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# A Note on the Convergence of the Solutions of a Linear Functional Differential Equation*

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## 1. INTRODUCTION

Consider the linear, autonomous, scalar differential equation

$$x'(t) = -\mu_0 x(t) + \int_{-\infty}^{0} x(t+s)\, d\mu(s), \tag{1}$$

where $\mu_0$ is a real number, $\mu: (-\infty, 0) \to \mathbf{R}$ is a nondecreasing function, and $\int_{-\infty}^{0} d\mu < \infty$.

Equation (1) occurs in certain biological applications (see e.g. [2]).

If $\mu_0 > \int_{-\infty}^{0} d\mu$ then any solution of (1) belonging to a bounded, continuous initial function tends to zero as $t \to \infty$ (see e.g. [9]).

If $\mu_0 < \int_{-\infty}^{0} d\mu$ then it is easy to see that there are unbounded solutions. More precisely, there is a positive real number $\lambda_0$ such that $e^{\lambda_0 t}$ is a solution of (1) on $\mathbf{R}$.

In the critical case $\mu_0 = \int_{-\infty}^{0} d\mu$ it is shown in [6] that under the condition

$$\int_{-\infty}^{0} |s|\, d\mu(s) < \infty \tag{2}$$

any solution of (1) belonging to a bounded, continuous initial function tends to a constant as $t \to \infty$.

The different methods of proving the convergence of solutions of differential equations with infinite delay have the same type of condition for the delay term as (2) [3–8]. The question arises whether Condition (2) is

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necessary for the convergence of all solutions of (1) belonging to bounded, continuous initial functions.

The purpose of this paper is to prove that (2) is a sufficient and necessary condition under which all solutions of (1) belonging to bounded, continuous initial functions converge to a finite limit as $t \to \infty$.

## 2. Preliminaries

Let $\mathbf{R}$, $\mathbf{R}^-$, $\mathbf{R}^+$ and $I$ be the intervals $(-\infty, \infty)$, $(-\infty, 0]$, $[0, \infty)$, and $[0, 1]$, respectively. Let BC denote the set of real-valued bounded, continuous functions on $\mathbf{R}^-$. Define $\mathrm{IC} = \{\varphi \in \mathrm{BC}\colon \varphi(s) \in I\}$. The function $x\colon \mathbf{R} \to \mathbf{R}$ is said to be a solution of (1) on $\mathbf{R}^+$ through $(0, \varphi)$, $\varphi \in \mathrm{BC}$, if $x(s) = \varphi(s)$ for $s \in \mathbf{R}^-$ and (1) holds on $\mathbf{R}^+$. It is easy to see that for any $\varphi \in \mathrm{BC}$ there is a unique solution of (1) through $(0, \varphi)$ on $\mathbf{R}^+$ [1]. In this paper, by a solution of (1) we always mean a solution through $(0, \varphi)$ for some $\varphi \in \mathrm{BC}$, and this solution is denoted by $x(\varphi)$.

For a given $\varphi \in \mathrm{BC}$ and $r \in \mathbf{R}^+$ let us define the set $\mathrm{IC}_{\varphi, r}$ by

$$\mathrm{IC}_{\varphi, r} = \{\psi \in \mathrm{IC}\colon \psi(s) = \varphi(s) \text{ for } s \in [-r, 0]\}.$$

Without loss of generality we may assume that $\mu$ is normalized such that it is continuous from the left, $\lim_{s \to -\infty} \mu(s) = 0$. Then, clearly, $\int_{-\infty}^{u} d\mu(s) = \mu(u)$ for $u \leqslant 0$. Moreover, if (2) holds, then by changing the order of integration we obtain

$$\int_{-\infty}^{0} |s|\, d\mu(s) = \int_{-\infty}^{0} \left(\int_{s}^{0} du\right) d\mu(s)$$
$$= \int_{-\infty}^{0} \left(\int_{-\infty}^{u} d\mu(s)\right) du = \int_{-\infty}^{0} \mu(u)\, du, \tag{3}$$

that is, $\mu$ is integrable on $\mathbf{R}^-$.

## 3. The Result

The sufficiency of Condition (2) is contained in a more general result in [6]. A different method is used in [5] to obtain the sufficiency of (2) whenever the initial functions are bounded and uniformly continuous on $\mathbf{R}^-$. For the sake of completeness, we also give here a proof of the sufficiency of (2) for arbitrary initial functions from BC by using the idea of [5].

