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# A characterization of 3-graded Lie algebras generated by a pair

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**Abstract**

We prove that any 3-graded Lie algebra generated by an element of degree $-1$ and another of degree 1 over a field $K$ of characteristic zero is isomorphic to a 3-graded Lie subalgebra of $\mathfrak{sl}_2(K[t]/(p(t)\cdot K[t]))$ endowed with its usual 3-gradation, for some $p(t)\in K[t]$. We also give a thorough description of the ideals in the free case. 

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## 1. Introduction

A $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ of the form

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \tag{1}$$

over a field $K$ is called a 3-*graded Lie algebra*.

Such algebras include the complexification of a semisimple Lie algebra of hermitian type (which is a 3-graded Lie algebra with respect to the Harish–Chandra decomposition), Heisenberg algebras and some infinite dimensional examples as well (see, for instance, [7,10]). They also generalize the so-called Kantor–Koecher–Tits algebras or KKT algebras for short, which correspond to the concept of Jordan pair written in terms

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of Lie algebras [8,9]. They are those 3-graded Lie algebras spanned by homogeneous elements of degrees $-1$ and 1 and whose center contains no nonzero homogeneous element of degree zero. This second condition, although crucial in many situations, is a strong assumption; relaxing it, one still obtains very interesting, even more natural and uniform results and the real role of such assumption in the global theory becomes clearer. This procedure behaves in much the same way as waiving the nondegeneracy property of the Poisson bivector in symplectic geometry to consider the more general setting of Poisson manifolds.

In this article we are going to deal with 3-graded Lie algebras spanned by a pair of generators, that is, by a homogeneous element of degree 1 and another of degree $-1$. Although, in our opinion, generalizations of $\mathfrak{sl}_2(K)$ of that kind do not need further motivations, we want to point out that they could be useful to study the general case, by analyzing its 3-graded subalgebras generated by pairs of elements conveniently chosen. This is done for instance in [1] where the free case in a pair of generators is utilized to prove commutations relations for the arbitrary case. Those relations have been applied to the description of reproducing kernels for Hilbert spaces of holomorphic functions associated with the holomorphic discrete series representations (see [2]).

If one considers a complex semisimple Lie algebra of hermitian type, one obtains a unitary representation for each of its 3-graded subalgebras by restriction of a unitary representation of the first. In this way, it would be interesting to examine a likely relation between the characterization proved in Theorem 16 and the Wallach set in unitarizable highest weight modules over the complex hermitian Lie algebra in question. Although the last subject has been studied extensively in, for instance, [4,6] and even considerably simplified in [5], it remains very technical in essence and perhaps a connection with 3-graded algebras could bring in some new elucidation on this matter.

Here, we have tried to advance the subject as far as possible without any additional hypothesis over $K$ besides $\operatorname{char} K = 0$; the case when $K$ is algebraically closed will be treated on another occasion. It is somewhat surprising how prolific and computable the theory reveals to be even under so feeble hypotheses.

The fact that a 3-graded Lie algebra $\mathfrak{g}$ in a pair of generators is realizable as a subalgebra of $\mathfrak{sl}_2(K[t]/(p(t)\cdot K[t]))$ for some $p(t) \in K[t]$ gives a measure of its center. For instance, the free case examined below has center zero since it can be seen as a 3-graded subalgebra of $\mathfrak{sl}_2(K[t])$. Of course, this is not the case of the tridimensional Heisenberg Lie algebra, for instance.

We reproduce here a direct proof of the embedding of the free case into $\mathfrak{sl}_2(K[t])$ without making use of KKT algebras as is carried out in [1]. For a comparison between the free KKT algebra and the free 3-graded Lie algebra, both in a pair of generators, when $\operatorname{char} K \neq 2, 3$, see [3] and Remark 3.

We assume $\operatorname{char} K = 0$ throughout the article. In order to prove embedding results for a 3-graded Lie algebra in a pair of generators, we start looking at the free example.

