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Construction of biorthogonal multiwavelets

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Abstract

There are perfect formulas for the constructions of biorthogonal uniwavelets. Let

$$\phi(x)=\sum_{k\in Z}p_k\phi(2x-k),\qquad \tilde\phi(x)=\sum_{k\in Z}\tilde p_k\tilde\phi(2x-k)$$

be a pair of biorthogonal uniscaling functions, then a pair of biorthogonal uniwavelet associated with the above biorthogonal uniscaling functions can be easily expressed as

$$\psi(x)=\sum_{k\in Z}(-1)^{k-1}\tilde p_{1-k}\phi(2x-k),\qquad \tilde\psi(x)=\sum_{k\in Z}(-1)^{k-1}p_{1-k}\tilde\phi(2x-k).$$

However, it seems that there is not such a good formula of similar structure for biorthogonal multiwavelets. In this paper, we will give a procedure for constructing compactly supported biorthogonal multiwavelets, which makes construction of biorthogonal multiwavelets easy like in the construction of biorthogonal uniwavelet. Our approach is also suitable for the case of compactly supported orthogonal multiwavelets. Four examples for constructing multiwavelets are given.

Keywords: Compactly supported function; Biorthogonal multiscaling functions; Biorthogonal multiwavelets; Two-scale matrix equation; Two-scale matrix sequence; Two-scale matrix symbol

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1. Introduction

Since Geronimo et al. [1] presented the examples of multiwavelets by using fractal interpolation functions, the construction of multiwavelets has been studied by many authors (see [2–5]). The compactly supported orthogonal multiscaling functions constructed by Geronimo et al. are both symmetric and continuous. One of the associated orthogonal multiwavelets is symmetric while the other is antisymmetric. However, as we know, so good properties are either impossible or incompatible with each other for a single compactly supported scaling function. It turns out that many researchers proceed to study multiwavelets. Multiwavelets with the above properties can be constructed easily. From this respect, applications of multiwavelets are more extensive than those of uniwavelet. Strela [6] shown that the multiwavelets are superior to the uniwavelet in the effects of GHM-multiwavelets and D_4-wavelet on image compression. The study of biorthogonal multiwavelets began in [7]; later, Hardin et al. [8] and Goh [9] followed. However, as yet there has not been a general method to obtain biorthogonal multiwavelets. The main objective of this paper is to give a procedure constructing compactly supported multiwavelets.

The paper is organized as follows: In Section 2, we briefly recall the concept of multiresolution analysis of multiplicity r. In Section 3, we give our main result, a constructive procedure of compactly supported biorthogonal multiwavelets by using biorthogonal compactly supported multiscaling functions. In Section 4, four examples are also given.

2. Multiresolution analysis of multiplicity *r*

The multiwavelets are associated with multiresolution analysis of multiplicity r; i.e., multiwavelets can be constructed by multiresolution analysis with multiplicity r.

Let $\mathbf{\Phi}(x) = (\phi_1, \phi_2, \ldots, \phi_r)^T$, $\phi_1, \phi_2, \ldots, \phi_r \in L^2(R)$,

$$\mathbf{\Phi}(x) = \sum_{k \in Z} P_k \mathbf{\Phi}(2x - k) \tag{1}$$

for some $r \times r$ matrices $\{P_k\}_{k \in Z}$ called the two-scale matrix sequence. $\mathbf{\Phi}(x)$ is called multiscaling function with multiplicity r.

(1) can be rewritten as

$$\hat{\mathbf{\Phi}}(w) = P(z)\hat{\mathbf{\Phi}}\left(\frac{w}{2}\right), \tag{2}$$

where $z = e^{-iw/2}$, and

$$P(z) = \frac{1}{2} \sum_{k \in Z} P_k z^k \tag{3}$$

called the two-scale matrix symbol of the two-scale matrix sequence $\{P_k\}_{k\in Z}$ of $\mathbf{\Phi}$.

Define a subspace $\mathbf{V}_j \subset L^2(R)$ by

$$\mathbf{V}_j = \mathrm{clos}_{L^2(R)}\langle \phi_{\ell:j,k}\colon\ 1 \leqslant \ell \leqslant r,\ k \in Z\rangle, \quad j \in Z; \tag{4}$$

here and afterwards, for $f_\ell \in L^2$, we will use the notation

$$f_{\ell:j,k} = 2^{j/2} f_\ell(2^j x - k).$$

As usual, $\mathbf{\Phi}(x)$ in (1) generates a multiresolution analysis $\{\mathbf{V}_j\}_{j\in Z}$ of $L^2(R)$, if $\{\mathbf{V}_j\}_{j\in Z}$ defined in (4) satisfy the following properties:

