Topology and its Applications 157 (2010) 102–107

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

On spaces which are linearly D

Guo Hongfeng [a,b,*,1], Heikki Junnila [c,2]

[a] School of Statistics and Mathematics, Shandong University of Finance, Jinan, PR China
[b] School of Mathematics, Shandong University, Jinan, PR China
[c] Department of Mathematics and Statistics, University of Helsinki, Helsinki, Finland

ARTICLE INFO

MSC:
54A25
54D20

Keywords:
D-space
SubmetaLindelöf
Linearly Lindelöf
Countably compact

ABSTRACT

We introduce a generalization of D-spaces, which we call linearly D-spaces. The following results are obtained for a T_1-space X.

- X is linearly Lindelöf if, and only if, X is a linearly D-space of countable extent.
- X is linearly D provided that X is submetaLindelöf.
- X is linearly D provided that X is the union of finitely many linearly D-subspaces.
- X is compact provided that X is countably compact and X is the union of countably many linearly D-subspaces.

0. Introduction

The D-spaces of E.K. van Douwen have been the object of intensive study in recent years, and many new classes of D-spaces have been discovered (see, for example, [8,2,9,17]). However, many basic problems on D-spaces remain unsolved. For instance, it is not known whether every regular Lindelöf space, or every paracompact Hausdorff space, is a D-space; these questions were already asked by van Douwen in the 1970's. More recently, A.V. Arhangel'skii has asked whether a space is D provided the space is the union of finitely many D-subspaces [2].

In this paper, we introduce a generalization of D-spaces which we call *linearly* D-spaces, and we show that the problems on D-spaces mentioned above have positive solutions for linearly D-spaces: every submetaLindelöf T_1-space is linearly D, and a space is linearly D provided it is the union of finitely many linearly D subspaces.

Terminology and notation

For a family $\mathcal{L}$ of sets and for a set A, we set $(\mathcal{L})_A = \{L \in \mathcal{L}\colon\ L \cap A \neq \emptyset\}$; if $A = \{x\}$, then we write $(\mathcal{L})_x$ in room of $(\mathcal{L})_A$.

A cover $\mathcal{L}$ of X is a *trivial cover* if $X \in \mathcal{L}$; otherwise, $\mathcal{L}$ is a *non-trivial cover*.

A family $\mathcal{L}$ of sets is *monotone* if $\mathcal{L}$ is linearly ordered by inclusion, and $\mathcal{L}$ is *well-monotone* if $\mathcal{L}$ is well-ordered by proper inclusion. We recall two well-known results on monotone covers: (i) every monotone cover has a well-monotone subcover; (ii) the least cardinal of a subcover of a monotone cover is a regular cardinal. These two results can be combined into one: every monotone cover contains subcover of the form $\{L_\alpha\colon\ \alpha < \kappa\}$, where κ is a regular cardinal and the family $\{L_\alpha\colon\ \alpha < \kappa\}$ is *strictly increasing*, that is, $L_\alpha \subsetneqq L_\beta$ whenever $\alpha < \beta < \kappa$.

* Corresponding author.
E-mail addresses: ghf@mail.sdu.edu.cn (H.F. Guo), heikki.junnila@helsinki.fi (H. Junnila).

[1] Supported by Doctor's Fundation of Shandong University of Finance 08BSJJ29. Work done during the author's visit to the University of Helsinki, supported by China Scholarship Council.
[2] Partially supported by Natural Science Foundation of China grant 10671173.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2009.04.044

A *neighbournet* (an *open neighbournet*) of a topological space X is a binary relation U on X such that, for every $x \in X$, the set $U\{x\}$ is a neighbourhood (an open neighbourhood) of x in X; a *transitive neighbournet* of X is a neighbournet of X which is transitive as a binary relation on X [18].

1. *D*-spaces and far points

A space X is a *D-space* provided that, for every neighbournet U of X, there exists a closed discrete subset D of X such that $UD = X$ [13].

Buzyakova, Tkachuk and Wilson call a subset A of X a *kernel* of a neighbournet U of X if $UA = X$ [10]. In this terminology, the defining property of a D-space is simply stated: *every neighbournet has a closed discrete kernel.*

D-spaces were introduced by van Douwen in the mid-1970's. During that time van Douwen was also studying special points of Čech–Stone remainders, and it is conceivable that the motivation for D-spaces arose in connection with his studies of "far points". We shall indicate below a connection between D-spaces and the non-existence of far points.