THEOREM. *Assume* $\mu_0 = \int_{-\infty}^{0} d\mu$. *Then all solutions of* (1) *converge to a finite limit as* $t \to \infty$ *if and only if* (2) *holds.*

*Proof.* *Sufficiency of* (2). Let $\varphi \in \mathrm{BC}$ be given and let $x = x(\varphi)$. Define $M = \sup_{s \leqslant 0} x(s)$ and $m = \inf_{s \leqslant 0} x(s)$. First we show that $x(t) \in [m, M]$ for $t \geqslant 0$, that is, $x$ is bounded. Let

$$u(t) = \max_{0 \leqslant s \leqslant t} \{x(t), M\}.$$

If $x(t) < u(t)$ then $D^+u(t) \stackrel{\text{def}}{=} \limsup_{h \to 0+} (1/h)(u(t+h) - u(t)) = 0$. If $x(t) = u(t)$ then clearly $D^+u(t) \leqslant \max\{0, x'(t)\}$. On the other hand, $x(t) = u(t)$ implies $x(t) \geqslant x(t+s)$ for all $s \leqslant 0$. Thus, from Eq. (1) we get that $x'(t) \leqslant 0$. Therefore $D^+u(t) \leqslant 0$ for all $t \geqslant 0$. Consequently, $x(t) \leqslant u(t) \leqslant M$ for all $t \geqslant 0$. Similarly, $x(t) \geqslant m$ for all $t \geqslant 0$, which was stated.

Let $a = \liminf_{t \to \infty} x(t)$, $b = \limsup_{t \to \infty} x(t)$. For the existence of $\lim_{t \to \infty} x(t)$ it suffices to prove that $a = b$. Suppose the contrary, i.e., $a < b$. Let $c \in (a, b)$ and let $\varepsilon > 0$ be given such that

$$b + \varepsilon + \tfrac{1}{2}(c - b - \varepsilon) \exp\left(-\int_{-\infty}^{0} \mu(s)\, ds\right) < b.$$

There exists such an $\varepsilon > 0$ by continuity, because the left hand side of the inequality is less than $b$ at $\varepsilon = 0$. The function $\mu$ is integrable on $R^-$ by (3).

Let $T_1$ be defined such that $t \geqslant T_1$ implies $x(t) \leqslant b + \varepsilon$. Choose $T_2 \geqslant T_1$ such that $x(T_2) = c$ and

$$(M - m) \int_{T_2}^{\infty} \mu(T_1 - u)\, du < \tfrac{1}{2}(b + \varepsilon - c) \exp\left(-\int_{-\infty}^{0} \mu(s)\, ds\right).$$

For $t \geqslant T_2$ let

$$v(t) = \max_{T_2 \leqslant s \leqslant t} x(s).$$

Now, $x(t) < v(t)$ implies $D^+v(t) = 0$, and $x(t) = v(t)$, $x'(t) \leqslant 0$ imply $D^+v(t) \leqslant 0$. Assume that $x(t) = v(t)$ and $x'(t) > 0$. Then $D^+v(t) \leqslant x'(t)$ and by using Eq. (1)

$$\begin{aligned} x'(t) &= \left(\int_{-\infty}^{T_1 - t} + \int_{T_1 - t}^{T_2 - t} + \int_{T_2 - t}^{0}\right) (x(t+s) - x(t))\, d\mu(s) \\ &\leqslant \int_{T_1 - t}^{T_2 - t} (b + \varepsilon - x(t))\, d\mu(s) + (M - m) \int_{-\infty}^{T_1 - t} d\mu(s) \\ &\leqslant \int_{-\infty}^{T_2 - t} d\mu(s)(b + \varepsilon - x(t)) + (M - m) \int_{-\infty}^{T_1 - t} d\mu(s). \end{aligned}$$

Since the right-hand side of this inequality is nonnegative for $t \geqslant T_1$, we have

$$D^+ v(t) \leqslant \int_{-\infty}^{T_2 - t} d\mu(s)(b + \varepsilon - v(t)) + (M - m) \int_{-\infty}^{T_1 - t} d\mu(s)$$

for all $t \geqslant T_2$. Using that $v(T_2) = c$ and a well-known differential inequality, one obtains for $t \geqslant T_2$ that

$$\begin{aligned} v(t) &\leqslant b + \varepsilon + (c - b - \varepsilon) \exp\left( -\int_{T_2}^{t} \int_{-\infty}^{T_2 - u} d\mu(s)\, du \right) \\ &\quad + \int_{T_2}^{t} (M - m) \int_{-\infty}^{T_1 - u} d\mu(s) \exp\left( -\int_{u}^{t} \int_{-\infty}^{T_2 - \tau} d\mu(s)\, d\tau \right) du \\ &\leqslant b + \varepsilon + (c - b - \varepsilon) \\ &\quad \times \exp\left( -\int_{T_2}^{t} \mu(T_2 - u)\, du \right) + \int_{T_2}^{t} (M - m)\, \mu(T_1 - u)\, du \\ &\leqslant b + \varepsilon + \tfrac{1}{2}(c - b - \varepsilon) \exp\left( -\int_{-\infty}^{0} \mu(s)\, ds \right) < b, \end{aligned}$$

which implies $\limsup_{t \to \infty} x(t) < b$, a contradiction. Thus, $\lim_{t \to \infty} x(t)$ exists.