(1) $\cdots \subset \mathbf{V}_0 \subset \mathbf{V}_1 \subset \mathbf{V}_2 \subset \cdots$;
(2) $\mathrm{clos}_{L^2(R)}(\bigcup_{j\in Z}\mathbf{V}_j) = L^2(R)$;
(3) $\bigcap_{j\in Z}\mathbf{V}_j = \{0\}$;
(4) $f(x) \in \mathbf{V}_j \Leftrightarrow f(2x) \in \mathbf{V}_{j+1}$, $j \in Z$;
(5) the family $\{\phi_{\ell:j,k}\colon 1 \leqslant \ell \leqslant r,\ k \in Z\}$ is a Riesz basis for $\mathbf{V}_j$.

Let $\mathbf{W}_j$, $j \in Z$, denote the complementary subspace of $\mathbf{V}_j$ in $\mathbf{V}_{j+1}$, and vector-valued function $\mathbf{\Psi}(x) = (\psi_1, \psi_2, \ldots, \psi_r)^T$, $\psi_\ell \in L^2$, $\ell = 1, 2, \ldots, r$, constitutes a Riesz basis for $\mathbf{W}_j$, i.e.,

$$\mathbf{W}_j = \mathrm{clos}_{L^2(R)}\langle \psi_{\ell:j,k}\colon\ 1 \leqslant \ell \leqslant r,\ k \in Z\rangle,\ j \in Z. \tag{5}$$

From condition (5), it is clear that $\psi_1(x), \psi_2(x), \ldots, \psi_r(x)$ are in $\mathbf{W}_0 \subset \mathbf{V}_1$. Hence there exists a sequence of matrices $\{Q_k\}_{k\in Z}$ such that

$$\mathbf{\Psi}(x) = \sum_{k\in Z} Q_k \mathbf{\Phi}(2x - k). \tag{6}$$

By the two-scale relation (6) of $\mathbf{\Psi}$, we have

$$\hat{\mathbf{\Psi}}(w) = Q(z)\hat{\mathbf{\Phi}}\left(\frac{w}{2}\right), \tag{7}$$

where

$$Q(z) = \frac{1}{2}\sum_{k\in Z} Q_k z^k. \tag{8}$$

As such, the family $\{\phi_{\ell:j,k}, \psi_{\ell:j,k}\colon\ 1 \leqslant \ell \leqslant r,\ k \in Z\}$ constitutes a Riesz basis of $\mathbf{V}_{j+1}$, i.e.,

$$\mathbf{V}_{j+1} = \mathbf{V}_j + \mathbf{W}_j. \tag{9}$$

3. Construction of biorthogonal multiwavelets

For column vector functions Λ and Γ with elements in $L^2(R)$, define

$$\langle \Lambda, \Gamma \rangle = \int_R \Lambda(x)\Gamma(x)^T \, dx.$$

We call $\mathbf{\Phi}(x) = (\phi_1, \phi_2, \ldots, \phi_r)^T$ and $\tilde{\mathbf{\Phi}}(x) = (\tilde{\phi}_1, \tilde{\phi}_2, \ldots, \tilde{\phi}_r)^T$ a pair of biorthogonal multiscaling functions, if

$$\big\langle \mathbf{\Phi}(\cdot), \tilde{\mathbf{\Phi}}(\cdot - n) \big\rangle = \delta_{0,n} I_r, \quad n \in Z. \tag{10}$$

$\mathbf{\Psi}(x) = (\psi_1, \psi_2, \ldots, \psi_r)^T$ and $\tilde{\mathbf{\Psi}}(x) = (\tilde{\psi}_1, \tilde{\psi}_2, \ldots, \tilde{\psi}_r)^T$ will be said to be a pair of biorthogonal multiwavelets associated with multiscaling functions $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$, if $\mathbf{\Phi}$, $\tilde{\mathbf{\Phi}}$ and $\mathbf{\Psi}$, $\tilde{\mathbf{\Psi}}$ satisfy the following equations:

$$\big\langle \mathbf{\Phi}(\cdot), \tilde{\mathbf{\Psi}}(\cdot - n) \big\rangle = \big\langle \mathbf{\Psi}(\cdot), \tilde{\mathbf{\Phi}}(\cdot - n) \big\rangle = O, \tag{11}$$

$$\big\langle \mathbf{\Psi}(\cdot), \tilde{\mathbf{\Psi}}(\cdot - n) \big\rangle = \delta_{0,n} I_r, \quad n \in Z, \tag{12}$$

where O and I_r denote the zero matrix and unity matrix, respectively.