Recall that van Douwen called a point p of the remainder $X^* = \beta X \setminus X$ a *far point* of (the Tihonov space) X provided that p is *not* in the closure, in βX, of any closed and discrete subset of X. He showed in [11, Lemma 3.1] that a normal space X has no far points if, and only if, for every closed filterbase $\mathcal{F}$ of X, there exists a closed discrete $S \subset X$ such that $S \cap F \neq \emptyset$ for every $F \in \mathcal{F}$. With the help of this result, it is easy to establish the following.

Proposition 1.1. *If a normal space has no far points, then the space is a D-space.*

Proof. Assume on the contrary that there exists a normal space X which is not a D-space and has no far points. Let U be an open neighbournet of X without a closed discrete kernel. We set $\mathcal{F} = \{X \setminus UD\colon\ D \subset X \text{ is closed and discrete}\}$, and we note that $\mathcal{F}$ is a family of non-empty closed subsets of X. Moreover, $\mathcal{F}$ is closed under finite intersections. To see this, let $D_1, \dots, D_n$ be closed discrete subsets of X. Then we have that

$$\bigcap_{i=1}^{n}(X \setminus UD_i) = X \Big\backslash \bigcup_{i=1}^{n} UD_i = X \setminus U\left(\bigcup_{i=1}^{n} D_i\right),$$

and it follows, since the set $\bigcup_{i=1}^{n} D_i$ is closed and discrete, that $\bigcap_{i=1}^{n}(X \setminus UD_i) \in \mathcal{F}$.

By the foregoing and [11, Lemma 3.1], there exists a closed discrete set D in X such that we have $D \cap F \neq \emptyset$ for every $F \in \mathcal{F}$. This, however, leads to a contradiction, since we have that $D \subset UD$ and $X \setminus UD \in \mathcal{F}$. □

The above result is not too useful, since the non-existence of far points is a very strong condition on a Tihonov space: even spaces like $\mathbb{R}$ or $\mathbb{Q}$ fail to satisfy this condition (see [11]). However, the result does suggest some more reasonable conditions to consider. We could, for instance, restrict the class of closed filterbases for which we require the existence of a closed discrete set meeting every member of the filterbase. If instead of all closed filterbases we consider only monotone families of non-empty closed sets, then we arrive at a property which turns out to be weaker than the D-space property. In the following, we shall study spaces with this property under the name "linearly D-spaces".

2. Linearly *D*-spaces

If we only require the existence of closed discrete kernels for some special kinds of neighbournets, then we obtain weakenings of the D-space property. One property obtained in this manner could be called the "transitive D-property": every transitive neighbournet of X has a closed discrete kernel. In this paper, we do not consider the transitive D-property, but another property, which turns out to be even weaker.

Definition 2.1. A neighbournet U of X is *monotone* provided that $\{U\{x\}\colon\ x \in X\}$ is a monotone family of sets. A space X is *linearly D* provided that every monotone neighbournet of X has a closed discrete kernel.

We note that linearly D-spaces need not satisfy any reasonable separation axioms since the set $\{0, 1\}$ equipped with the Sierpinski topology $\{\emptyset, \{0, 1\}, \{1\}\}$ is a D-space. It is easy to see that the following property holds in a linearly D-space: every non-empty closed subset contains a closed singleton set. In the following, we often need the property that *every* singleton set is closed, and therefore practically all the results below are stated for T_1-spaces.

Let $\mathcal{L}$ be a family of sets. We say that a set A is *$\mathcal{L}$-small* if there exists $L \in \mathcal{L}$ such that $A \subset L$; if A is not $\mathcal{L}$-small, then A is *$\mathcal{L}$-big*.

In our first result, we characterize linearly D T_1-spaces by the condition that every non-trivial monotone open cover $\mathcal{U}$ of the space has a closed discrete $\mathcal{U}$-big set. It is easy to see that this condition is equivalent with the condition mentioned in the previous section that for every monotone family of non-empty closed sets, there exists a closed discrete set meeting every member of the family.

