

## MATHEMATICAL LOGIC

### ON THE COMPLETENESS OF THE CLASSICAL SENTENTIAL LOGIC

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(Communicated by Prof. A. HEYTING at the meeting of May 31, 1958)

We consider a version of the classical sentential logic in which *atoms*  $p, q, r, \dots$ , *negation*  $\neg$ , and *implication*  $\rightarrow$  are used. It is well known that this subject may be approached in three ways:

(A) We define the notion of a *logical identity* (tautology, or valid formula) by means of truth-tables for the connectives  $\neg$  and  $\rightarrow$ .

(B) We set up a calculus of sequents.

(C) We select an axiom system and rules of deduction, and we consider the formulas deducible from the axioms.

The *completeness theorem* expresses the fact that, essentially, the methods (A), (B) and (C) lead to the same result. In particular, *a formula is deducible if and only if it is a logical identity*.

The equivalence of the approaches under (A) and (B) is a relatively simple matter; it is also rather easy to prove that every formula deducible from the axioms is a logical identity. Therefore, it will not be necessary here to go into these matters.

However, the problem of proving that every logical identity is deducible is not quite as simple. In fact, several proofs of this most substantial part of the completeness theorem have been given, but they are either rather tedious or they rely on metamathematical results of a relatively advanced kind, which in turn would become much more easily accessible if the completeness proof were given in advance. Therefore, the following discussion may still present some interest.

In some earlier publications, I have introduced the *method of semantic tableaux* for testing whether a given formula  $X$  is, or is not, a logical identity. For our present purpose, it will be sufficient to mention the following rules for the construction and closure of such a tableau.

(i) The formula  $X$  to be tested is inserted in the right column as the only initial formula.

(ij) If  $\bar{U}$  appears in a left (right) column, then  $U$  is inserted in the *conjugate* right (left) column [*i.e.*, in the right (left) column of the same (sub-)tableau].

(ii<sup>a</sup>) If  $U \rightarrow V$  appears in a left column, then the (sub-)tableau splits up into two sub-tableaux; in the right column of one sub-tableau we insert  $U$  and in the left column of the other,  $V$ .

(ii<sup>b</sup>) If  $U \rightarrow V$  appears in a right column, then we insert  $V$  in the same column, and  $U$  in the conjugate left column.

(iv) If the same formula presents itself in both columns of the same (sub-)tableau, then that (sub-)tableau is closed; if the two sub-tableaux of a (sub-)tableau are closed, then that (sub-)tableau is also closed.

(v) The initial formula  $X$  is a logical identity, if and only if the semantic tableau as described is closed.

It will be clear that the method of semantic tableaux represents the above approach under (A). With a view to the approach under (C), we may select some axiom system for the classical sentential logic in  $\neg$  and  $\rightarrow$ , combined with substitution and *modus ponens*. We wish to show that a formula  $X$  which is a logical identity is also deducible on the basis of that axiom system.

So let  $X$  be a logical identity. Then the semantic tableau for  $X$  must be closed. Suppose the tableau starts as follows:

True		False	
.		$X$	
		.	
		.	
$U \rightarrow V$		$U$	
	$V$		

(I) We suppose, specifically, that  $U \rightarrow V$  is the first implication to appear in the left column and thus demanding a splitting of the tableau. Let us now consider the formulas:

$$(a) \quad (\overline{U \rightarrow V}) \rightarrow X; \quad (b) \quad \overline{U} \rightarrow X; \quad (c) \quad V \rightarrow X.$$

The semantic tableaux for these formulas look as follows:

True	False	True	False	True	False
$\overline{U \rightarrow V}$	(a)	$\overline{U}$	(b)	$V$	(c)
.	$X$	.	$X$	.	$X$
.		.		.	.
$U \rightarrow V$	$U \rightarrow V$	$U$			.
	.	.			.
	.				.

The tableau for (a) is closed without splitting; the tableau for (b) can be extended in accordance with the left sub-tableau in the semantic tableau for  $X$  and thus will be closed; and the tableau for (c) can be extended in accordance with the right sub-tableau and hence it will also be closed. Thus, if instead of the original formula  $X$ , we consider the formulas (a), (b) and (c), we eliminate the first splitting in the semantic tableau for  $X$ .