Similar to (1) and (6), $\tilde{\mathbf{\Phi}}$ and $\tilde{\mathbf{\Psi}}$ also satisfy the following two-scale matrix equations:

$$\tilde{\mathbf{\Phi}}(x) = \sum_{k \in Z} \tilde{P}_k \tilde{\mathbf{\Phi}}(2x - k), \tag{13}$$

$$\tilde{\mathbf{\Psi}}(x) = \sum_{k \in Z} \tilde{Q}_k \tilde{\mathbf{\Phi}}(2x - k). \tag{14}$$

By taking Fourier transform for the both sides of (13) and (14), we have

$$\hat{\tilde{\mathbf{\Phi}}}(w) = \tilde{P}(z) \hat{\tilde{\mathbf{\Phi}}}\left(\frac{w}{2}\right), \tag{15}$$

$$\hat{\tilde{\mathbf{\Psi}}}(w) = \tilde{Q}(z) \hat{\tilde{\mathbf{\Phi}}}\left(\frac{w}{2}\right), \tag{16}$$

where

$$\tilde{P}(z) = \frac{1}{2} \sum_{k \in Z} \tilde{P}_k z^k, \tag{17}$$

$$\tilde{Q}(z) = \frac{1}{2} \sum_{k \in Z} \tilde{Q}_k z^k. \tag{18}$$

In the case of uniwavelet (i.e., $r = 1$), there is a simple procedure for finding uniwavelet if uniscaling function is known. In the case of multiwavelets, however, it seems that a simple approach obtaining multiwavelets has not been discovered yet. In the following, we will proceed to investigate the construction of multiwavelets and give a approach for constructing a pair of biorthogonal multiwavelets associated with a given pair of biorthogonal multiscaling functions. The given method is very simple like in the case of uniwavelet.

To study the biorthogonal multiwavelets, we need the following

Lemma 1. *Let $\eta = (\eta_1, \eta_2, \ldots, \eta_r)^T$, $\tilde{\eta} = (\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_r)^T$, where $\eta_1, \eta_2, \ldots, \eta_r$, $\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_r \in L^2$. Then η and $\tilde{\eta}$ are a family of biorthogonal functions if and only if*

$$\sum_{k\in Z} \hat{\eta}(w + 2k\pi)\hat{\tilde{\eta}}(w + 2k\pi)^* = I_r, \quad |z| = 1; \tag{19}$$

here and throughout, the asterisk denotes complex conjugation of transpose.

Proof. Let η and $\tilde{\eta}$ are a family of biorthogonal functions. For every $n \in Z$, we have

$$\begin{aligned}
\delta_{0,n} I_r &= \langle \eta(\cdot), \tilde{\eta}(\cdot - n)\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\eta}(w)\hat{\tilde{\eta}}(w)^* e^{inw}\, dw \\
&= \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} \int_{2k\pi}^{2(k+1)\pi} \hat{\eta}(w)\hat{\tilde{\eta}}(w)^* e^{inw}\, dw \\
&= \frac{1}{2\pi} \int_{0}^{2k\pi} \left[\sum_{k\in Z} \hat{\eta}(w + 2k\pi)\hat{\tilde{\eta}}(w + 2k\pi)^*\right] e^{inw}\, dw
\end{aligned}$$

which implies (19) holds. The converse is obvious. □

Theorem 1. *Let $\mathbf{\Phi}(x)$ and $\tilde{\mathbf{\Phi}}(x)$ defined in* (1) *and* (13)*, respectively, be a pair of biorthogonal multiscaling functions, $P(z)$ and $\tilde{P}(z)$ defined in* (3) *and* (17)*, respectively, be two-scale matrix symbols. Then $P(z)$ and $\tilde{P}(z)$ satisfy the identity*

$$P(z)\tilde{P}(z)^* + P(-z)\tilde{P}(-z)^* = I_r, \quad |z| = 1. \tag{20}$$

Equivalently, the two-scale matrix sequences $\{P_k\}$, $\{\tilde{P}_k\}$ satisfy

$$\sum_{i\in Z} P_i \tilde{P}_{i+2k}^T = 2\delta_{k,0} I_r, \quad |z| = 1. \tag{21}$$

Further, suppose $\mathbf{\Psi}$ and $\tilde{\mathbf{\Psi}}$ are a pair of biorthogonal multiwavelets associated with $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$, respectively, and $Q(z)$ and $\tilde{Q}(z)$ are two-scale matrix symbols. Then

$$\begin{cases}
P(z)\tilde{Q}(z)^* + P(-z)\tilde{Q}(-z)^* = O, \\
\tilde{P}(z)Q(z)^* + \tilde{P}(-z)Q(-z)^* = O, \\
Q(z)\tilde{Q}(z)^* + Q(-z)\tilde{Q}(-z)^* = I_r.
\end{cases} \tag{22}$$

Equivalently,

$$\sum_{i\in Z} P_i \tilde{Q}^T_{i+2k} = O, \tag{23}$$

$$\sum_{i\in Z} \tilde{P}_i Q^T_{i+2k} = O, \tag{24}$$

$$\sum_{i\in Z} Q_i \tilde{Q}'^T_{i+2k} = 2\delta_{0,k} I_r. \tag{25}$$

Theorem 1 can be proved easily by Lemma 1.