Theorem 2.2. *The following are equivalent for a T_1-space X:*

A. *X is linearly D.*
B. *For every non-trivial monotone open cover $\mathcal{U}$ of X, there exists a closed discrete $\mathcal{U}$-big set in X.*
C. *For every subset $A \subset X$ of uncountable regular cardinality κ, there is a closed discrete subset B of X, such that for every neighbourhood U of B, we have $|U \cap A| = \kappa$.*

Proof. A $\Rightarrow$ B: Assume that X is linearly D. To show that Condition B holds, let $\mathcal{U}$ be a monotone open cover of X with $X \notin \mathcal{U}$. Let $\prec$ be a well-ordering of $\mathcal{U}$, and define a monotone neighbournet W by setting $W\{x\} = \min_\prec(\mathcal{U})_x$. Let D be a closed discrete kernel for W. Then D is $\mathcal{U}$-big, because for every $U \in \mathcal{U}$, we have that $WU \subset \min_\prec\{V \in \mathcal{U}\colon U \subset V\} \neq X$.

B $\Rightarrow$ A: Assume that X satisfies Condition B. To show that X is linearly D, let U be a monotone neighbournet of X. If there exists $p \in X$ with $U\{p\} = X$, then $\{p\}$ is a closed discrete kernel for U. Assume that there is no such p. Then the family $\mathcal{U} = \{\operatorname{Int} U\{x\}\colon x \in X\}$ is a non-trivial monotone open cover of X. Since B holds, there exists a $\mathcal{U}$-big closed discrete set D in X. We show that D is a kernel of U. Let $x \in X$. Since D is $\mathcal{U}$-big and $\operatorname{Int} U\{x\} \in \mathcal{U}$, there exists $d \in D$ such that $d \notin \operatorname{Int} U\{x\}$. Since we have that $d \in \operatorname{Int} U\{d\} \setminus \operatorname{Int} U\{x\}$ and since the neighbournet U is monotone, we must have that $U\{x\} \subset U\{d\}$. As a consequence, we have that $x \in U\{d\} \subset UD$. We have shown that D is a kernel for U.

B $\Rightarrow$ C: Assume that B holds, and let A be a subset of X with no complete accumulation point such that the cardinal $\kappa = |A|$ is uncountable and regular. Write $A = \{x_\alpha\colon \alpha < \kappa\}$ with $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$. For every $\beta < \kappa$, let $U_\beta = X \setminus \overline{\{x_\alpha\colon \alpha \geqslant \beta\}}$. Note that the family $\mathcal{U} = \{U_\beta\colon \beta < \kappa\}$ is well-monotone and open. Moreover, $\mathcal{U}$ covers X. To see this, let $x \in X$. Since x is not a point of complete accumulation for A, there exists a neighbourhood O of X and an ordinal $\beta < \kappa$ such that $O \cap A \subset \{x_\alpha\colon \alpha < \beta\}$. We now have that $x \in U_\beta$.

Since X satisfies Condition B, there exists a closed discrete $\mathcal{U}$-big set F in X. To complete the proof of B $\Rightarrow$ C, we show that we have $|G \cap A| = \kappa$ for every open neighbourhood G of F. Assume on the contrary that there exists an open neighbourhood G of F such that $|G \cap A| < \kappa$. Let $\beta < \kappa$ be such that $G \cap A \subset \{x_\alpha\colon \alpha < \beta\}$. Then we have that $F \subset U_\beta$, but this is a contradiction, since F is $\mathcal{U}$-big.

C $\Rightarrow$ B: Assume that Condition C holds, and let $\mathcal{U}$ be a non-trivial monotone open cover of X. The monotone cover $\mathcal{U}$ has a strictly increasing subcover $\mathcal{V} = \{U_\alpha\colon \alpha < \kappa\}$ where the cardinal κ is regular. Every $\mathcal{V}$-big set is $\mathcal{U}$-big, and hence it suffices to show that there exists a closed discrete $\mathcal{V}$-big set in X. Note that if $\kappa = \omega$, then any choice of points $x_n \in U_{n+1} \setminus U_n$ gives a closed discrete $\mathcal{V}$-big set $\{x_n\colon n < \omega\}$. Assume that κ is uncountable. For every $\beta < \alpha$, let $x_\alpha \in U_{\alpha+1} \setminus U_\alpha$. Let $A = \{x_\alpha\colon \alpha < \kappa\}$. Since Condition C holds, there exists a closed discrete subset B in X such that we have $|G \cap A| = \kappa$ for every open neighbourhood G of B. The set B is $\mathcal{V}$-big, because otherwise there would exist $\beta < \kappa$ with $B \subset U_\beta$, and we would have that $|U_\beta \cap A| = |\{x_\alpha\colon \alpha < \beta\}| < \kappa$. $\square$