*Necessity of* (2). First we show the following simple statements.

*Claim* 1. If $\varphi \in \mathrm{IC}$ then $0 \leqslant x(\varphi)(t) \leqslant 1$ for $t \geqslant 0$.

*Proof.* This is the same as that of the boundedness in the proof of the sufficiency of (2).

*Claim* 2. If $\varphi \in \mathrm{BC}$ and $\varphi$ has compact support (i.e., $\varphi$ is zero outside of a compact subset of $R^-$), then for $t \geqslant 0$ we have

$$\begin{aligned} x(\varphi)(t) &+ \int_{-\infty}^{0} \int_{t+s}^{t} x(\varphi)(u)\, du\, d\mu(s) \\ &= \varphi(0) + \int_{-\infty}^{0} \int_{s}^{0} \varphi(u)\, du\, d\mu(s). \end{aligned} \tag{4}$$

*Proof.* (4) can be obtained from Eq. (1) by integrating it from 0 to $t$ and using the fact that

$$\int_{-\infty}^{0} (x(\varphi)(t)-x(\varphi)(t+s))\,d\mu(s)$$

$$=\int_{-\infty}^{0} \frac{d}{dt}\int_{t+s}^{t} x(\varphi)(u)\,du\,d\mu(s)$$

$$=\frac{d}{dt}\int_{-\infty}^{0}\int_{t+s}^{t} x(\varphi)(u)\,du\,d\mu(s),$$

since $\varphi$ has compact support.

*Claim* 3. If $\varphi \in \mathrm{IC}$ and $\varphi$ has compact support, then either $x(\varphi)(t) \to 0$ or $x(\varphi)(t)$ does not converge as $t \to \infty$. Similarly, if $\varphi \in \mathrm{IC}$ and $1-\varphi$ has compact support, then either $x(\varphi)(t) \to 1$ or $x(\varphi)(t)$ does not converge as $t \to \infty$.

*Proof.* By Claim 1, $x(\varphi)(t) \in I$ for $t \geqslant 0$. Suppose the contrary, i.e., $x(\varphi)(t) \to \alpha$, $\alpha \in (0, 1]$. If $T$ is so large that $x(\varphi)(t) \geqslant \alpha/2$ for $t \geqslant T$, then for $t \geqslant T$

$$x(\varphi)(t)+\int_{-\infty}^{0}\int_{t+s}^{t} x(\varphi)(u)\,du\,d\mu(s) \geqslant \frac{\alpha}{2}\int_{T-t}^{0} |s|\,d\mu(s).$$

Hence and from $\int_{-\infty}^{0} |s|\,d\mu(s)=\infty$, the left-hand side of (4) tends to infinity as $t \to \infty$. This is impossible since $\varphi$ has compact support and thus the right-hand side of (4) is a finite constant. The second part can be shown from the first one by applying it for $1-\varphi$ and using that $x(1-\varphi)(t)=1-x(\varphi)(t)$.

*Claim* 4. For any $T>0$, $\varepsilon>0$ there exists $r=r(T, \varepsilon)>0$ such that for all $\varphi \in \mathrm{IC}$ and $\psi \in \mathrm{IC}_{\varphi, r}$

$$|x(\varphi)(t)-x(\psi)(t)|<\varepsilon \quad (t \in [0, T]).$$

*Proof.* Let $y(t)=x(\varphi)(t)-x(\psi)(t)$ for $t \in R$. Then $y(t)=0$ for $t \in [-r, 0]$ and we have

$$y'(t)=-\mu_0\, y(t)+f(r, t)+\int_{-t}^{0} y(t+s)\,d\mu(s) \qquad (t \geqslant 0),$$

where $f(r, t)=\int_{-\infty}^{-t-r} y(t+s)\,d\mu(s)$. From $\varphi, \psi \in \mathrm{IC}$ it follows that $f(r, t) \to 0$ as $r \to \infty$ uniformly in $t$ on $[0, T]$. Let $\delta>0$ be given and let $r$ be so large that $|f(r, t)|<\delta$ for $t \in [0, T]$. Define $z(t)=|y(t)|$, $t \in R$. It is easy to see that

$$D^{+}z(t) \leqslant -\mu_0 z(t)+\delta+\int_{-t}^{0} z(t+s)\,d\mu(s) \qquad (t \in [0, T]).$$

