(U_k + V_k + w) = I(U_k) + \{ D(I(U_k), V_k + w) + \frac{1}{2}(\mathcal{L}_k(V_k + w), V_k + w) \]

\[ + \frac{1}{2} \int_{\mathbb{R}^N} \sum_{ij=1}^N (a_{ij}(x, U_k + w) - a_{ij}(x, U_k)) D_i w D_j w \, dx + R, \]

where \( R \) satisfies

\[ |R| \leq \epsilon \left( \|V_k\| + \|w\| \right)^2 + C_\epsilon \left( \|V_k\| + \|w\| \right)^{\frac{2N}{N-2}}, \]

and Lemma 6.2 follows. \( \Box \)

The following lemma is a counterpart of Lemma 5.2.

**Lemma 6.3.** It holds that

1. \( (\mathcal{L}_k V_k, V_k) \leq o_k(1) \|V_k\|^2, \) \( (\mathcal{L}_k V_k, w) \leq o_k(1) \|V_k\| \|w\|. \)
2. There exist \( \mu, K, \delta > 0 \) such that \( (\mathcal{L}_k(z)w, w) \geq \mu \|w\|^2, \) for \( \|z\| \leq \delta, |k| \geq K. \)
Proof. (1) We estimate \((L_k V_k, V_k)\) only. The term \((L_k V_k, w)\) can be estimated in a similar way.

\[
(L_k V_k, V_k) = \int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij}(x, u_0 + k * u_0) D_i (v + k * \tilde{v}) D_j (v + k * \tilde{v}) \, dx \\
+ \int_{\mathbb{R}^N} \sum_{i,j=1}^N D_s a_{ij}(x, u_0 + k * u_0) D_i (u_0 + k * u_0) D_j (v + k * \tilde{v})^2 \, dx \\
+ \int_{\mathbb{R}^N} \left( \frac{1}{2} \sum_{i,j=1}^N D_s^2 a_{ij}(x, u_0 + k * u_0) + V(x) \\
- D_s g(x, u_0 + k * u_0) \right) (v + k * \tilde{v})^2 \, dx.
\]

For \(|k| \geq 3R\), we estimate the integrals over \(B_R(0), B_R(k)\) and \(\Omega = \mathbb{R}^N \setminus (B_R(0) \cup B_R(k))\) respectively. First we have

\[
\int_{\mathbb{R}^N \setminus B_R(0)} \sum_{i,j=1}^N a_{ij}(x, u_0 + k * u_0) D_i v D_j v \, dx \leq C \int_{\mathbb{R}^N \setminus B_R(0)} |Dv|^2 \, dx \leq o_R(1) \|v\|^2.
\]

This estimate is uniform in \(v \in V\), since \(V\) is a finite dimensional subspace. Next we have

\[
\left| \int_{B_R(0)} \sum_{i,j=1}^N a_{ij}(x, u_0 + k * u_0) (2 D_i v D_j (k * \tilde{v}) + D_i (k * \tilde{v}) D_j (k * \tilde{v})) \, dx \right| \\
\leq o_R(1) (\|v\|_{H^1(\mathbb{R}^N)} \|\tilde{v}\|_{H^1(\mathbb{R}^N)} + \|\tilde{v}\|^2_{H^1(\mathbb{R}^N)}),
\]

\[
\left| \int_{B_R(0)} \sum_{i,j=1}^N (a_{ij}(x, u_0 + k * u_0) - a_{ij}(x, u_0)) D_i v D_j v \, dx \right| \\
\leq c \int_{B_R(0)} |k * u_0| |Dv|^2 \, dx \leq o_R(1) \|v\|^2_{H^1(\mathbb{R}^N)},
\]

and

\[
\int_{B_R(0)} \sum_{i,j=1}^N a_{ij}(x, u_0 + k * u_0) D_i (v + k * \tilde{v}) D_j (v + k * \tilde{v}) \, dx \\
= \int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij}(x, u_0) D_i v D_j v \, dx - \int_{\mathbb{R}^N \setminus B_R(0)} \sum_{i,j=1}^N a_{ij}(x, u_0) D_i v D_j v \, dx
\]
\[ + \int_{B_R(0)} \sum_{ij=1}^{N} a_{ij}(x, u_0 + k \ast u_0) \left( 2 D_i v D_j (k \ast \tilde{v}) + D_i (k \ast \tilde{v}) D_j (k \ast \tilde{v}) \right) \, dx \]

\[ + \int_{B_R(0)} \sum_{ij=1}^{N} (a_{ij}(x, u_0 + k \ast u_0) - a_{ij}(x, u_0)) D_i v D_j v \, dx \]

\[ = \int_{\mathbb{R}^N} \sum_{ij=1}^{N} a_{ij}(x, u_0) D_i v D_j v \, dx + o_R(1) \left( \|v\|^2 + \|	ilde{v}\|^2 \right). \]

Similarly we have

\[ \int_{B_R(0)} \left\{ \sum_{ij=1}^{N} D_s a_{ij}(x, u_0 + k \ast u_0) D_i (u_0 + k \ast u_0) D_j (v + k \ast \tilde{v})^2 \right\} \, dx \]

\[ = \int_{\mathbb{R}^N} \sum_{ij=1}^{N} D_s a_{ij}(x, u_0) D_i u_0 D_j v^2 \, dx - \int_{\mathbb{R}^N \setminus B_R(0)} \sum_{ij=1}^{N} D_s a_{ij}(x, u_0) D_i u_0 D_j v^2 \, dx \]

\[ + \int_{B_R(0)} \sum_{ij=1}^{N} D_s a_{ij}(x, u_0 + k \ast u_0) D_i u_0 D_j (2v(k \ast \tilde{v}) + (k \ast v)^2) \, dx \]

\[ + \int_{B_R(0)} \sum_{ij=1}^{N} (D_s a_{ij}(x, u_0 + k \ast u_0) - D_s a_{ij}(x, u_0)) D_i u_0 D_j v^2 \, dx \]

\[ = \int_{\mathbb{R}^N} \sum_{ij=1}^{N} D_s a_{ij}(x, u_0) D_i u_0 D_j v^2 \, dx + o_R(1) \left( \|v\|^2 + \|	ilde{v}\|^2 \right) \]

and

\[ \int_{B_R(0)} \left\{ \frac{1}{2} \sum_{ij=1}^{N} D_s^2 a_{ij}(x, u_0 + k \ast u_0) D_i (u_0 + k \ast u_0) D_j (u_0 + k \ast u_0) + V(x) \right. \]