Remark. By the proof of C $\Rightarrow$ B above, we see that in Condition B, it would be enough to require that any well-monotone open cover $\mathcal{U}$ of X without a countable subcover has a closed discrete $\mathcal{U}$-big set.

With the help of this remark, we can show that a T_1-space is linearly D provided that every "well-monotone neighbournet" has a closed discrete kernel.

Corollary 2.3. *A T_1-space X is linearly D if, and only if, for every well-monotone open cover $\mathcal{U}$ of X, the neighbournet U of X, defined by the condition $U\{x\} = \bigcap(\mathcal{U})_x$, has a closed discrete kernel.*

Proof. *Necessity* of the condition is clear, since a neighbournet U as above is monotone.

Sufficiency follows using the above remark and the observation that, if $\mathcal{U}$ and U are as above and $X \notin \mathcal{U}$, then a kernel of the neighbournet U is $\mathcal{U}$-big. $\square$

Since a "well-monotone neighbournet" is clearly transitive, it follows that a space is linearly D provided that the space is "transitively D" (as defined at the beginning of this section).

The conclusion for the set A in Condition C of Theorem 2.2 can be expressed by saying that A has a closed and discrete "set of complete accumulation". This condition on all uncountable subsets of regular cardinality can also be stated in the form of a dichotomy between *points* of complete accumulation and certain closed discrete sets.

Proposition 2.4. *A T_1-space X is linearly D if, and only if, for every set $A \subset X$ of uncountable regular cardinality, either the set A has a complete accumulation point or there exists a closed discrete set D of size $|A|$ and a disjoint family $\{A_d\colon d \in D\}$ of subsets of A such that $d \in \overline{A_d}$ for every $d \in D$.*

Proof. *Sufficiency.* The stated condition clearly implies Condition C of Theorem 2.2.

Necessity. Assume that X is linearly D and $A \subset X$ is a set of uncountable regular cardinality κ which does not have a complete accumulation point. Let B satisfy Condition C of Theorem 2.2, with respect to the set A. For every $b \in B$, let V_b be a neighbourhood of b such that $|A \cap V_b| < \kappa$. Then, for every $E \subset B$ with $|E| < \kappa$, we have that $|A \cap \bigcup_{e \in E} V_e| < \kappa$ and it follows, since B is a "set of complete accumulation for A", that the set $B \cap \overline{A \setminus \bigcup_{e \in E} V_e}$ is non-empty. With the help of this observation and transfinite recursion, we can define distinct points d_α, $\alpha < \kappa$, of B such that, for every $\alpha < \kappa$, the point d_α

is in the closure of the set $H_\alpha = A \setminus \bigcup_{\beta<\alpha} V_{d_\beta}$. If we set $D = \{d_\alpha\colon\ \alpha < \kappa\}$ and, for every $\alpha < \kappa$, $A_{d_\alpha} = H_\alpha \cap V_{d_\alpha}$, then the set D and the family $\{A_d\colon\ d \in D\}$ satisfy the condition in the proposition. □

There are two related, and more natural, dichotomies for sets of regular uncountable cardinality involving complete accumulation points and closed discrete sets, but we do not know if either one of these conditions is equivalent with the one above. We do not even know whether every D-space satisfies the stronger one of these two conditions.

Problem 2.5. Let X be a T_1 (linearly) D-space and let $A \subset X$ have uncountable regular cardinality. Does A either have a complete accumulation point or a subset of size $|A|$ which is closed and discrete in X?

According to Proposition 2.11 below, every submetaLindelöf T_1-space is linearly D, and the proof of the proposition actually shows that submetaLindelöf T_1-spaces satisfy the stronger property indicated in Problem 2.5. Moreover, it is easy to see that also every strongly collectionwise Hausdorff linearly D-space satisfies this stronger property and, by [15, Proposition 2.5], so does every "thickly covered" T_1-space (and hence, in particular, every subspace of a Banach space with weak topology).