Multiply this inequality by $e^{\mu_0 t}$ and define $v(t) = e^{\mu_0 t} z(t)$, $t \in R$. Obviously

$$D^+ v(t) \leqslant e^{\mu_0 T}\delta + e^{\mu_0 T} \int_{-t}^{0} v(t+s)\, d\mu(s) \qquad (t \in [0, T]).$$

By integrating on $[0, t]$, $0 \leqslant t \leqslant T$, changing the order of integration, and using $v(0) = 0$, we get

$$\begin{aligned} v(t) &\leqslant e^{\mu_0 T}\, \delta t + e^{\mu_0 T} \int_0^t \int_{-u}^{0} v(u+s)\, d\mu(s)\, du \\ &= e^{\mu_0 T}\, \delta t + e^{\mu_0 T} \int_{-t}^{0} \int_0^{t+s} v(u)\, du\, d\mu(s) \\ &\leqslant e^{\mu_0 T} T\delta + \mu_0 e^{\mu_0 T} \int_0^t v(u)\, du \qquad (t \in [0, T]). \end{aligned}$$

Applying Gronwall's inequality, one obtains that

$$v(t) \leqslant e^{\mu_0 T} T \exp(\mu_0 T e^{\mu_0 T}) \cdot \delta \ (t \in [0, T]),$$

which completes the proof of Claim 4.

Now, we give a function $\varphi \in \mathrm{IC}$ such that $x(\varphi)(t)$ does not converge as $t \to \infty$. Two cases can be distinguished by Claim 3.

*Case* 1. There is a function $\varphi \in \mathrm{IC}$ such that either $\varphi$ or $1 - \varphi$ has compact support and $x(\varphi)(t)$ does not converge as $t \to \infty$.

*Case* 2. If $\varphi \in \mathrm{IC}$ and $\varphi$ has compact support, then $x(\varphi)(t) \to 0$ as $t \to \infty$. If $\varphi \in \mathrm{IC}$ and $1 - \varphi$ has compact support, then $x(\varphi)(t) \to 1$ as $t \to \infty$.

If Case 1 holds, there is nothing to prove. Assume that Case 2 is satisfied. Let $\psi_0$ be given by

$$\psi_0(s) = \begin{cases} s+1 & \text{if } s \in [-1, 0] \\ 0 & \text{if } s < -1 \end{cases}$$

(see Fig. 1). Then $x(\psi_0)(t) \to 0$ as $t \to \infty$. So there exists $t_0 > 0$ such that $x(\psi_0)(t_0) < 1/3$. Claim 4 implies that there is $r_0 > 1$ such that for any $\psi \in \mathrm{IC}_{\psi_0, r_0}$ one has $x(\psi)(t_0) < 1/3$.

Let $\psi_1 \in \mathrm{IC}_{\psi_0, r_0}$ be defined by

$$\psi_1(s) = \begin{cases} \psi_0(s) & \text{if } s \in [-r_0, 0] \\ -s - r_0 & \text{if } s \in [-r_0 - 1, -r_0] \\ 1 & \text{if } s < -r_0 - 1 \end{cases}$$

(see Fig. 1). Then $1-\psi_1$ has compact support. Therefore, $x(\psi_1)(t)\to 1$ as $t\to\infty$. It follows that there is $t_1>\max\{1, t_0\}$ such that $x(\psi_1)(t_1)>2/3$. By Claim 4 there exists $r_1>r_0+1$ such that for all $\psi\in \mathrm{IC}_{\psi_1,r_1}$ one has $x(\psi)(t_1)>2/3$.

Suppose that $r_0, r_1, ..., r_{2k-1}, t_0, t_1, ..., t_{2k-1}$ and $\psi_0, \psi_1, ..., \psi_{2k-1}$ are given, $k\geqslant 1$.