\[ - D_s g(x, u_0 + k \ast u_0) \left( v + k \ast \tilde{v} \right)^2 \, dx \]

\[ = \int_{\mathbb{R}^N} \left\{ \frac{1}{2} \sum_{ij=1}^{N} D_s^2 a_{ij}(x, u_0) D_i u_0 D_j u_0 + V(x) - D_s g(x, u_0) \right\} v^2 \, dx \]

\[ - \int_{\mathbb{R}^N \setminus B_R(0)} \left\{ \frac{1}{2} \sum_{ij=1}^{N} D_s^2 a_{ij}(x, u_0) D_i u_0 D_j u_0 + V(x) - D_s g(x, u_0) \right\} v^2 \, dx \]
As in the proof of Lemma 5.2, we claim that there exist \( \delta, K, \mu \) such that (2) of the lemma. If the claim is not true, there exist sequences \( \{z_n\} \subset \mathbb{Z}^N \) and \( \{w_n\} \subset W \), such that \( \|z_n\|_{L^2} \leq \frac{1}{n} \), \( |k_n| \geq n \), \( \|w_n\|_{L^2} = 1 \) and \( \lim_{n \to \infty} \langle \tilde{L}_k(z_n)w_n, w_n \rangle \leq 0 \). This implies \( \|w_n\|_{H^1} \leq C \). Assume \( w_n \to w \) in \( H^1 \). Take \( |k_n| > 3R \). Since \( k_n \to 0 \), \( z_n \to 0 \), \( w_n \to w \) in \( B_R \), by lower semi-continuity and Lebesgue’s dominated convergence theorem

\[
\left\{ \int_{B_R(0)} \sum_{ij=1}^N a_{ij}(x, u_0 + k_n * u_0 + z_n)D_i w_n D_j w_n \, dx \right\} \to \int_{B_R(0)} \sum_{ij=1}^N a_{ij}(x, u_0 + k_n * u_0)D_i u_0 D_j u_0 \, dx
\]

In the above, we have used the uniform continuity of \( D_s a_{ij} \) and \( D_s g \) to estimate the last two terms. Altogether the integral over \( B_R(0) \) is equal to \( (L v, v) + o_k(1)(\|v\|^2 + \|\tilde{v}\|^2) \). In the same way, the integral over \( B_R(k) \) is equal to \( (L \tilde{v}, \tilde{v}) + o_k(1)(\|v\|^2 + \|\tilde{v}\|^2) \). Thus we have

\[
(L_k V_k, V_k) = (Lv, v) + (L \tilde{v}, \tilde{v}) + \epsilon(|k|)(\|v\|^2 + \|\tilde{v}\|^2) \leq o_k(1)(\|v\|^2 + \|\tilde{v}\|^2).
\]

In a similar way,

\[
(L_k V_k, w) = (Lv, v) + (L(k * \tilde{v}), w) + o_k(1)(\|v\|^2 + \|\tilde{v}\|^2 + \|w\|^2)
\]

\[
\leq o_k(1)(\|v\|^2 + \|\tilde{v}\|^2 + \|w\|^2).
\]

As in the proof of Lemma 5.2, we claim that there exist \( \delta, K, \mu \) such that

\[
(L_k(z)w, w) \geq \mu \|w\|^2_{L^2}, \quad \text{if } \|z\| \leq \delta, \quad |k| \geq K \text{ and } w \in W_k,
\]

which implies (2) of the lemma. If the claim is not true, there exist sequences \( \{z_n\} \subset H^1 \), \( \{k_n\} \subset \mathbb{Z}^N \) and \( \{w_n\} \subset W \), such that \( \|z_n\|_{H^1} \leq \frac{1}{n} \), \( |k_n| \geq n \), \( \|w_n\|_{L^2} = 1 \) and \( \lim_{n \to \infty} \langle \tilde{L}_k(z_n)w_n, w_n \rangle \leq 0 \). This implies \( \|w_n\|_{H^1} \leq C \). Assume \( w_n \to w \) in \( H^1 \). Take \( |k_n| > 3R \). Since \( k_n \to 0 \), \( z_n \to 0 \), \( w_n \to w \) in \( B_R \), by lower semi-continuity and Lebesgue’s dominated convergence theorem.
\[ + \int_{B_R(0)} \left( \frac{1}{2} \sum_{i=1}^{N} D^2 a_{ij}(x, u_0 + k_n \ast u_0) D_i (u_0 + k_n \ast u_0) D_j (u_0 + k_n \ast u_0) \right. \]
\[ + V(x) - D_s g(x, u_0 + k \ast u_0) w_n^2 \right) \ dx \} \]
\[ \geq \int_{B_R(0)} \sum_{i=1}^{N} a_{ij}(x, u_0) D_i w D_j w \ dx + \int_{B_R(0)} \sum_{i=1}^{N} D_s a_{ij}(x, u_0) D_i u_0 D_j w^2 \ dx \]
\[ + \int_{B_R(0)} \left( \frac{1}{2} \sum_{i=1}^{N} D^2 a_{ij}(x, u_0) D_i u_0 D_j u_0 + V(x) - D_s g(x, u_0) \right) w^2 \ dx \]
\[ = (L_w, w) + o_R(1) \]
\[ \geq \mu \| w \|_{L^2(\mathbb{R}^N)}^2 + o_R(1) \]
\[ \geq \mu \lim_{n \to \infty} \int_{B_R(0)} w_n^2 \ dx + o_R(1). \]

Similarly
\[ \lim_{n \to \infty} \left\{ \int_{B_R(k_n)} \sum_{i=1}^{N} a_{ij}(x, u_0 + k_n \ast u_0 + z_n) D_i w_n D_j w_n \ dx \right. \]
\[ + \int_{B_R(k_n)} \sum_{i=1}^{N} D_s a_{ij}(x, u_0 + k_n \ast u_0) D_i (u_0 + k_n \ast u_0) D_j w_n^2 \ dx \]
\[ + \int_{B_R(k_n)} \left( \frac{1}{2} \sum_{i=1}^{N} D^2 a_{ij}(x, u_0 + k_n \ast u_0) D_i (u_0 + k_n \ast u_0) D_j (u_0 + k_n \ast u_0) \right. \]
\[ + V(x) - D_s g(x, u_0 + k \ast u_0) w_n^2 \right) \ dx \} \]
\[ \geq (L \tilde{w}, \tilde{w}) + o_R(1) \]
\[ \geq \mu \lim_{n \to \infty} \int_{B_R(0)} w_n^2 \ dx + o_R(1) \]
\[ = \mu \lim_{n \to \infty} \int_{B_R(k_n)} w_n^2 (x) \ dx + o_R(1). \]
Note that, in the domain \( \Omega = \mathbb{R}^N \setminus (B_R(0) \cup B_R(k_n)) \), \( D(u_0 + k_n \ast u_0) \) and \( u_0 + k_n \ast u_0 \) are small, and by \((f_1)\), \( D_s g(x, u_0 + k_n \ast u_0) \) is also small. Thus we have for \( c > 0 \) from \((V)\)