By weakening the second condition of the above dichotomy, we obtain a property satisfied by all linearly D-spaces. However, we do not know whether this property yields a characterization of linearly D-spaces.

Problem 2.6. Is a T_1-space X linearly D provided that, for every set $A \subset X$ of uncountable regular cardinality, either A has a complete accumulation point or $\overline{A}$ has a subset of size $|A|$ which is closed and discrete in X?

We shall next consider properties and examples of linearly D-spaces. First we record two observations, which can be verified by standard arguments.

Proposition 2.7.

A. *A closed subspace of a linearly D-space is linearly D.*
B. *The continuous image of a linearly D-space under a closed mapping is linearly D.*

Recall that a space X is *linearly Lindelöf* if every monotone open cover of X has a countable subcover. This is known to be equivalent to the statement that every subset of X of uncountable regular cardinality has a complete accumulation point. The *extent* $e(X)$ of a space X is the smallest infinite cardinal number τ such that we have $|D| \leqslant \tau$ for every closed discrete subset D of X.

Theorem 2.8. *A T_1-space is linearly Lindelöf if, and only if, the space is linearly D and has countable extent.*

Proof. *Necessity*. It is well known that every linearly Lindelöf space has countable extent and the remark made after the proof of Theorem 2.2 shows that every linearly Lindelöf T_1-space is linearly D.

Sufficiency. Assume that X is linearly D and $e(X) \leqslant \omega$. To show that X is linearly Lindelöf, let $\mathcal{U}$ be a monotone open cover of X. Assume that $X \notin \mathcal{U}$. Since X is linearly D, there exists a closed discrete $\mathcal{U}$-big set F. Since X has countable extent, the set F is countable. For every $x \in F$, let $U_x \in (\mathcal{U})_x$. Then $\{U_x\colon\ x \in F\}$ is a countable subcover of $\mathcal{U}$. □

Since every countably compact linearly Lindelöf space is compact, we have the following consequence of Theorem 2.8.

Corollary 2.9. *A T_1-space is compact if, and only if, the space is countably compact and linearly D.*

Next we exhibit some spaces which are not linearly D.

Example 2.10. (a) The space Γ constructed by van Douwen and Wicke in [14] fails to be linearly D. This follows from Theorem 2.8, since Γ has countable extent and is not linearly Lindelöf. Countable extent (under the name "ω_1-compactness") of Γ was established in [14], where it was also shown that Γ is locally countable. It follows from local countability that no uncountable subset of Γ has a complete accumulation point; therefore Γ is not linearly Lindelöf.

(b) A stationary subset S of an uncountable regular cardinal is not linearly D. To see this, let $U_x = \{y \in S\colon\ y < x\}$ for each $x \in S$, and let $\mathcal{U} = \{U_x\colon\ x \in S\}$. Then $\mathcal{U}$ is a monotone open cover of S, and every $\mathcal{U}$-big set is cofinal in S. The proof of [12, Proposition 2.2] shows that the stationary set S does not contain a closed discrete cofinal subset. As a consequence, S is not linearly D.

In our next result, we indicate a substantial class of linearly D-spaces. Note that the proof below actually establishes the strengthening of the linear D-property indicated in Problem 2.5 above.

SubmetaLindelöf spaces were defined by C.E. Aull in [6]. Recall that X is *submetaLindelöf* (or *$\delta\theta$-refinable*) if every open cover of X has an open refinement $\bigcup_{n\in\omega}\mathcal{V}_n$ with the property that each $\mathcal{V}_n$ is a cover and for each $x\in X$, there is an $n\in\omega$ with $|(\mathcal{V}_n)_x|\leqslant\omega$.

Proposition 2.11. *Every submetaLindelöf T_1-space is linearly D.*

Proof. Suppose that X is a submetaLindelöf T_1-space. We use Condition C of Theorem 2.2 to show that X is linearly D. Let A be a subset of X with $|A|=\kappa$, where κ is an uncountable regular cardinal. Assume that the set A has no point of complete accumulation. We show that there exists a set $D\subset A$ with $|D|=\kappa$ such that D is closed and discrete in X.