### 1.1. The free case

A 3-graded Lie Algebra $\mathfrak{g}(x, y)$ over $K$ generated by variables $x$ of degree 1 and $y$ of degree $-1$ is *free* if the following property holds: given any 3-graded Lie algebra $\mathfrak{h}$

over $K$ and elements $x' \in \mathfrak{h}_1$ and $y' \in \mathfrak{h}_{-1}$, there is a unique homomorphism of graded Lie algebras $\psi : \mathfrak{g}(x, y) \to \mathfrak{h}$ such that $\psi(x) = x'$ and $\psi(y) = y'$.

The uniqueness of $\mathfrak{g}(x, y)$ is immediate. In order to prove its existence, we consider $\mathfrak{FL}(x, y)$ the free Lie algebra over $K$ in two variables $x$ and $y$ with the natural bigradation given by the number $n_x$ of occurrences of $x$ and $n_y$ of $y$ in a Lie monomial (recall that the elements of $\mathfrak{FL}(x, y)$ are linear combinations of expressions of the form $[x_1, \dots, [x_{m-2}, [x_{m-1}, x_m]] \dots]$ with $x_i = x$ or $y$, $m = 1, 2, \dots$ , called *Lie monomials*) and let $\mathfrak{I}$ be the ideal of $\mathfrak{FL}(x, y)$ spanned by the homogeneous elements of $\mathfrak{FL}(x, y)$ such that $|n_x - n_y| \geqslant 2$.

$\mathfrak{I}$ is a graded ideal since it is spanned by homogeneous elements. Therefore, $\mathfrak{g} = \mathfrak{FL}(x, y)/\mathfrak{I}$ inherits the bigradation of $\mathfrak{FL}(x, y)$ and the natural projection $\tau : \mathfrak{FL}(x, y) \to \mathfrak{g}$ becomes a homomorphism of bigraded Lie algebras.

From its bigradation, we obtain the $\mathbb{Z}$-gradation for $\mathfrak{g}$: to a homogeneous element of bidegree $(m, n)$, we assign the degree $m - n$. For this $\mathbb{Z}$-gradation, we finally have

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

which is a 3-gradation of $\mathfrak{g}$ such that $\deg(x + \mathfrak{I}) = 1$ and $\deg(y + \mathfrak{I}) = -1$.

Now, given an arbitrary 3-graded Lie algebra $\mathfrak{h}$ over $K$ and elements $x' \in \mathfrak{h}_1$ and $y' \in \mathfrak{h}_{-1}$, let $\bar{\psi} : \mathfrak{FL}(x, y) \to \mathfrak{h}$ be the homomorphism defined by $\bar{\psi}(x) = x'$ and $\bar{\psi}(y) = y'$. Since $\bar{\psi}(\mathfrak{I}) = 0$, $\bar{\psi}$ descends to a homomorphism $\psi : \mathfrak{g} \to \mathfrak{h}$ such that $\psi \circ \tau = \bar{\psi}$. From the fact that $x + \mathfrak{I}, y + \mathfrak{I}$ are generators for $\mathfrak{g}$, it is immediate that $\psi$ is a homomorphism of graded Lie algebras with respect to the 3-gradations of $\mathfrak{g}$ and $\mathfrak{h}$ and that such a homomorphism is unique.

### 1.2. Commutativity of iterated brackets

Let $\mathfrak{g}$ be any 3-graded Lie algebra over $K$. For $z \in \mathfrak{g}_1$, $w \in \mathfrak{g}_{-1}$ define

$$[[z, w]]^1 = [z, w], \quad [[z, w]]^i = [z, [w, [[z, w]]^{i-1}]], \;\; 1 = 2, 3, \dots .$$

Let $I : \mathfrak{g} \to \mathfrak{g}$ be the identity map, $A_i : \mathfrak{g} \to \mathfrak{g}$, $i = 0, 1, \dots$ such that $A_0 = I$ on $\mathfrak{g}_1, -I$ on $\mathfrak{g}_{-1}$ and zero on $\mathfrak{g}_0$, $A_i = \mathrm{ad}[[z, w]]^i$, $i = 1, 2, \dots$ and we write for short $A = A_1$.