\[
\lim_{n \to \infty} \left\{ \int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(x, u_0 + k_n \ast u_0 + z_n) D_i w_n D_j w_n \, dx \\
+ \int_{\Omega} \sum_{i,j=1}^{N} D_s a_{ij}(x, u_0 + k_n \ast u_0) D_i (u_0 + k_n \ast u_0) D_j w_n^2 \, dx \\
+ \int_{\Omega} \left( \frac{1}{2} \sum_{i,j=1}^{N} D_s^2 a_{ij}(x, u_0 + k_n \ast u_0) D_i (u_0 + k_n \ast u_0) D_j (u_0 + k_n \ast u_0) \right) \, dx \\
+ \int_{\Omega} (V(x) - D_s g(x, u_0 + k \ast u_0)) w_n^2 \, dx \right\}
\geq \lim_{n \to \infty} \left( \int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(x, u_0 + k_n \ast u_0 + z_n) D_i w_n D_j w_n \, dx + \frac{1}{2} \int_{\Omega} V(x) w_n^2 \, dx \right)
+ o_R(1) \| w_n \|_{H^1}^2
\geq c/2 \lim_{n \to \infty} \int_{\Omega} w_n^2 \, dx + o_R(1).
\]

Since \( R \) is arbitrary, \( \lim_{n \to \infty} (\tilde{L}_{k_n}(z_n) w_n, w_n) \geq \min(\mu, c/2) \lim_{n \to \infty} \int_{\mathbb{R}^N} w_n^2 \, dx = \min\{\mu, c/2\} \), a contradiction. \( \square \)

The following theorem is a counterpart of Theorem 5.1.

**Theorem 6.4.** There exist positive constants \( \delta_1, \delta_2, K \) such that for \( v, \tilde{v} \in V, \| v \|, \| \tilde{v} \| \leq \delta_1 \) and \( |k| \geq K \), the following statements (a), (b), (c), and (d) hold.

(a) There exists \( w \in W_k := V \perp \cap (k \ast V) \perp \) such that \( \| w \| \leq \delta_2 \), \( w \in W^{1,\infty}(\mathbb{R}^N) \) and \( u = u_0 + k \ast u_0 + v + k \ast \tilde{v} + w \) satisfies

\[
\{ DI(u), z \} = 0, \quad \text{for } z \in W_k \cap L^\infty(\mathbb{R}^N).
\]  
(6.2)

(b) Let \( \phi(v) \in W \equiv V \perp \) be the unique solution, obtained in Theorem 5.1, of the following equation:

\[
\{ DI(u_0 + v + \phi(v)), z \} = 0, \quad \forall z \in W \cap L^\infty(\mathbb{R}^N).
\]  
(6.3)

Then any solution \( w \) of (6.2) satisfies

\[
\| w - \phi(v) - k \ast \phi(\tilde{v}) \|_{H^1(\mathbb{R}^N)} + \| w - \phi(v) - k \ast \phi(\tilde{v}) \|_{L^\infty(\mathbb{R}^N)} = o_R(1).
\]  
(6.4)
(c) Let \( v_1, \tilde{v}_1, v_2, \tilde{v}_2 \in V \cap B_{\delta_1} \) and \( w_1, w_2 \in W_k \cap B_{\delta_2} \) such that \( u_i = u_0 + k * u_0 + v_i + k * \tilde{v}_i + w_i, \ i = 1, 2 \), satisfies
\[
\{ DI(u_i), z \} = 0, \quad \forall z \in W_k \cap L^\infty(\mathbb{R}^N).
\] (6.5)

Then for some constant \( C > 0 \)
\[
\| w_1 - w_2 \|_{H^1(\mathbb{R}^N)} + \| w_1 - w_2 \|_{L^\infty(\mathbb{R}^N)} \leq C(\| v_1 - v_2 \|_{H^1(\mathbb{R}^N)} + \| \tilde{v}_1 - \tilde{v}_2 \|_{H^1(\mathbb{R}^N)}).
\] (6.6)

Consequently, given \( v, \tilde{v} \in V \cap B_{\delta_1} \), the solution of (6.2) is unique, we denote by \( \phi(v, \tilde{v}) \) this unique solution.

(d) The function \( H(v, \tilde{v}) = I(u_0 + k * u_0 + v + k * \tilde{v} + \phi(v, \tilde{v})) \) is differentiable and
\[
(D_e H(v, \tilde{v}), z) = \{ DI(u), z \}, \quad (D_\tilde{v} H(v, \tilde{v}), z) = \{ DI(u), k * z \}, \quad z \in V.
\] (6.7)

Hence \( (v, \tilde{v}) \) is a critical point of \( H \) if and only if \( u = u_0 + k * u_0 + v + k * \tilde{v} + \phi(v, \tilde{v}) \) is a critical point of \( I \).

**Proof of part (a) of Theorem 6.4 (existence and regularity).** Let \( v, \tilde{v} \in V, w \in W_k \). By Lemmas 6.1, 6.2, 6.3 for \( \epsilon > 0 \) there exist \( \delta_1, \delta_2, K \) such that if \( \| v \|, \| \tilde{v} \| \leq \delta_1, \| w \| \leq \delta_2, |k| \geq K \), then on one hand

\[
I(u_0 + k * u_0 + v + k * \tilde{v} + w)
\geq I(u_0 + k * u_0) + \{ DI(u_0 + k * u_0), v + k * \tilde{v} + w \} + \frac{1}{2} (\mathcal{L}_k(v + k * \tilde{v}), v + k * \tilde{v})
+ (\tilde{\mathcal{L}}(v + k * v), w) + \frac{1}{2} (\mathcal{L}_k(w)w, w) - \epsilon (\| v \|^2 + \| \tilde{v} \|^2 + \| w \|^2)
\geq I(u_0 + k * u_0) + \{ DI(u_0 + k * u_0), v + k * \tilde{v} \} - \epsilon_1 \| w \|
- C_1 (\| v \|^2 + \| \tilde{v} \|^2) - \epsilon_2 (\| v \| + \| \tilde{v} \|) \| w \| + \mu \| w \|^2 - \epsilon (\| v \|^2 + \| \tilde{v} \|^2 + \| w \|^2)
\geq I(u_0 + k * u_0) + \{ DI(u_0 + k * u_0), v + k * \tilde{v} \} + \mu_1 \| w \|^2 - \epsilon_1 \| w \| - C_1 (\| v \|^2 + \| \tilde{v} \|^2)
\]