For every $x\in X$, there exists an open neighbourhood U_x such that $|A\cap U_x|<\kappa$. The open cover $\mathcal{U}=\{U_x\colon x\in X\}$ of X admits an open refinement $\bigcup_{n\in\omega}\mathcal{V}_n$ as in the definition of submetaLindelöfness above. Since the cardinal $\kappa=|A|$ is regular, there exists $k\in\omega$ such that the set $B=\{x\in A\colon |(\mathcal{V}_k)_x|\leqslant\omega\}$ has cardinality κ. Let D be a maximal subset of B with the property that $D\cap St(x,\mathcal{V}_k)=\{x\}$ for every $x\in D$. Since $\mathcal{V}_k$ is an open cover of X, the set D is closed and discrete in X. By maximality of D, we have that $B\subset\bigcup(\mathcal{V}_k)_D$. Since κ is regular and since $|V\cap B|\leqslant|V\cap A|<\kappa$ for every $V\in\mathcal{V}_k$, we have that $|(\mathcal{V}_k)_D|\geqslant\kappa$. As a consequence, $|D|=\kappa$. □

Arhangel'skii and Buzyakova [5] call a space X an *aD-space* provided that, for each closed set $F\subset X$ and each open cover $\mathcal{U}$ of X, there exist a closed and discrete set $A\subset F$ and, for every $a\in A$, a set $U_a\in(\mathcal{U})_a$ such that the family $\{U_a\colon a\in A\}$ covers the set F.

In [3, Theorem 1.15], Arhangel'skii shows that every submetaLindelöf T_1-space is aD.

Not all linearly D-spaces are aD. To see this, recall that there exist linearly Lindelöf spaces which are not Lindelöf ([21, 4]; for a locally compact example, see [19]). Any such space is linearly D but not aD, because every aD-space of countable extent is Lindelöf.

The following problem is still open.

Problem 2.12. Is every aD-space linearly D?

We have a partial solution for the above problem. Recall that a space X is *monolithic* if for every $A\subset X$, the subspace $\overline{A}$ has a network of size $\leqslant|A|$ (see [1]).

Theorem 2.13. *Every T_1 monolithic aD-space is linearly D.*

Proof. Suppose that X is a T_1 monolithic aD-space. We use Theorem 2.2 to show that X is linearly D. Let $\mathcal{O}$ be a non-trivial monotone open cover of X. Then $\mathcal{O}$ has a strictly increasing subcover $\mathcal{U}=\{U_\alpha\colon \alpha<\kappa\}$ where κ is an infinite regular cardinal. For every $\alpha<\kappa$, let $x_\alpha\in U_{\alpha+1}\setminus U_\alpha$. Set $A=\{x_\alpha\colon \alpha<\kappa\}$. Since X is aD, there exists a closed discrete set $B\subset\overline{A}$ and a mapping $\varphi: B\to\mathcal{U}$, such that $\overline{A}\subset\varphi(B)=\bigcup\{\varphi(x)\colon x\in B\}$ and $x\in\varphi(x)$ for each $x\in B$. Since $|A|=\kappa$ and $|\varphi(x)\cap A|<\kappa$ for every $x\in B$, it follows by regularity of κ that $|B|=\kappa$.

We show that B is $\mathcal{U}$-big. Assume on the contrary that there exists $\gamma<\kappa$ such that $B\subset U_\gamma$. Let $C=\{x_\alpha\colon \alpha<\gamma\}$, and note that $B\subset\overline{C}$. Since X is monolithic, the subspace $\overline{C}$ has a network of size $\leqslant|C|<\kappa$. This is a contradiction, since $|B|=\kappa$ and B is closed and discrete in the subspace $\overline{C}$. It follows that B is $\mathcal{U}$-big and hence also $\mathcal{O}$-big. □

The above proof shows that Theorem 2.13 remains valid if we replace "monolithic" by the weaker property that $e(\overline{A})\leqslant|A|$ for every subset A.

We close this section with a result which generalizes [12, Theorem 1.2] and [3, Theorem 2.1]. With the help of a remarkable result of Z. Balogh and M.E. Rudin [7], we can prove our result by a simple modification of [12, proof of Theorem 2.1].