Define two matrices $M_{P,Q}(z)$, $\tilde{M}_{\tilde{P},\tilde{Q}}(z)$ by

$$M_{P,Q}(z) = \begin{bmatrix} P(z) & P(-z) \\ Q(z) & Q(-z) \end{bmatrix}, \qquad \tilde{M}_{\tilde{P},\tilde{Q}}(z) = \begin{bmatrix} \tilde{P}(z) & \tilde{P}(-z) \\ \tilde{Q}(z) & \tilde{Q}(-z) \end{bmatrix}. \tag{26}$$

Then (20) and (22) are equivalent to the following single equation:

$$M_{P,Q}(z)\tilde{M}_{\tilde{P},\tilde{Q}}(z) = I_{2r}, \quad |z| = 1. \tag{27}$$

Lemma 2. *Let $\mathbf{\Phi}(x)$ and $\tilde{\mathbf{\Phi}}(x)$, a pair of compactly supported biorthogonal multiscaling functions with multiplicity r, satisfy the following equations*:

$$\mathbf{\Phi}(x) = \sum_{k=0}^{M} P_k \mathbf{\Phi}(2x-k), \tag{28}$$

$$\tilde{\mathbf{\Phi}}(x) = \sum_{k=0}^{M} \tilde{P}_k \tilde{\mathbf{\Phi}}(2x-k). \tag{29}$$

Set $\mathbf{\Phi}'(x) = (\mathbf{\Phi}(2x)^T, \mathbf{\Phi}(2x-1)^T)^T$, $\tilde{\mathbf{\Phi}}'(x) = (\tilde{\mathbf{\Phi}}(2x)^T, \tilde{\mathbf{\Phi}}(2x-1)^T)^T$. Then

(1) $\mathbf{\Phi}'(x)$ *and* $\tilde{\mathbf{\Phi}}'(x)$ *are also a pair of compactly supported biorthogonal multiscaling functions with multiplicity* $2r$, *and* $\operatorname{supp}\mathbf{\Phi}'(x) \subset [0, \lceil M/2\rceil]$, $\tilde{\mathbf{\Phi}}'(x) \subset [0, \lceil M/2\rceil]$, *where* $\lceil x\rceil = \inf\{n\colon n \geqslant x,\ n \in Z\}$;

(2) $\mathbf{\Phi}'(x)$ *and* $\tilde{\mathbf{\Phi}}'(x)$ *satisfy the following two-scale matrix equations*:

$$\mathbf{\Phi}'(x) = \sum_{k=0}^{\lceil M/2\rceil} \begin{bmatrix} P_{2k} & P_{2k+1} \\ P_{2k-2} & P_{2k-1} \end{bmatrix} \mathbf{\Phi}'(2x-k), \tag{30}$$

$$\tilde{\mathbf{\Phi}}'(x) = \sum_{k=0}^{\lceil M/2\rceil} \begin{bmatrix} \tilde{P}_{2k} & \tilde{P}_{2k+1} \\ \tilde{P}_{2k-2} & \tilde{P}_{2k-1} \end{bmatrix} \tilde{\mathbf{\Phi}}'(2x-k). \tag{31}$$

Proof. (1) The biorthogonality of $\mathbf{\Phi}'(x)$ and $\tilde{\mathbf{\Phi}}'(x)$ is clear.

(2) By the definition of $\mathbf{\Phi}'(x)$, we have

$$\begin{aligned}
\mathbf{\Phi}'(x) &= \begin{bmatrix} \mathbf{\Phi}(2x) \\ \mathbf{\Phi}(2x-1) \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{M} P_k \mathbf{\Phi}(4x-k) \\ \sum_{k=0}^{M} P_k \mathbf{\Phi}(4x-k-2) \end{bmatrix} \\
&= \begin{bmatrix} \sum_{k=0}^{\lceil M/2 \rceil} P_{2k} \mathbf{\Phi}(4x-2k) + \sum_{k=0}^{\lceil M/2 \rceil} P_{2k+1} \mathbf{\Phi}(4x-k-1) \\ \sum_{k=0}^{\lceil M/2 \rceil} P_{2k} \mathbf{\Phi}(4x-2(k+1)) + \sum_{k=0}^{\lceil M/2 \rceil} P_{2k+1} \mathbf{\Phi}(4x-2k-3) \end{bmatrix} \\
&= \sum_{k=0}^{\lceil M/2 \rceil} \begin{bmatrix} P_{2k} & P_{2k+1} \\ P_{2k-2} & P_{2k-1} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}(2(2x)-k) \\ \mathbf{\Phi}(2(2x-1)-k) \end{bmatrix} \\
&= \sum_{k=0}^{\lceil M/2 \rceil} \begin{bmatrix} P_{2k} & P_{2k+1} \\ P_{2k-2} & P_{2k-1} \end{bmatrix} \mathbf{\Phi}'(2x-k).
\end{aligned}$$