**Lemma 1.** $[[z, w], [[z, w]]^m] = 0$, $z \in \mathfrak{g}_1$, $w \in \mathfrak{g}_{-1}$, *m a positive integer.*

**Proof.** By induction. For $m = 1$ it is trivial. Suppose the result is valid for $m \leqslant p$, for some positive integer $p$. First we prove some auxiliary relations:

(i) $A_{i+1} z = A A_i z$, $p \geqslant i \geqslant 0$.
The case $i = 0$ is trivial and for $p \geqslant i \geqslant 1$:

$$A_{i+1} z = -[[z, A_i w], z] = -[z, [A_i w, z]] = -[A_i z, [z, w]] = A A_i z.$$

(ii) $A_{i+1} w = -A A_i w$, $i \geqslant 0$.

$$A_{i+1} w = -[[z, A_i w], w] = -[[z, w], A_i w] = -A A_i w.$$

(iii) For integers $i,j \geqslant 0$, if $i+j=p$ then

$$[A_{i+1}z, A_j w] = A[z, A_p w] + [A_i z, A_{j+1} w].$$

In fact, suppose $p \geqslant 2$.

$$[A_{i+1}z, A_j w] = [AA_i z, A_j w] = A[A_i z, A_j w] + [A_i z, A_{j+1} w].$$

Now $AA_i[z, A_j w] = 0$. This follows from induction hypothesis if $j < p$. For $j = p$, one has $i = 0$ and $AA_i[z, A_j w] = 0$ since $A_i$ is zero on $\mathfrak{g}_0$.

Hence, $A[A_i z, A_j w] = A[A_i A_j w, z] + AA_i[z, A_j w] = A[A_i A_j w, z] = (-1)^{j-1} A[A_i A^j w, z] = (-1)^{j-1} A[A^j A_i w, z] = (-1)^{i+j} A[A^j A^i w, z] = -A[A_p w, z] = A[z, A_p w]$.

Therefore, $[A_{i+1}z, A_j w] = A[z, A_p w] + [A_i z, A_{j+1} w]$, which is clearly valid for $p = 1$.

Back to the proof of the lemma, we have for $m = p+1$:

$$\begin{aligned} A_{p+1}[z,w] &= [A_{p+1}z, w] + [z, A_{p+1}w] = -[A_{p+1}z, A_0 w] + [z, A_{p+1}w] \\ &= -(p+1)A[z, A_p w] - [A_0 z, A_{p+1} w] + [z, A_{p+1} w] \quad \text{(by (iii))} \\ &\Rightarrow (p+2)A_{p+1}[z,w] = -[z, A_{p+1}w] + [z, A_{p+1}w] = 0, \end{aligned}$$

finishing the proof. □

Let

$$K[t^n] \equiv \{q(t^n) \mid q(t) \in K[t]\}.$$

Given $p(t) \in K[t]$ and $S \subset K[t]$, we denote

$$p(t) \cdot S \equiv S \cdot p(t) \equiv \{p(t) \cdot q(t) \mid q(t) \in S\},$$

$$\mathfrak{sl}_2(p(t)K[t]) \equiv \left\{ \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \in M_2(p(t)K[t]) \mid a(t) + d(t) = 0 \right\}.$$

Recall that $\mathfrak{sl}_2(K[t])$ is also a 3-graded Lie algebra over $K$ with respect to the gradation

$$\mathfrak{sl}_2(K[t])_{-1} = \left\{ \begin{pmatrix} 0 & 0 \\ p(t) & 0 \end{pmatrix} \middle| \; p(t) \in K[t] \right\},$$

$$\mathfrak{sl}_2(K[t])_0 = \left\{ \begin{pmatrix} p(t) & 0 \\ 0 & -p(t) \end{pmatrix} \middle| \; p(t) \in K[t] \right\},$$

$$\mathfrak{sl}_2(K[t])_1 = \left\{ \begin{pmatrix} 0 & p(t) \\ 0 & 0 \end{pmatrix} \middle| \; p(t) \in K[t] \right\},$$

$$\mathfrak{sl}_2(K[t])_i = \{0\} \subset \mathfrak{sl}_2(K[t]), \quad i \in \mathbb{Z} \setminus \{-1, 0, 1\}$$

and $\mathfrak{sl}_2(p(t)K[t])$ is a 3-graded subalgebra of $\mathfrak{sl}_2(K[t])$, $p(t) \in K[t]$.