Theorem 2.14. *Every monotonically normal linearly D-space is paracompact.*

Proof. Assume on the contrary that there exists a non-paracompact monotonically normal linearly D-space X. By a result from [7], there exists a stationary subset S of a regular uncountable cardinal such that S is homeomorphic to a closed subspace of X. By Proposition 2.7, the space S is linearly D; this, however, contradicts Example 2.10(b). □

Similarly, we can show that every monotonically normal aD-space is paracompact.

3. Unions of linearly *D*-spaces

It is still an open problem whether the finite union of D-spaces is D, but the following result holds for linearly D-spaces.

Theorem 3.1. *If a space X is the union of finitely many linearly D-subspaces, then X is linearly D.*

Proof. It suffices to prove the result for two factors. Suppose that $X = Y \cup Z$, where the subspaces Y and Z of X are linearly D.

To show that X is linearly D, let $\mathcal{U}$ be a monotone open cover of X with $X \notin \mathcal{U}$. Without loss of generality, suppose that no element of $\mathcal{U}$ contains Y. Let $\mathcal{V} = \{U \cap Y\colon\ U \in \mathcal{U}\}$. Since Y is linearly D, there is a closed discrete subset E of Y, which is $\mathcal{V}$-big. Note that E is $\mathcal{U}$-big. Let $F = \overline{E} \setminus E$. Then F is closed in X and F is contained in Z. As a closed subspace of a linearly D-space, the space F is linearly D.

Note that if there exists $U \in \mathcal{U}$ such that $F \subset U$, then $E \setminus U$ is a closed discrete $\mathcal{U}$-big set in X. Assume that no member of $\mathcal{U}$ contains F. Then the family $\mathcal{W} = \{U \cap F\colon\ U \in \mathcal{U}\}$ is a monotone open cover of F and $F \notin \mathcal{W}$. Since F is linearly D, there exists a closed discrete $\mathcal{W}$-big set K in the subspace F. Clearly, K is a closed discrete $\mathcal{U}$-big set in X. $\square$

Theorems 2.8 and 3.1 give a generalization of the result of Gruenhage that if X is of countable extent and X can be written as the union of finitely many D-subspaces, then X is linearly Lindelöf [17, Theorem 4.2].

The space Γ constructed by van Douwen and Wicke in [14] is the union of countably many discrete subspaces. This shows that Theorem 3.1 cannot be extended to countable unions. However, as we shall show next, the countable sum theorem for linearly D-spaces does hold in a countably compact space.

Arhangel'skii asked in [2] whether a countably compact space is compact provided the space is the union of countably many D-subspaces. This problem was solved by Gerlitz, Juhasz and Szentmiklossy in [16] and later, independently, by L.-X. Peng in [20]. In the proof of their result, Gerlitz, Juhasz and Szentmiklossy (implicitly) obtained the following result, but for the sake of completeness, we include a proof here.

Theorem 3.2. *A countably compact T_1-space is compact provided that the space is the union of countably many linearly D-subspaces.*

Proof. Assume that $X = \bigcup_{n\in\mathbb{N}} A_n$, where X is countably compact and T_1, and each A_n is linearly D. We may choose $A_0 = \emptyset$ and, by Theorem 3.1, we can assume that $A_n \subset A_{n+1}$ for every $n \in \mathbb{N}$. To show that X is compact, it suffices, by Corollary 2.9, to show that X is linearly D. Suppose on the contrary that X is not linearly D. It follows by Theorem 2.2 that X has a non-trivial monotone open cover $\mathcal{U}$ which admits no closed discrete $\mathcal{U}$-big set. Note that the cover $\mathcal{U}$ has no countable subcover.

By induction, we shall define natural numbers $n_0 < n_1 < \cdots$ and $\mathcal{U}$-big closed sets $F_0 \supset F_1 \supset \cdots$ such that $F_k \cap A_{n_k} = \emptyset$ for every $k \in \mathbb{N}$. We set $F_0 = X$ and $n_0 = 0$. Assume that the $\mathcal{U}$-big closed set F_k and the number n_k have been defined. Since $\mathcal{U}$ has no countable subcover, there exists $\ell \in \mathbb{N}$ such that the set $F_k \cap A_\ell$ is $\mathcal{U}$-big. We can assume that $\ell > n_k$. The closed subspace $F_k \cap A_\ell$ of the linearly D subspace A_ℓ is linearly D, and hence there exists a $\mathcal{U}$-big set E which is closed and discrete in $F_k \cap A_\ell$ and hence in A_ℓ. Note that $(\overline{E} \setminus E) \cap A_\ell = \emptyset$. Since E is discrete, the set $\overline{E} \setminus E$ is closed. Moreover, the set $\overline{E} \setminus E$ is $\mathcal{U}$-big, because if there would exist $U \in \mathcal{U}$ such that $\overline{E} \setminus E \subset U$, then $E \setminus U = \overline{E} \setminus U$ would be a closed (in X) and discrete $\mathcal{U}$-big set. The foregoing shows that we can set $n_{k+1} = \ell$ and $F_{k+1} = \overline{E} \setminus E$ to complete the inductive step.