This means (30) holds. By (30), we obtain that $\operatorname{supp} \mathbf{\Phi}'(x) \subset [0, \lceil M/2 \rceil]$. Similarly, (31) can be deduced. □

According to Lemma 2, without loss of generality, we only discuss the problems about construction of multiwavelets with 3-coefficient. We give the main result of this paper below.

Theorem 2. *Let $\mathbf{\Phi}(x)$ and $\tilde{\mathbf{\Phi}}(x)$ be a pair of 3-coefficient compactly supported biorthogonal multiscaling functions satisfying the following equations*:

$$\mathbf{\Phi}(x) = P_0 \mathbf{\Phi}(2x) + P_1 \mathbf{\Phi}(2x-1) + P_2 \mathbf{\Phi}(2x-2), \tag{32}$$

$$\tilde{\mathbf{\Phi}}(x) = \tilde{P}_0 \tilde{\mathbf{\Phi}}(2x) + \tilde{P}_1 \tilde{\mathbf{\Phi}}(2x-1) + \tilde{P}_2 \tilde{\mathbf{\Phi}}(2x-2). \tag{33}$$

Assume that there exists an integer i, $0 \leqslant i \leqslant 2$, such that the matrix D defined in the following equation is invertible:

$$D^2 = \left(2I_r - P_i \tilde{P}_i^T\right)^{-1} P_i \tilde{P}_i^T. \tag{34}$$

Define

$$\begin{cases} Q_j = D P_j, & j \neq i, \\ Q_j = -D^{-1} P_j, & j = i, \\ \tilde{Q}_j = D^T \tilde{P}_j, & j \neq i, \\ \tilde{Q}_j = -(D^T)^{-1} \tilde{P}_j, & j = i, \end{cases} \quad i, j \in \{0, 1, 2\}. \tag{35}$$

Then the following equations defining $\mathbf{\Psi}(x)$ and $\tilde{\mathbf{\Psi}}(x)$ are a pair of biorthogonal multiwavelets associated with $\mathbf{\Phi}(x)$ and $\tilde{\mathbf{\Phi}}(x)$:

$$\mathbf{\Psi}(x) = Q_0 \mathbf{\Phi}(2x) + Q_1 \mathbf{\Phi}(2x-1) + Q_2 \mathbf{\Phi}(2x-2),$$

$$\tilde{\mathbf{\Psi}}(x) = \tilde{Q}_0 \tilde{\mathbf{\Phi}}(2x) + \tilde{Q}_1 \tilde{\mathbf{\Phi}}(2x-1) + \tilde{Q}_2 \tilde{\mathbf{\Phi}}(2x-2).$$

Proof. For convenience, let $i = 1$. By (23)–(25), it suffices to show that $\{Q_0, Q_1, Q_2, \tilde{Q}_0, \tilde{Q}_1, \tilde{Q}_2\}$ satisfy the following equations:

$$P_0 \tilde{Q}_2{}^T = O, \tag{36}$$

$$P_0 \tilde{Q}_0^T + P_1 \tilde{Q}_1^T + P_2 \tilde{Q}_2^T = O, \tag{37}$$

$$\tilde{P}_0 Q_2^T = O, \tag{38}$$

$$\tilde{P}_0 Q_0^T + \tilde{P}_1 Q_1^T + \tilde{P}_2 Q_2^T = O, \tag{39}$$

$$Q_0 \tilde{Q}_2^T = O, \tag{40}$$

$$\tilde{Q}_0 Q_2^T = O, \tag{41}$$

$$Q_0 \tilde{Q}_0^T + Q_1 \tilde{Q}_1^T + Q_2 \tilde{Q}_2^T = 2I_r. \tag{42}$$

If $\{Q_0, Q_1, Q_2, \tilde{Q}_0, \tilde{Q}_1, \tilde{Q}_2\}$ are given by (35), then Eqs. (36), (38), (40) and (41) are obtained immediately from (21).