for some \( \mu_1, C_1 > 0 \) and \( \epsilon_1 \to 0 \) as \( k \to \infty \). On the other hand

\[
I(u_0 + k * u_0 + v + k * \tilde{v})
\leq I(u_0 + k * u_0) + \{ DI(u_0 + k * u_0), v + k * \tilde{v} \}
+ \frac{1}{2} (\mathcal{L}_k(v + k * \tilde{v}), v + k * \tilde{v}) + \epsilon (\| v \|^2 + \| \tilde{v} \|^2)
\leq I(u_0 + k * u_0) + \{ DI(u_0 + k * u_0), v + k * \tilde{v} \} + C_1 (\| v \|^2 + \| \tilde{v} \|^2).
\]

Let \( \epsilon_1 \leq \frac{1}{4} \mu_1 \delta_2, 4C_1 \delta_1^2 \leq \frac{1}{4} \mu_1 \delta_2^2 \). Consider the function \( g(w) = I(u_0 + k * u_0 + v + k * \tilde{v} + w), \ w \in W_k, \| w \| \leq \delta_2 \). If \( \| w \| = \delta_2 \), then
\[
g(w) \geq I(u_0 + k \ast u_0) + \{DI(u_0 + k \ast u_0), v + k \ast \bar{v}\} + \mu_1 \delta_2^2 - \epsilon_1 \delta_2 - 2C_1 \delta_1^2
\]
\[
\geq I(u_0 + k \ast u_0) + \{DI(u_0 + k \ast u_0), v + k \ast \bar{v}\} + \frac{1}{2} \mu_1 \delta_2^2
\]
\[
> I(u_0 + k \ast u_0) + \{DI(u_0 + k \ast u_0), v + k \ast \bar{v}\} + 2C_1 (\|v\|^2 + \|\bar{v}\|^2)
\]
\[
\geq g(0).
\]

So if \(w \in W_k\), \(\|w\| \leq \delta_2\), assumes the minimum of \(g\), then \(\|w\| < \delta_2\) and solves the equation \(\langle DI(u), z \rangle = 0\), for \(z \in W_k \cap L^\infty(\mathbb{R}^N)\) with \(u = u_0 + k \ast u_0 + v + k \ast \bar{v} + w\).

Now let \(\{w_n\} \subset W_k\), \(\|w_n\| \leq \delta_2\) be a minimizing sequence. Assume \(w_n \rightharpoonup w\) in \(H^1(\mathbb{R}^N)\), \(w_n \to w\) in \(L^p_{loc}(\mathbb{R}^N)\), \(2 \leq p < \frac{2N}{N-2}\) and \(w_n(x) \rightharpoonup w(x)\), a.e. \(x \in \mathbb{R}^N\). Choose \(R\) large such that

\[
\|u_0 + k \ast u_0 + v + k \ast \bar{v}\|_{H^1(\mathbb{R}^N \setminus B_R(0))} \leq \epsilon.
\]

We have for \(u = u_0 + k \ast u_0 + v + k \ast \bar{v} + w_n\)

\[
\frac{1}{2} \int_{\mathbb{R}^N \setminus B_R(0)} \sum_{i,j=1}^N a_{ij}(x,u) D_i u D_j u \, dx + \frac{1}{2} \int_{\mathbb{R}^N \setminus B_R(0)} V(x) u^2 \, dx - \int_{\mathbb{R}^N \setminus B_R(0)} G(x,u) \, dx
\]
\[
\geq c \int_{\mathbb{R}^N \setminus B_R(0)} (|Dv|^2 + u^2) \, dx - \int_{\mathbb{R}^N \setminus B_R(0)} (\epsilon |u|^2 + C_\epsilon |u|^{\frac{2N}{N-2}}) \, dx
\]
\[
\geq c \int_{\mathbb{R}^N \setminus B_R(0)} (|Dv|^2 + u^2) \, dx - \epsilon \int_{\mathbb{R}^N \setminus B_R(0)} u^2 \, dx - C_\epsilon \left( \int_{\mathbb{R}^N \setminus B_R(0)} (|Dv|^2 + u^2) \, dx \right)^{\frac{N}{N-2}}
\]
\[
\geq 0
\]

provided \(\delta_2\) is small enough. Therefore for \(u_n = u_0 + k \ast u_0 + v + k \ast \bar{v} + w_n\)

\[
\lim_{n \to \infty} I(u_n) \geq \lim_{n \to \infty} \left\{ \frac{1}{2} \int_{B_R(0)} \sum_{i,j=1}^N a_{ij}(x,u_n) D_i u_n D_j u_n \, dx \right. 
\]
\[
+ \frac{1}{2} \int_{B_R(0)} V(x) u_n^2 \, dx - \int_{B_R(0)} G(x,u_n) \left. \right\}
\]
\[
\geq \frac{1}{2} \int_{B_R(0)} \sum_{i,j=1}^N a_{ij}(x,u) D_i u D_j u \, dx + \frac{1}{2} \int_{B_R(0)} V(x) u^2 \, dx - \int_{B_R(0)} G(x,u) \, dx
\]

where \(u = u_0 + k \ast u_0 + v + k \ast \bar{v} + w\). Letting \(R \to \infty\), we get \(g(w) = I(u) = \inf\{g(w) | w \in W_k, \|w\| \leq \delta_2\}\), and that \(w\) is a minimizer and solves \(\langle DI(u), z \rangle = 0, \forall z \in W_k \cap L^\infty(\mathbb{R}^N)\).
Now let $w$ be any solution of the problem (6.2) and $u = u_0 + k \ast u_0 + v + k \ast \tilde{v} + w$. Let $f \in V \oplus k \ast V$ satisfy

$$\langle DI(u), z \rangle = \int_{\mathbb{R}^N} f z \, dx, \quad \forall z \in V \oplus k \ast V.$$ 

Then $u$ satisfies

$$\langle DI(u), z \rangle = \int_{\mathbb{R}^N} f z \, dx, \quad \forall z \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N).$$ 