**Theorem 2.** *Let $\mathfrak{g}(x,y)$ be the free 3-graded Lie algebra over $K$ generated by variables $x$ of degree 1 and $y$ of degree $-1$.*

*The homomorphism $\mathfrak{i}_{m,n}:\mathfrak{g}(x,y)\to\mathfrak{sl}_2(K[t])$ defined by*

$$\mathfrak{i}_{m,n}(x)=\begin{pmatrix}0 & t^m\\ 0 & 0\end{pmatrix},\quad \mathfrak{i}_{m,n}(y)=\begin{pmatrix}0 & 0\\ t^n & 0\end{pmatrix}$$

*is a monomorphism of 3-graded Lie algebras for each pair $(m,n)$ of positive integers.*

**Proof.** Let $i_{m,n}:\mathfrak{g}(x,y)\to\mathfrak{sl}_2(K[t])$ be the homomorphism defined by

$$\mathfrak{i}_{m,n}(x)=\begin{pmatrix}0 & t^m\\ 0 & 0\end{pmatrix},\quad \mathfrak{i}_{m,n}(y)=\begin{pmatrix}0 & 0\\ t^n & 0\end{pmatrix}.$$

It is a homomorphism of 3-graded Lie algebras. The set $\mathscr{B}$ formed by $x,y,[[x,y]]^i$, $[[[x,y]]^i,x]$, $[[[x,y]]^i,y]$, $i=1,2,\ldots$, spans $\mathfrak{g}(x,y)$ by Lemma 1. On the other hand, the image of these elements by $\mathfrak{i}_{m,n}$ in $\mathfrak{sl}_2(K[t])$ consists of linearly independent matrices. Therefore $\mathscr{B}$ is a basis for $\mathfrak{g}(x,y)$ and $\mathfrak{i}_{m,n}$ is a monomorphism of 3-graded Lie algebras. □

Let

$$\mathfrak{i}=\mathfrak{i}_{1,1}$$

or, in other words, the homomorphism $\mathfrak{i}:\mathfrak{g}(x,y)\to\mathfrak{sl}_2(K[t])$ defined by

$$\mathfrak{i}(x)=\begin{pmatrix}0 & t\\ 0 & 0\end{pmatrix},\quad \mathfrak{i}(y)=\begin{pmatrix}0 & 0\\ t & 0\end{pmatrix}.$$

It is easy to see that this monomorphism is also a monomorphism of graded Lie algebras with respect to the total gradation of $\mathfrak{g}(x,y)$ and the $\mathbb{N}$-gradation of $\mathfrak{sl}_2(K[t])$ where the homogeneous elements of degree $n$ are the zero trace matrices whose entries have the form $ct^n$, $c\in K$. It is called *canonical embedding* of $\mathfrak{g}(x,y)$.

**Remark 3.** Notice that these results are different from embeddings for the free KKT algebra in a pair of generators. In the latter case, since a KKT algebra is the Lie algebra associated with a Jordan pair, only the adjoint action of $\mathfrak{g}(x,y)_0$ is regarded, not $\mathfrak{g}(x,y)_0$ itself, which makes the proof a lot easier. We have proved that both Lie algebras coincide in $\operatorname{char} K=0$. It is known from the theory of Jordan pairs that the free KKT algebra in a pair of generators can be embedded into $\mathfrak{sl}_2(K[t])$ if $\operatorname{char} K\neq 2,3$. However, this does not hold for $\mathfrak{g}(x,y)$ if $\operatorname{char} K>3$. We refer the reader to [3] for more details.