For (37), we have from (21) that

$$\begin{aligned}
P_0 \tilde{Q}_0^T + P_1 \tilde{Q}_1^T + P_2 \tilde{Q}_2^T &= P_0 \tilde{P}_0^T D - P_1 \tilde{P}_1^T D^{-1} + P_2 \tilde{P}_2^T D \\
&= \left[P_0 \tilde{P}_0^T + P_2 \tilde{P}_2^T\right] D - P_1 \tilde{P}_1^T D^{-1} \\
&= \left[2I_r - P_1 \tilde{P}_1^T\right] D - P_1 \tilde{P}_1^T D^{-1} \\
&= \left[\left(2I_r - P_1 \tilde{P}_1^T\right) D^2 - P_1 \tilde{P}_1^T\right] D^{-1} = O.
\end{aligned}$$

Similarly, (39) can be derived. Finally, since

$$\begin{aligned}
Q_0 \tilde{Q}_0^T + Q_1 \tilde{Q}_1^T + Q_2 \tilde{Q}_2^T &= D P_0 \tilde{P}_0^T D + D^{-1} P_1 \tilde{P}_1^T D^{-1} + D P_2 \tilde{P}_2^T D \\
&= D\left[P_0 \tilde{P}_0^T + P_2 \tilde{P}_2^T\right] D + D^{-1} P_1 \tilde{P}_1^T D^{-1} \\
&= D\left[2I_r - P_1 \tilde{P}_1^T\right] D + D^{-1} P_1 \tilde{P}_1^T D^{-1} \\
&= D^{-1}\left[D^2\left(2I_r - P_1 \tilde{P}_1^T\right) D^2 - P_1 \tilde{P}_1^T\right] D^{-1} \\
&= D^{-1}\left[D^2 P_1 \tilde{P}_1^T + P_1 \tilde{P}_1^T\right] D^{-1} \\
&= D^{-1}\left[D^2 + I_r\right] P_1 \tilde{P}_1^T D^{-1} \\
&= D\left[P_1 \tilde{P}_1^T + D^{-2} P_1 \tilde{P}_1^T\right] D^{-1} \\
&= D 2I_r D^{-1} = 2I_r,
\end{aligned}$$

(42) then follows. This completes the proof of Theorem 2. □

Corollary 1. *Let* $\mathbf{\Phi}(x)$ *defined in* (32) *be* 3*-coefficient compactly supported orthogonal multiscaling function. Assume that there exists an integer* i, $0 \leqslant i \leqslant 2$, *such that the matrix* H *defined in the following equation is invertible and symmetric*:

$$H^2 = \left(2I_r - P_i P_i^T\right)^{-1} P_i P_i^T. \tag{43}$$

Let

$$\begin{cases} Q_j = HP_j, & j \neq i, \\ Q_j = -H^{-1}P_j, & j = i, \end{cases} \quad i, j \in \{0, 1, 2\}. \tag{44}$$

Then the following equation defining $\mathbf{\Psi}(x)$ *is orthogonal multiwavelet associated with* $\mathbf{\Phi}(x)$:

$$\mathbf{\Psi}(x) = Q_0\mathbf{\Phi}(2x) + Q_1\mathbf{\Phi}(2x-1) + Q_2\mathbf{\Phi}(2x-2).$$

Remark. According to the matrix theory, if B is an invertible matrix satisfying $A^2 = B$, then matrix A is not unique. Hence, matrices D and H defined in (34) and (43), respectively, are not unique. Further, by Theorem 1 (or Corollary 1) we declare that there exist many distinct multiwavelets associated with a pair of biorthogonal multiscaling functions (or an orthogonal multiscaling functions).

4. Examples

We will illustrate by some examples how to use our method to construct biorthogonal (or orthogonal) multiwavelets.

Example 1 (Construction of biorthogonal multiwavelets). Let $\mathbf{\Phi}(x) = (\phi_1, \phi_2)^T$ and $\tilde{\mathbf{\Phi}}(x) = (\tilde{\phi}_1, \tilde{\phi}_2)^T$, $\operatorname{supp}\mathbf{\Phi}(x) = \operatorname{supp}\tilde{\mathbf{\Phi}}(x) = [-1, 1]$, be a pair of 3-coefficient biorthogonal multiscaling functions satisfying the following equations [9]:

$$\mathbf{\Phi}(x) = P_{-1}\mathbf{\Phi}(2x+1) + P_0\mathbf{\Phi}(2x) + P_1\mathbf{\Phi}(2x-1),$$
$$\tilde{\mathbf{\Phi}}(x) = \tilde{P}_{-1}\tilde{\mathbf{\Phi}}(2x+1) + \tilde{P}_0\tilde{\mathbf{\Phi}}(2x) + \tilde{P}_1\tilde{\mathbf{\Phi}}(2x+1),$$