We have reached a contradiction with countable compactness of X, because $\langle F_k \rangle_{k=0}^{\infty}$ is a decreasing sequence of non-empty closed sets with $\bigcap_{k=0}^{\infty} F_k \subset X \setminus \bigcup_{k=0}^{\infty} A_k = \emptyset$. $\square$

Corollary 3.3. *([16,20]) A countably compact space is compact if the space is the union of countably many D-subspaces.*

References

[1] A.V. Arhangel'skii, Topological Function Spaces, Kluwer Academic Publishers, Dordrecht, 1992.
[2] A.V. Arhangel'skii, D-spaces and finite unions, Proc. Amer. Math. Soc. 132 (2004) 2163–2170.
[3] A.V. Arhangel'skii, D-spaces and covering properties, Topology Appl. 146/147 (2005) 437–449.
[4] A.V. Arhangel'skii, R.Z. Buzyakova, On linearly Lindelöf and strongly discretely Lindelöf spaces, Proc. Amer. Math. Soc. 127 (1999) 2449–2458.
[5] A.V. Arhangel'skii, R.Z. Buzyakova, Addition theorems and D-spaces, Comment. Math. Univ. Carolin. 43 (2002) 653–663.
[6] C.E. Aull, A generalization of a theorem of Aquaro, Bull. Austral. Math. Soc. 9 (1973) 105–108.
[7] Z. Balogh, M.E. Rudin, Monotone normality, Topology Appl. 47 (1992) 115–127.
[8] C.R. Borges, A.C. Wehrly, A study of D-spaces, Topology Proc. 16 (1991) 7–15.
[9] R.Z. Buzyakova, Hereditary D-property of function spaces over compacta, Proc. Amer. Math. Soc. 132 (2004) 2171–2181.
[10] R.Z. Buzyakova, V.V. Tkachuk, V.V. Wilson, A quest for nice kernels of neighbourhood assignments, Comment. Math. Univ. Carolin. 48 (2007) 689–697.
[11] E.K. van Douwen, Why certain Čech–Stone remainders are not homogeneous, Collect. Math. 41 (1979) 45–52.
[12] E.K. van Douwen, D.J. Lutzer, A note on paracompactness in generalized ordered spaces, Proc. Amer. Math. Soc. 125 (1997) 1237–1245.
[13] E.K. van Douwen, W. Pfeffer, Some properties of the Sorgenfrey line and related spaces, Pacific J. Math. 81 (1979) 371–377.
[14] E.K. van Douwen, H.H. Wicke, A real, weird topology on the reals, Houston J. Math. 13 (1977) 141–152.
[15] A. Dow, H.J.K. Junnila, J. Pelant, Coverings, networks and weak topologies, Mathematika 53 (2006) 287–320.
[16] J. Gerlitz, I. Juhasz, Z. Szentmiklossy, Two improvements of Tkacenko's addition theorem, Comment. Math. Univ. Carolin. 46 (2005) 705–710.
[17] G. Gruenhage, A note on D-spaces, Topology Appl. 153 (2006) 2229–2240.
[18] H.J.K. Junnila, Neighbournets, Pacific J. Math. 76 (1978) 83–108.
[19] K. Kunen, Locally compact linearly Lindelöf spaces, Comment. Math. Univ. Carolin. 43 (2002) 155–158.
[20] L.-X. Peng, A note on D-spaces and infinite unions, Topology Appl. 154 (2007) 2223–2227.
[21] A.S. Mischenko, Finally compact spaces, Soviet Math. Dokl. 145 (1962) 1199–1202.