(II) As the formula:

$$[(\overline{U \rightarrow V}) \rightarrow X] \rightarrow \{(\overline{U} \rightarrow X) \rightarrow [(V \rightarrow X) \rightarrow X]\}$$

is deducible, it follows that the deducibility of the formulas (a), (b) and (c) entails the deducibility of the formula  $X$ . We know that (a), (b) and (c) are logical identities, but we still have to prove that this fact entails their deducibility.

(III) It will, however, be clear that in the same way we can eliminate the first splittings in the semantic tableaux for (b) and (c), and so on. Thus our problem is reduced to proving that, if *the semantic tableau for a formula is closed without splitting, then the formula is deducible*.

So let  $X$  be any formula of this kind. The closure of its semantic tableau will result from the appearance, in the left and the right column, of formulas  $Y$  and  $Z$  which are *equiform* (i.e., typographically alike); these formulas are produced by (the decomposition of) the formula  $X$  under consideration.

Working upward, we have formulas  $Y_1$  (or  $Y$ ),  $Y_2, \dots, Y_j$  which produce  $Y$ , formulas  $Z_1$  (or  $Z$ ),  $Z_2, \dots, Z_k$  which produce  $Z$ , and formulas  $X_1, X_2, \dots, X_m$  (or  $X$ ) which produce both  $X$  and  $Y$ . It will be clear that  $X_1$  is either  $Y_j \rightarrow Z_k$  or  $Z_k \rightarrow Y_j$  and that  $X_1$  must appear in the right column.

Let  $X_n^*$  be  $\bar{X}_n$  or  $X_n$ , according as  $X_n$  appears in the left or in the right column; let  $Y_n^*$  be  $Y_n$  or  $\bar{Y}_n$ , according as  $Y_n$  appears in the left or in the right column; and let  $Z_n^*$  be  $\bar{Z}_n$  or  $Z_n$ , according as  $Z_n$  appears in the left or in the right column. In particular,  $Y_1^*$  is  $Y_1$ , or  $Y$ ,  $Z_1^*$  is  $Z_1$ , or  $Z$ , and  $X_m^*$  is  $X_m$ , or  $X$ .

(1)  $Y_1^* \rightarrow Z_1^*$ , or  $Y \rightarrow Z$ , is deducible, as  $Y$  and  $Z$  are equiform.

(2)  $Y_{n+1}^* \rightarrow Y_n^*$  is deducible. — Suppose, first, that  $Y_{n+1}^*$  is  $Y_{n+1}$ . Then  $Y_{n+1}$  appears in the left column and so must be  $\bar{Y}_n$ . It follows that  $Y_n$  appears in the right column, and hence  $Y_n^*$  is  $\bar{Y}_n$ . So  $Y_{n+1}^* \rightarrow Y_n^*$  must be  $\bar{Y}_n \rightarrow \bar{Y}_n$ , and this formula is clearly deducible.

Next, suppose that  $Y_{n+1}^*$  is  $\bar{Y}_{n+1}$ . Then  $Y_{n+1}$  appears in the right column, and so it can be either  $\bar{Y}_n$  or  $Y_n \rightarrow U$  or  $U \rightarrow Y_n$ . In the first and second case,  $Y_n$  will appear in the left column and  $Y_n^*$  will be  $Y_n$ ; in the third case,  $Y_n$  will appear in the right column and  $Y_n^*$  will be  $\bar{Y}_n$ . Accordingly,

$Y_{n+1}^* \rightarrow Y_n^*$  is either  $\overline{\overline{Y}_n} \rightarrow Y_n$  or  $(\overline{Y_n \rightarrow U}) \rightarrow Y_n$  or  $(\overline{U_n \rightarrow Y_n}) \rightarrow \overline{Y}_n$ , and each of these formulas is deducible.

(3) In a similar manner, we show that  $Z_n^* \rightarrow Z_{n+1}^*$  is deducible.

(4) From the results under (1)–(3) it follows that  $Y_j^* \rightarrow Z_k^*$  is deducible. Now, according as  $X_1$  is  $Y_j \rightarrow Z_k$  or  $Z_k \rightarrow Y_j$ ,  $Y_j^* \rightarrow Z_k^*$  will be  $Y_j \rightarrow Z_k$  or  $\overline{Y}_j \rightarrow \overline{Z}_k$ . It follows that  $X_1$ , or  $X_1^*$ , is deducible.

(5) By the method, applied under (2), we prove that  $X_n^* \rightarrow X_{n+1}^*$  is deducible.

(6) From the results under (4) and (5) it follows that  $X_m^*$ , or  $X$ , is deducible. — This completes our proof.

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