By Moser’s iteration, we have

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leq C \|f\|_{L^\infty(\mathbb{R}^N)} \leq C \|u\|_{H^1(\mathbb{R}^N)}.$$ 

By the regularity theory, 

$$\|Du\|_{L^2(\mathbb{R}^N)} \leq C \|u\|_{L^\infty(\mathbb{R}^N)} \leq C \|u\|_{H^1(\mathbb{R}^N)}.$$ 

Proof of part (b) of Theorem 6.4. Denote $u = u_0 + v + \phi(v)$, $\tilde{u} = u_0 + \tilde{v} + \phi(\tilde{v})$. Since $w = \phi(v)$ has exponential decay uniform in $v \in V$, $\|v\| \leq \delta_1$, we can prove

$$\langle DI(u + k \ast \tilde{u}), z \rangle = \langle DI(u), z \rangle + \langle DI(k \ast \tilde{u}), z \rangle + o_R(1)\|z\|, \quad \forall z \in H^1(\mathbb{R}^N). \quad (6.8)$$ 

As before, we choose $R > 0$, $|k| \geq 3R$, and estimate the integrals over $B_R(0)$, $B_R(k)$ and $\Omega = \mathbb{R}^N \setminus (B_R(0) \cup B_R(k))$ respectively.

The integrals over $B_R(0)$ and $B_R(k)$ are equal to $\langle DI(u), z \rangle + o_R(1)\|z\|$ and $\langle DI(k \ast \tilde{u}), z \rangle + o_R(1)\|z\|$, respectively, and the integrals over $\Omega$ is small. As an example, we estimate the integral

$$\int_{B_R(0)} \sum_{i,j=1}^N a_{ij}(x, u + k \ast \tilde{u})D_i(u + k \ast \tilde{u})D_j z \, dx$$

$$= \int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij}(x, u)D_i u D_j z \, dx - \int_{\mathbb{R}^N \setminus B_R(0)} \sum_{i,j=1}^N a_{ij}(x, u)D_i u D_j z \, dx$$

$$+ \int_{B_R(0)} \sum_{i=1}^N a_{ij}(x, u + k \ast \tilde{u})D_i(k \ast \tilde{u})D_j z \, dx$$

$$+ \int_{B_R(0)} \sum_{i,j=1}^N (a_{ij}(x, u + k \ast u) - a_{ij}(x, u))D_i u D_j z \, dx$$

$$= \int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij}(x, u)D_i u D_j z \, dx + o_R(1)\|z\|.$$
Similarly
\[
\int_{\mathbb{R}^N} \sum_{ij=1}^{N} a_{ij}(x, u + k \ast \tilde{u}) D_i(u + k \ast \tilde{u}) D_j z \, dx
\]
\[
= \int_{\mathbb{R}^N} \sum_{ij=1}^{N} a_{ij}(x, k \ast \tilde{u}) D_i(k \ast \tilde{u}) D_j z \, dx + o_R(1) \|z\|,
\]
\[
\int_{\Omega} \sum_{ij=1}^{N} a_{ij}(x, u + k \ast \tilde{u}) D_i(u + k \ast \tilde{u}) D_j z \, dx = o_k(1) \|z\|,
\]
and
\[
\int_{\mathbb{R}^N} \sum_{ij=1}^{N} a_{ij}(x, u + k \ast \tilde{u}) D_i(u + k \ast \tilde{u}) D_j z \, dx
\]
\[
= \int_{\mathbb{R}^N} \sum_{ij=1}^{N} a_{ij}(x, u) D_i u D_j z \, dx + \int_{\mathbb{R}^N} \sum_{ij=1}^{N} a_{ij}(x, k \ast u) D_i(k \ast u) D_j z \, dx + o_k(1) \|z\|.
\]

Let \( \{\varphi_1, \varphi_2, \ldots, \varphi_d\} \) be an orthogonal base of \( V \subset L^2 \). Let \( f = \sum_{j=1}^{d} \varphi_j \langle DI(u), \varphi_j \rangle \), \( \tilde{f} = \sum_{j=1}^{d} \varphi_j \langle DI(\tilde{u}), \varphi_j \rangle \). Moreover choose \( F \in V \oplus k \ast V \) such that \( \langle DI(U), \varphi \rangle = \int_{\mathbb{R}^N} F \varphi \, dx \), \( \forall \varphi \in V \oplus k \ast V \), where \( U = u_0 + v + k \ast (u_0 + \tilde{v}) + w \). If the spaces \( V \) and \( k \ast V \) were orthogonal, then \( F \) would be equal to
\[
\hat{F} = \sum_{j=1}^{d} \varphi_j \langle DI(U), \varphi_j \rangle + \sum_{j=1}^{d} k \ast \varphi_j \langle DI(U), k \ast \varphi_j \rangle.
\]

Though \( V \) and \( k \ast V \) are not exactly orthogonal, it is easy to check that the difference \( F - \hat{F} \) is small:
\[
F = \sum_{j=1}^{d} \varphi_j \langle DI(U), \varphi_j \rangle + \sum_{j=1}^{d} k \ast \varphi_j \langle DI(U), k \ast \varphi_j \rangle + \epsilon(|k|), \quad (6.9)
\]
where \( \|\epsilon(|k|)\|_{L^\infty(\mathbb{R}^N)} \to 0 \), as \( |k| \to \infty \). Thus
\[
F - f - k \ast \tilde{f}
\]
\[
= \sum_{j=1}^{d} \varphi_j \langle DI(U) - DI(u), \varphi_j \rangle + \sum_{j=1}^{d} k \ast \varphi_j \langle DI(U) - DI(k \ast \tilde{u}), k \ast \varphi_j \rangle + \epsilon(|k|)
\]
\[
= \sum_{j=1}^{d} \varphi_j \langle DI(U) - DI(u) - DI(k \ast \tilde{u}), \varphi_j \rangle
\]
\[ + \sum_{j=1}^{d} k \ast \varphi_j(DI(U) - DI(u) - DI(k \ast \tilde{u}), k \ast \varphi_j) + \epsilon(|k|) \]
\[ = \sum_{j=1}^{d} \varphi_j(DI(U) - DI(u + k \ast \tilde{u}), \varphi_j) \]
\[ + \sum_{j=1}^{d} k \ast \varphi_j(DI(U) - DI(u + k \ast \tilde{u}), k \ast \varphi_j) + \epsilon(|k|). \]

Hence
\[ \|F - f - k \ast \tilde{f}\|_{L^\infty(\mathbb{R}^N)} \leq C \|U - (u + k \ast \tilde{u})\|_{H^1(\mathbb{R}^N)} + o_k(1) \]
\[ = C \|\phi(v, \tilde{v}) - \phi(v) - \phi(\tilde{v})\|_{H^1(\mathbb{R}^N)} + o_k(1). \quad (6.10) \]