**Lemma 4.**

$$\mathfrak{i}(\mathfrak{g}(x,y))=\left\{\begin{pmatrix}p(t) & q(t)\\ r(t) & -p(t)\end{pmatrix}\,\middle|\, q(t),r(t)\in tK[t^2],\ p(t)\in t^2K[t^2]\right\}.$$

**Proof.** We know that set $\mathscr{B}$ formed by $(\operatorname{ad} x \operatorname{ad} y/2)^i[x,y]$, $(\operatorname{ad} x \operatorname{ad} y/2)^i x$, $(\operatorname{ad} y \operatorname{ad} x/2)^i y, i=0,1,\ldots,$ is a basis of $\mathfrak{g}(x,y)$ by Lemma 1 and Theorem 2. The proof is now a straightforward computation. $\square$

*Notational remarks*: Given a ring $R$, with unit element $1 \in R$, we write

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

By means of the monomorphism $\mathfrak{i} : \mathfrak{g}(x,y) \to \mathfrak{s}l_2(K[t])$, we identify

$$x = tE, \quad y = tF \quad \text{and} \quad \mathfrak{g}(x,y) = \mathfrak{i}(\mathfrak{g}(x,y))$$

as described above in Lemma 4, unless otherwise specified.

Let $\theta : \mathfrak{g}(x,y) \to \mathfrak{g}(x,y)$ denote the isomorphism of Lie algebras defined by

$$\theta(x) = y, \quad \theta(y) = x,$$

which turns out to be an involution i.e. $\theta^2(z) = z$, $z \in \mathfrak{g}(x,y)$ and an isomorphism of graded Lie algebras between $\mathfrak{g}(x,y)$ and $\mathfrak{g}(x,y)^{\mathrm{op}}$ (recall that $\mathfrak{g}(x,y)^{\mathrm{op}} = \mathfrak{g}(x,y)$ as Lie algebras but $\mathfrak{g}(x,y)_i^{\mathrm{op}} = \mathfrak{g}(x,y)_{-i}$). Hence $\mathfrak{g}(x,y)$ is *symmetric* with respect to $\theta$ (i.e. a 3-graded Lie algebra endowed with an involution that reverses degrees).

## 2. Ideals of the free algebra

**Example 5.** Here we list some examples of ideals of $\mathfrak{g}(x,y)$ generated by one or two elements which will be useful later in this article.

(1) $\mathscr{I} = \langle a, b \rangle,\ a = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix},\ b = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}.$

Of course, in this case $\mathscr{I} = \mathfrak{g}(x,y)$.

(2) $\mathscr{I}_d = \langle a \rangle,\ a = \begin{pmatrix} t^2 & 0 \\ 0 & -t^2 \end{pmatrix}.$

$$\mathscr{I}_d = \left\{ \begin{pmatrix} p(t) & q(t) \\ r(t) & -p(t) \end{pmatrix} \middle| \ p(t) \in t^2K[t^2],\ q(t), r(t) \in t^3K[t^2] \right\}.$$

(3) $\mathscr{I}_l = \langle a \rangle,\ a = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}.$

$$\mathscr{I}_l = \left\{ \begin{pmatrix} p(t) & q(t) \\ r(t) & -p(t) \end{pmatrix} \middle| \ r(t) \in tK[t^2],\ p(t) \in t^2K[t^2],\ q(t) \in t^3K[t^2] \right\}.$$

(4) $\mathscr{I}_u = \langle a \rangle,\ a = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}.$

$$\mathscr{I}_u = \left\{ \begin{pmatrix} p(t) & q(t) \\ r(t) & -p(t) \end{pmatrix} \middle|\ q(t) \in tK[t^2],\ p(t) \in t^2K[t^2],\ r(t) \in t^3K[t^2] \right\}.$$

(5) $\mathscr{I}_c = \langle a \rangle,\ a = \begin{pmatrix} 0 & t \\ ct & 0 \end{pmatrix},\ c \in K^* \equiv K - \{0\}.$

$$\mathscr{I}_c = K \cdot a \oplus \mathscr{I}_d, \quad \mathscr{I}_d \text{ as in case (2).}$$