where

$$P_{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \\ -1 & -\frac{2}{5} \end{bmatrix}, \qquad P_0 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \qquad P_1 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{5} \\ 1 & -\frac{2}{5} \end{bmatrix},$$
$$\tilde{P}_{-1} = \begin{bmatrix} \frac{1}{2} & \frac{5}{4} \\ -\frac{7}{16} & -\frac{35}{32} \end{bmatrix}, \qquad \tilde{P}_0 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \qquad \tilde{P}_1 = \begin{bmatrix} \frac{1}{2} & -\frac{5}{4} \\ \frac{7}{16} & -\frac{35}{32} \end{bmatrix}.$$

Suppose $i = 0$. Using (34) and (35) we obtain

$$D = \begin{bmatrix} 1 & 0 \\ 0 & \frac{\sqrt{7}}{7} \end{bmatrix},$$
$$Q_{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \\ -\frac{\sqrt{7}}{7} & -\frac{2\sqrt{7}}{35} \end{bmatrix}, \qquad Q_0 = \begin{bmatrix} -1 & 0 \\ 0 & -\frac{\sqrt{7}}{2} \end{bmatrix},$$
$$Q_1 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{5} \\ \frac{\sqrt{7}}{7} & -\frac{2\sqrt{7}}{35} \end{bmatrix}, \qquad \tilde{Q}_{-1} = \begin{bmatrix} \frac{1}{2} & \frac{5}{4} \\ -\frac{\sqrt{7}}{16} & -\frac{5\sqrt{7}}{32} \end{bmatrix},$$

$$\tilde{Q}_0 = \begin{bmatrix} -1 & 0 \\ 0 & -\frac{\sqrt{7}}{2} \end{bmatrix}, \qquad \tilde{Q}_1 = \begin{bmatrix} \frac{1}{2} & -\frac{5}{4} \\ \frac{\sqrt{7}}{16} & -\frac{5\sqrt{7}}{32} \end{bmatrix}.$$

From Theorem 2 we conclude that

$$\mathbf{\Psi}(x) = \sum_{k=-1}^{1} Q_k \mathbf{\Phi}(2x-k), \qquad \tilde{\mathbf{\Psi}}(x) = \sum_{k=-1}^{1} \tilde{Q}_k \tilde{\mathbf{\Phi}}(2x-k)$$

are a pair of biorthogonal multiwavelets associated with $\mathbf{\Phi}(x)$, $\tilde{\mathbf{\Phi}}(x)$.

Example 2 (Construction of orthogonal multiwavelets). Let $\mathbf{\Phi}(x) = (\phi_1, \phi_2)^T$, $\operatorname{supp} \mathbf{\Phi}(x) = [0, 2]$, be 3-coefficient orthogonal multiscaling function satisfying the following equation [10]:

$$\mathbf{\Phi}(x) = P_0 \mathbf{\Phi}(2x) + P_1 \mathbf{\Phi}(2x-1) + P_2 \mathbf{\Phi}(2x-2),$$

where

$$P_0 = \begin{bmatrix} 0 & \frac{2+\sqrt{7}}{4} \\ 0 & \frac{2-\sqrt{7}}{4} \end{bmatrix}, \qquad P_1 = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}, \qquad P_2 = \begin{bmatrix} \frac{2-\sqrt{7}}{4} & 0 \\ \frac{2+\sqrt{7}}{4} & 0 \end{bmatrix}.$$

Suppose $i = 1$. Using (43) and (44) in Corollary 1, we obtain

$$H = \begin{bmatrix} \frac{7+\sqrt{7}}{14} & \frac{7-\sqrt{7}}{14} \\ \frac{7-\sqrt{7}}{14} & \frac{7+\sqrt{7}}{14} \end{bmatrix},$$

$$Q_0 = \begin{bmatrix} 0 & \frac{3}{4} \\ 0 & \frac{1}{4} \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} -\frac{2+\sqrt{7}}{4} & -\frac{2-\sqrt{7}}{4} \\ -\frac{2-\sqrt{7}}{4} & -\frac{2+\sqrt{7}}{4} \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{3}{4} & 0 \end{bmatrix}.$$

Form Corollary 1,

$$\mathbf{\Psi}(x) = \sum_{k=0}^{2} Q_k \mathbf{\Phi}(2x-k)$$

is orthogonal multiwavelet associated with $\mathbf{\Phi}(x)$.