Let us define $\psi_{2k}\in \mathrm{IC}_{\psi_{2k-1},r_{2k-1}}$ as

$$\psi_{2k}(s)=\begin{cases}\psi_{2k-1}(s) & \text{if } s\in[-r_{2k-1},0]\\ s+r_{2k-1}+1 & \text{if } s\in[-r_{2k-1}-1,-r_{2k-1}]\\ 0 & \text{if } s<-r_{2k-1}-1\end{cases}$$

(see Fig. 1). Then $x(\psi_{2k})(t)\to 0$ as $t\to\infty$. Consequently, one can choose $t_{2k}>\max\{2k, t_{2k-1}\}$ such that $x(\psi_{2k})(t_{2k})<1/3$. From Claim 4 it follows that there exists $r_{2k}>r_{2k-1}+1$ such that for all $\psi\in \mathrm{IC}_{\psi_{2k},r_{2k}}$ we have $x(\psi)(t_{2k})<1/3$.

Let $\psi_{2k+1}\in \mathrm{IC}_{\psi_{2k},r_{2k}}$ be defined by

$$\psi_{2k+1}(s)=\begin{cases}\psi_{2k}(s) & \text{if } s\in[-r_{2k},0]\\ -s-r_{2k} & \text{if } s\in[-r_{2k}-1,-r_{2k}]\\ 1 & \text{if } s<-r_{2k}-1\end{cases}$$

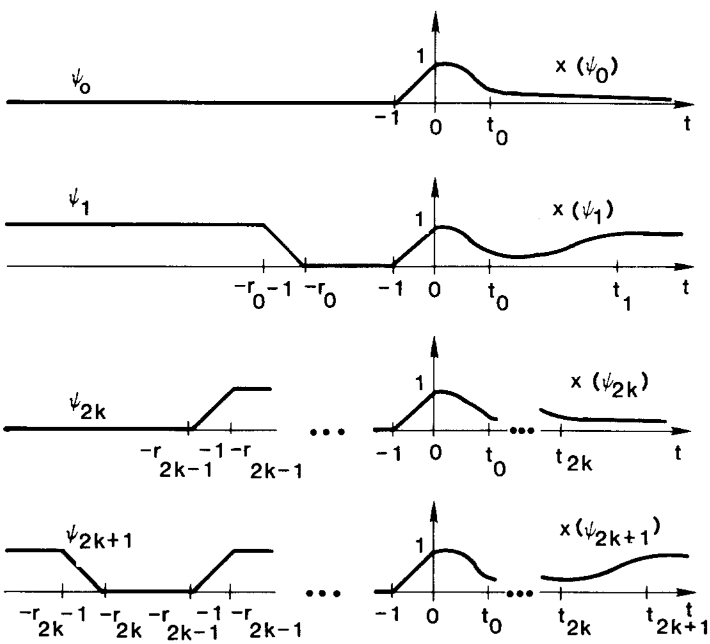


FIGURE 1

(see Fig. 1). Now $x(\psi_{2k+1})(t) \to 1$ as $t \to \infty$, since $1 - \psi_{2k+1}$ has compact support. Then there is $t_{2k+1} > \max\{2k+1, t_{2k}\}$ with $x(\psi_{2k+1})(t_{2k+1}) > 2/3$. Claim 4 implies that we can find $r_{2k+1} > r_{2k} + 1$ such that $x(\psi)(t_{2k+1}) > 2/3$ for all $\psi \in \mathrm{IC}_{\psi_{2k+1}, r_{2k+1}}$.

This induction defines the sequences $\{\psi_n\}_{n=0}^{\infty}$, $\{t_n\}_{n=0}^{\infty}$, $\{r_n\}_{n=0}^{\infty}$. It is easy to see that $\bigcap_{k=0}^{\infty} \mathrm{IC}_{\psi_k, r_k}$ contains a unique function $\varphi$ given by

$$\varphi(s) = \begin{cases} 0 & \text{if } s \in \bigcup_{k=0}^{\infty} [-r_{2k}, -r_{2k-1} - 1] \\ 1 & \text{if } s \in \bigcup_{k=0}^{\infty} [-r_{2k+1}, -r_{2k} - 1] \\ s + r_{2k-1} + 1 & \text{if } s \in [-1 - r_{2k-1}, -r_{2k-1}] \quad (k = 0, 1, ...) \\ -s - r_{2k} & \text{if } s \in [-1 - r_{2k}, -r_{2k}] \quad (k = 0, 1, ...), \end{cases}$$

where $r_{-1} = 0$.

Since $\varphi \in \bigcap_{k=0}^{\infty} \mathrm{IC}_{\psi_k, r_k}$, we have

$$x(\varphi)(t_{2k}) < 1/3 \qquad (k = 0, 1, ...)$$

and

$$x(\varphi)(t_{2k+1}) > 2/3 \qquad (k = 0, 1, ...).$$

Therefore, $x(\varphi)(t)$ does not converge as $t \to \infty$.

The proof is complete.

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