Now
\[ (DI(u_0 + v + k*(u_0 + \tilde{v}) + w) - DI(u_0 + v + k*(u_0 + \tilde{v}) + \phi(v) + k \ast \phi(\tilde{v}), z) \]
\[ = \int_{\mathbb{R}^N} (F - f - k \ast \tilde{f})z \, dx + \int_{\mathbb{R}^N} \epsilon_j D_j z \, dx + \int_{\mathbb{R}^N} \epsilon z \, dx = 0, \quad z \in H^1(\mathbb{R}^N) \quad (6.11) \]

where \( \|\epsilon_j\|_{H^1(\mathbb{R}^N)}, \|\epsilon_j\|_{L^\infty(\mathbb{R}^N)}, \|\epsilon\|_{H^1(\mathbb{R}^N)}, \|\epsilon\|_{L^\infty(\mathbb{R}^N)} \to 0 \) as \( |k| \to \infty \). Linearizing the left-hand of the above equation we get a linear elliptic equation for \( w - \phi(v) - k \ast \phi(\tilde{v}) \). We have
\[ \|w - \phi(v) - k \ast \phi(\tilde{v})\|_{L^\infty(\mathbb{R}^N)} \leq C \|w - \phi(v) - k \ast \phi(\tilde{v})\|_{H^1(\mathbb{R}^N)} + \|F - f - k \ast \tilde{f}\|_{L^\infty(\mathbb{R}^N)} \]
\[ + \sum_{j=1}^{N} \|\epsilon_j\|_{L^\infty(\mathbb{R}^N)} + \|\epsilon\|_{L^\infty(\mathbb{R}^N)} \]
\[ \leq C \|w - \phi(v) - k \ast \phi(\tilde{v})\|_{H^1(\mathbb{R}^N)} + o_k(1). \quad (6.12) \]

Since \( u_0 \) is a critical point of \( I \), we have \( \phi(0) = 0 \). It follows from the above estimate, (5.14) and (5.16) that
\[ \|w\|_{L^\infty(\mathbb{R}^N)} \leq C \left( \|w\|_{H^1(\mathbb{R}^N)} + \delta_1 \right) + o_k(1). \quad (6.13) \]

Note that \( \|w\|_{H^1(\mathbb{R}^N)} \leq \delta_2 \). We can assume \( \|w\|_{L^\infty(\mathbb{R}^N)} \) to be small, for suitable choice of \( \delta_1, \delta_2 \), and \( k \). The above estimate holds for any solution \( w \) of (6.2). We will use this fact in proving (d) of Theorem 6.4.

We need to estimate \( \|w - \phi(v) - k \ast \phi(\tilde{v})\|_{H^1(\mathbb{R}^N)} \). Let \( Q \) be the orthogonal projection to \( V \oplus k \ast V \subset L^2(\mathbb{R}^N) \). Since \( V \) and \( k \ast V \) are “almost” orthogonal in the sense that \( \int_{\mathbb{R}^N} v k \ast \tilde{v} \, dx = \epsilon(|k|) \to 0 \) as \( |k| \to \infty \) uniformly in \( v, \tilde{v} \in V, \|v\|, \|\tilde{v}\| \leq 1 \), we have
\[ P \phi(v) = \phi(v) - Q(\phi(v)) = \phi(v) + o_k(1), \]
\[ P k \ast \phi(\tilde{v}) = k \ast \phi(\tilde{v}) - Q(k \ast \phi(\tilde{v})) = k \ast \phi(\tilde{v}) + o_k(1). \]
By Lemma 6.3, we have
\[
\mu \| w - \phi(v) - k \phi(\bar{v}) \|^2_{H^1(\mathbb{R}^N)} \\
\leq \mu \| w - P(\phi(v) + k \phi(\bar{v})) \|^2 + o_k(1) \\
\leq C_k(w - \phi(v) - k \phi(\bar{v}), w - \phi(v) - k \phi(\bar{v})) + o_k(1) \\
- \langle DI(u_0 + v + k(u_0 + \bar{v}) + w), DI(u_0 + v + \phi(v)) \rangle \\
- \langle DI(k(u_0 + \bar{v}) + k \phi(\bar{v})), w - \phi(v) - k \phi(\bar{v}) \rangle + o_k(1) \\
= \langle D^2I(u_0 + k(u_0))(\phi(v, \bar{v}) - \phi(v) - k \phi(v)) - DI(u_0 + v + k(u_0 + \bar{v}) + w), \rangle \\
- DI(u_0 + v + k(u_0 + \bar{v}) + \phi(v) + k \phi(\bar{v})), w - \phi(v) - k \phi(\bar{v}) \rangle + o_k(1) \\
\leq \epsilon_0 \| w - \phi(v) - k \phi(\bar{v}) \|^2_{H^1(\mathbb{R}^N)} + \| w - \phi(v) - k \phi(\bar{v}) \|^2_{L^\infty(\mathbb{R}^N)} + o_k(1)
\]
where \( \epsilon_0 \) depends on \( m = \| v \|_{H^1(\mathbb{R}^N)} + \| \bar{v} \|_{H^1(\mathbb{R}^N)} + \| \phi(v, \bar{v}) \|_{H^1(\mathbb{R}^N)} + \| \phi(v, \bar{v}) \|_{L^\infty(\mathbb{R}^N)} \) and
as \( m \to 0 \) (that is the case when \( \delta_1, \delta_2 \to 0 \), \( \epsilon_0 \to 0 \). It follows that
\[
\| w - \phi(v) - k \phi(\bar{v}) \|_{H^1(\mathbb{R}^N)} \leq \epsilon \| w - \phi(v) - k \phi(\bar{v}) \|_{L^\infty(\mathbb{R}^N)} + o_k(1), \tag{6.14}
\]
where \( \epsilon \) can be arbitrarily small. Finally, (b) follows from the above estimates (6.12), (6.14). \( \square \)