Example 3 (Construction of orthogonal multiwavelets). Let $\mathbf{\Phi}(x) = (\phi_1, \phi_2)^T$ be 3-coefficient orthogonal multiscaling function satisfying the following equation [11,12]:

$$\mathbf{\Phi}(x) = P_0 \mathbf{\Phi}(2x) + P_1 \mathbf{\Phi}(2x-1) + P_2 \mathbf{\Phi}(2x-2),$$

where

$$P_0 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ \sqrt{2}\sin\theta & \sqrt{2}\sin\theta \end{bmatrix}, \qquad P_1 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2}\cos\theta \end{bmatrix},$$

$$P_2 = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -\sqrt{2}\sin\theta & \sqrt{2}\sin\theta \end{bmatrix}.$$

Suppose $i = 1$. Using (43) and (44) in Corollary 1, we have

$$H = \begin{bmatrix} 1 & 0 \\ 0 & \tan\theta \end{bmatrix}, \qquad H^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \coth\theta \end{bmatrix},$$

where θ satisfies that $\sin\theta \neq 0$, $\cos\theta \neq 0$,

$$Q_0 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ \sqrt{2}\cos\theta & \sqrt{2}\cos\theta \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} -1 & 0 \\ 0 & -\sqrt{2}\sin\theta \end{bmatrix},$$

$$Q_2 = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -\sqrt{2}\cos\theta & \sqrt{2}\cos\theta \end{bmatrix}.$$

By Corollary 1,

$$\mathbf{\Psi}(x) = \sum_{k=0}^{2} Q_k \mathbf{\Phi}(2x - k)$$

is orthogonal multiwavelet associated with $\mathbf{\Phi}(x)$.

It is clear that our method is simpler than that of the papers [11,12], while the obtained orthogonal multiwavelets coincide.

Example 4 (Trivial example—construction of orthogonal uniwavelet). Let ϕ_3^D be Daubechies scaling function [13], i.e.,

$$\phi_3^D(x) = \frac{1+\sqrt{3}}{4}\phi_3^D(2x) + \frac{3+\sqrt{3}}{4}\phi_3^D(2x-1) + \frac{3-\sqrt{3}}{4}\phi_3^D(2x-2) + \frac{1-\sqrt{3}}{4}\phi_3^D(2x-3).$$

Since $\phi_3^D(x)$ is a 4-coefficient orthogonal scaling function, from Lemma 2, let $\mathbf{\Phi}(x) = (\phi_3^D(2x), \phi_3^D(2x-1))^T$. Then

$$\mathbf{\Phi}(x) = \begin{bmatrix} \frac{1+\sqrt{3}}{4} & \frac{3+\sqrt{3}}{4} \\ 0 & 0 \end{bmatrix}\mathbf{\Phi}(2x) + \begin{bmatrix} \frac{3-\sqrt{3}}{4} & \frac{1-\sqrt{3}}{4} \\ \frac{1+\sqrt{3}}{4} & \frac{3+\sqrt{3}}{4} \end{bmatrix}\mathbf{\Phi}(2x-1) + \begin{bmatrix} 0 & 0 \\ \frac{3-\sqrt{3}}{4} & \frac{1-\sqrt{3}}{4} \end{bmatrix}\mathbf{\Phi}(2x-2)$$

is an orthogonal multiscaling function with multiplicity 2. Using Corollary 1, we obtain

$$\mathbf{\Psi}(x) = \begin{bmatrix} \psi_3^D(2x) \\ \psi_3^D(2x-1) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}-1}{4} & \frac{3-\sqrt{3}}{4} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \phi_3^D(4x) \\ \phi_3^D(4x-1) \end{bmatrix} + \begin{bmatrix} -\frac{3+\sqrt{3}}{4} & \frac{1+\sqrt{3}}{4} \\ -\frac{\sqrt{3}-1}{4} & -\frac{3-\sqrt{3}}{4} \end{bmatrix}\begin{bmatrix} \phi_3^D(4x-2) \\ \phi_3^D(4x-3) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{3+\sqrt{3}}{4} & \frac{1+\sqrt{3}}{4} \end{bmatrix}\begin{bmatrix} \phi_3^D(4x-4) \\ \phi_3^D(4x-5) \end{bmatrix}.$$

Hence

$$\psi_3^D(x) = \frac{\sqrt{3}-1}{4}\phi_3^D(2x) + \frac{3-\sqrt{3}}{4}\phi_3^D(2x-1) - \frac{3+\sqrt{3}}{4}\phi_3^D(2x-2) + \frac{1+\sqrt{3}}{4}\phi_3^D(2x-3).$$

The orthogonal wavelet constructed by using our method is the same as the wavelet obtained by using method in [13]. Of course, the method in the literature [13] is simpler than our method. But in the case of multiwavelets, the larger multiplicity r is, the more advantageous our method is.

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