**Proof of part (c) of Theorem 6.4 (uniqueness and continuous dependence).** For \( i = 1, 2 \), let
\( v_i, \bar{v}_i \in V; \| v_i \|, \| \bar{v}_i \| \leq \delta_1; w_i \in W_k, \| w_i \| \leq \delta_2, u_i = u_0 + k * u_0 + v_i + k * \bar{v}_i + w_i \). Suppose for \( i = 1, 2 \)
\[
\{ DI(u_i), z \} = 0, \quad \forall z \in W_k \cap L^\infty(\mathbb{R}^N).
\]
Let \( f_i \in V \otimes k * V \) and satisfy \( (DI(u_i), z) = \int_{\mathbb{R}^N} f_i z \, dx \), for \( z \in V \otimes k * V \). Then
\[
\{ DI(u_i), z \} = \int_{\mathbb{R}^N} f_i z \, dx, \quad \forall z \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N).
\]
We have
\[
\{ DI(u_2) - DI(u_1), z \} = \int_{\mathbb{R}^N} (f_2 - f_1) z \, dx, \quad \forall z \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N).
\]
As in Section 5, we have the following linear elliptic equation for \( u_2 - u_1 \)
\[
\int_{\mathbb{R}^N} \sum_{i,j=1}^N A_{ij}(x) D_i(D(u_2 - u_1)D_j z \, dx + \int_{\mathbb{R}^N} \sum_{j=1}^n B_j(x) D_j((u_2 - u_1)z) \, dx + \int_{\mathbb{R}^N} D(u_2 - u_1)z \, dx \\
= \int_{\mathbb{R}^N} (f_2 - f_1) z \, dx, \quad \text{for } z \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N).
\]
Since \( \| f_2 - f_1 \|_{L^\infty(\mathbb{R}^N)} \leq C \| u_2 - u_1 \|_{H^1(\mathbb{R}^N)} \), we have

\[
\| w_2 - w_1 \|_{L^\infty(\mathbb{R}^N)} \leq C \big( \| w_2 - w_1 \|_{H^1(\mathbb{R}^N)} + \| v_2 - v_1 \|_{H^1(\mathbb{R}^N)} + \| \tilde{v}_2 - \tilde{v}_1 \|_{H^1(\mathbb{R}^N)} \big), \tag{6.15}
\]

where \( C = C(\| A_{ij} \|_{L^\infty}, \| B_j \|_{L^\infty}, \| D \|_{L^\infty}) = C(\| u_1 \|_{H^1(\mathbb{R}^N)}, \| u_2 \|_{H^1(\mathbb{R}^N)}). \) We use Lemma 6.3 to estimate \( \| w_2 - w_1 \|_{H^1(\mathbb{R}^N)} \) as follows;

\[
\mu \| w_2 - w_1 \|^2_{H^1(\mathbb{R}^N)} \\
\leq (\mathcal{L}_k(w_2 - w_1), (w_2 - w_1)) \\
= (\mathcal{L}_k(u_2 - u_1), (w_2 - w_1)) - (\mathcal{L}_k(v_2 + k \ast \tilde{v}_2 - v_1 - k \ast \tilde{v}_1), (w_2 - w_1)) \\
= (\mathcal{L}_k(u_2 - u_1) - (DI(u_2) - DI(u_1)), (w_2 - w_1)) \\
- (\mathcal{L}_k(v_2 + k \ast \tilde{v}_2 - v_1 - k \ast \tilde{v}_1), (w_2 - w_1)), \\
\big| (\mathcal{L}_k(v_2 + k \ast \tilde{v}_2 - v_1 - k \ast \tilde{v}_1), (w_2 - w_1)) \big| \leq o_R(1)(\| v_2 - v_1 \|_{H^1(\mathbb{R}^N)} + \| \tilde{v}_2 - \tilde{v}_1 \|_{H^1(\mathbb{R}^N)}) \| w_2 - w_1 \|_{H^1(\mathbb{R}^N)} \\
\leq o_R(1)(\| v_2 - v_1 \|^2_{H^1(\mathbb{R}^N)} + \| \tilde{v}_2 - \tilde{v}_1 \|^2_{H^1(\mathbb{R}^N)} + \| v_2 - v_1 \|^2_{H^1(\mathbb{R}^N)} + \| w_2 - w_1 \|^2_{H^1(\mathbb{R}^N)}), \\
(\mathcal{L}_k(u_2 - u_1) - (DI(u_2) - DI(u_1)), (w_2 - w_1)) \\
= \int_0^1 ((D^2I(u_0 + k \ast u_0) - D^2I(tu_2 + (1-t)u_1))(u_2 - u_1), w_2 - w_1) \, dt.
\]

Notice that

\[
(tu_2 + (1-t)u_1) - (u_0 + k \ast u_0) = t(v_2 + k \ast \tilde{v}_2 + w_2) + (1-t)(v_1 + k \ast \tilde{v}_1 + w_1),
\]

\[
\big| (\mathcal{L}_k(u_2 - u_1) - (DI(u_2) - DI(u_1)), (w_2 - w_1)) \big| \leq \epsilon(\| u_2 - u_1 \|_{H^1(\mathbb{R}^N)} + \| u_2 - u_1 \|_{L^\infty(\mathbb{R}^N)})(\| w_2 - w_1 \|_{H^1(\mathbb{R}^N)} + \| w_2 - w_1 \|_{L^\infty(\mathbb{R}^N)}) \\
\leq \epsilon(\| v_2 - v_1 \|^2_{H^1(\mathbb{R}^N)} + \| \tilde{v}_2 - \tilde{v}_1 \|^2_{L^\infty(\mathbb{R}^N)} + \| v_2 - v_1 \|^2_{H^1(\mathbb{R}^N)} + \| w_2 - w_1 \|^2_{L^\infty(\mathbb{R}^N)}),
\]

where \( \epsilon \to 0 \) as \( \| v_1 \|_{H^1(\mathbb{R}^N)}, \| \tilde{v}_1 \|_{H^1(\mathbb{R}^N)}, \| v_2 \|_{H^1(\mathbb{R}^N)}, \| \tilde{v}_2 \|_{H^1(\mathbb{R}^N)}, \| v_1 \|_{L^\infty(\mathbb{R}^N)}, \| w_2 \|_{L^\infty(\mathbb{R}^N)} \) tend to zero. From (6.13) in the proof of (b), Theorem 6.4, we know \( \| w_i \|_{L^\infty(\mathbb{R}^N)} \leq C(\| v_j \|_{H^1(\mathbb{R}^N)} + \delta_1) + \epsilon(|k|). \) Therefore for suitable choice of \( \delta_1, \delta_2 \) and \( K, \epsilon \) can be made arbitrary small. It follows from the above estimates that

\[
\| w_2 - w_1 \|_{H^1(\mathbb{R}^N)} \\
\leq C \epsilon(\| v_2 - v_1 \|_{H^1(\mathbb{R}^N)} + \| \tilde{v}_2 - \tilde{v}_1 \|_{H^1(\mathbb{R}^N)} + \| v_2 - v_1 \|_{L^\infty(\mathbb{R}^N)}). \tag{6.16}
\]

Thus the part (c) Theorem 6.4 follows from (6.15) and (6.16). \( \square \)
Proof of part (d) Theorem 6.4 (finite dimensional reduction). By (a), (c) of Theorem 6.4, we have a Lipschitz continuous map \((v, \tilde{v}) \mapsto w = \phi(v, \tilde{v})\), which satisfies
\[
\{DI(u_0 + k \ast u_0 + v + k \ast \tilde{v} + \phi(v, \tilde{v})), \mathbb{R}_+\} = 0, \quad \forall z \in W_k \cap L^\infty(\mathbb{R}^N).
\] (6.17)

Let \(H(v, \tilde{v}) = I(u_0 + k \ast u_0 + v + k \ast \tilde{v} + \phi(v, \tilde{v}))\). We calculate \(D_vH, D_{\tilde{v}}H\). For \(h \in V\),
\[
H(v + h, \tilde{v}) - H(v, \tilde{v})
= I(u_0 + k \ast u_0 + v + h + k \ast \tilde{v} + \phi(v + h, \tilde{v})) - I(u_0 + k \ast u_0 + v + k \ast \tilde{v} + \phi(v, \tilde{v}))
= DI(u_0 + k \ast u_0 + v + k \ast \tilde{v} + \phi(v, \tilde{v})) + \langle D_vH, h \rangle + o(1)\|h\|_{L^1(\mathbb{R}^N)},
\]
where \(u = u_0 + k \ast u_0 + v + k \ast \tilde{v} + \phi(v, \tilde{v})\). Hence \(\langle D_vH(v, \tilde{v}), h \rangle = DI(u, h)\). In the same way, \(\langle D_{\tilde{v}}H(v, \tilde{v}), h \rangle = DI(u, k \ast h)\), for \(h \in V\). If \((v, \tilde{v})\) is a critical point of \(H\), then \(\langle DI(u), h \rangle = 0\) for \(h \in V \oplus k \ast V\), which, together with (6.17), implies that \(u\) is a critical point of \(I\). \(\square\)

Proof of Theorem B (Existence of two-bump solutions). Let \(u_0\) be an isolated critical point of \(I\) and the \(q\)-th critical group of \(u_0, C_q(I, u_0)\) is nontrivial. Consider the function \(h(v) = I(u_0 + v + \phi(v)), v \in B_{\delta_1} = \{v \in V | \|v\| \leq \delta_1\}, v = 0\) is an isolated critical point of \(h\) in \(B_{\delta_1}^V\), let \(\psi\) be a pseudo-gradient vector field of \(h\) such that
\[
\|\psi(v)\| \leq 2\|Dh(v)\|, \quad (Dh(v), \psi(v)) \geq \|\psi(v)\|^2.
\] (6.18)

Consider a family of functions
\[
H_s(v, \tilde{v}) = (1 - s)H_0(v, \tilde{v}) + sH(v, \tilde{v}), \quad s \in [0, 1]
\] (6.19)
where \(H_0(v, \tilde{v}) = h(v) + h(\tilde{v})\). We have
\[
(D_vH(v, \tilde{v}), \psi(v))
= (1 - s)(Dh(v), \psi(v)) + s(D_vH(v, \tilde{v}), \psi(v))
= (1 - s)(Dh(v), \psi(v)) + s\{DI(u_0 + v + k \ast u_0 + k \ast \tilde{v} + \phi(v, \tilde{v})), \psi(v)\}
= (1 - s)(Dh(v), \psi(v)) + s\{DI(u_0 + v + k \ast u_0 + k \ast \tilde{v} + \phi(v) + k \ast \phi(\tilde{v}) + o_k(1)), \psi(v)\}
= (1 - s)(Dh(v), \psi(v)) + s\{DI(u_0 + v + \phi(v)), \psi(v)\} + o_k(1)
= (Dh(v), \psi(v)) + o_k(1)
\geq \|\psi(v)\|^2 + o_k(1).
\]
Similarly
\[(D\tilde{v}H(v, \tilde{v}), \psi(\tilde{v})) \geq \|\psi(\tilde{v})\|^2 + o_k(1).\] (6.20)

Denote \(DH_s(v, \tilde{v}) = (DvH_s(v, \tilde{v}), D\tilde{v}H_s(v, \tilde{v})), \psi(v, \tilde{v}) = (\psi(v), \psi(\tilde{v})).\) Then
\[(DH_s(v, \tilde{v}), \psi(v, \tilde{v})) \geq \|\psi(v, \tilde{v})\|^2 + o_k(1),\]
\[\|DH_s(v, \tilde{v})\|^2 \geq \|\psi(v, \tilde{v})\|^2 + o_k(1).\] (6.21)

Consequently, for \(|k|\) large enough, \(B^V_{\delta_1} \times B^V_{\delta_1}\) is an isolated neighborhood block for all flows generated by the pseudo gradient vector fields of \(H_s, s \in [0, 1].\) Let \(S\) be the maximal invariant set of \(H\) in \(B^V_{\delta_1} \times B^V_{\delta_1} \). Then
\[
\text{Ind}(S) = \text{Ind}(\{(0, 0)\})
\]
where \(\text{Ind}\) denotes the Conley index \([14]\), and \(S_0 = \{(0, 0)\}\) is the maximal invariant set of \(H_0\) in \(B^V_{\delta_1} \times B^V_{\delta_1}\). Since the \(2q\)-th homological group of \(\text{Ind}(S_0)\) is nontrivial, \(S_0\) is nontrivial. Consequently, \(S\) is nontrivial and must contain critical points of \(H\).

**Proof of Theorem C.** We assume that the condition \((Z^*)\) holds, otherwise we are done. By Theorem A, the functional \(I = I_M\) has a nontrivial critical point of mountain pass type with \(C_1(I, u_0) \neq 0.\) By Theorem B, \(I\) has a two-bump critical point near \(u_0 + k* u_0\) for all \(k \in Z\) with \(|k|\) large enough. By the \(L^\infty\) estimates (Lemma 2.4, Theorem 5.3, Theorem 6.4) these critical points of \(I\) are critical points of \(J\) too, that is, they are solutions of the quasilinear equations (1.1).

**Acknowledgments**

The authors would like to thank the referees for carefully reading an earlier version of the paper and thoughtful suggestions. The research of the first and the third authors is supported by NSFC (11171171).

**References**