**Proof.** Let us prove, for instance, that the ideal (4) has the explicit form shown above.

Let $\mathscr{I}$ be an ideal of $\mathfrak{g}(x,y)$ generated by an element $a \in \mathfrak{g}(x,y)$. It consists of linear combinations of elements of the form

$$[x_1, \ldots, [x_{m-1}, [x_m, a]] \cdots], \quad x_i \in \mathfrak{g}(x,y). \tag{2}$$

In fact, if $\mathscr{I}'$ is the subspace of $\mathfrak{g}(x,y)$ spanned by such elements, clearly it is also an ideal of $\mathfrak{g}(x,y)$ which contains $a$ and is contained in $\mathscr{I}$. Hence one must have $\mathscr{I}' = \mathscr{I}$.

In the present situation,

$$a = tE$$

and it is easy to see that

$$t^{2n}H, t^{2n+1}F, t^{2n-1}E \in \mathscr{I}_u, \quad n \geqslant 1.$$

Let

$$X = \begin{pmatrix} p(t) & q(t) \\ r(t) & -p(t) \end{pmatrix} \in \mathscr{I}_u, \quad r(t), q(t) \in tK[t^2], \quad p(t) \in t^2K[t^2].$$

Since every element (2) but $a$ is equal to zero $\bmod M_2(t^2K[t])$, one has

$$X \equiv k \cdot a \bmod M_2(t^2K[t]), \quad k \in K,$$

and hence

$$r(t) \equiv 0 \bmod t^2K[t],$$

justifying the above description of $\mathscr{I}_u$. The proof in the other cases is similar. □

**Lemma 6.** *The quotient* $\mathfrak{g}(x,y)$ *by any nonzero* (*not necessarily graded*) *ideal is finite dimensional. In particular*, *if* $\mathfrak{g}$ *is an infinite dimensional* 3-*graded Lie algebra in a pair of generators over* $K$ *then*

$$\mathfrak{g} \cong \mathfrak{g}(x,y).$$

**Proof.** Let

$$\mathfrak{g} \cong \mathfrak{g}(x,y)/\mathscr{I}$$

where $\mathscr{I}$ is a nonzero ideal of $\mathfrak{g}(x,y)$ and

$$A = \begin{pmatrix} p(t) & q(t) \\ r(t) & -p(t) \end{pmatrix}, \quad r(t), q(t) \in tK[t^2], \quad p(t) \in t^2K[t^2]$$

a nonzero element of $\mathscr{I}$. Suppose $q(t) \neq 0$. Since

$$t^{2+2n}q(t)F, t^{3+2n}q(t)H, t^{4+2n}q(t)E \in \mathscr{I}, \quad n \geqslant 0,$$

it is clear that $\mathfrak{g}$ is finite dimensional. Similarly, $p(t) \neq 0$ or $r(t) \neq 0$ implies $\mathfrak{g}$ is finite dimensional as well. □

Of course, the ideal $\mathscr{I} = \mathfrak{g}(x,y)$ is generated by $x = tE$ and $y = tF$. The next lemma tells us that it is not spanned by just *one* element.

**Lemma 7.** *The ideal* $\mathscr{I} = \mathfrak{g}(x,y)$ *cannot be spanned by one element.*

**Proof.** Suppose otherwise that

$$\mathfrak{g}(x,y) = \langle a \rangle, \quad a = \begin{pmatrix} p(t) & q(t) \\ r(t) & -p(t) \end{pmatrix}, \quad q(t), r(t) \in tK[t^2], \quad p(t) \in t^2K[t^2].$$

Recall that any element of $\langle a \rangle$ has the form $k \cdot a + c$, where $k \in K$ and $c$ is linear combination of elements

$$[x_1, \ldots, [x_{m-1}, [x_m, a]] \ldots], \quad x_i \in \mathfrak{g}(x,y)$$

with $m \geqslant 1$. Hence for $x = tE$ we have

$$tE \equiv k \cdot a \bmod M_2(t^2K[t^2]), \quad \text{for some } k \in K$$

and

$$t \equiv k \cdot q(t) \bmod t^2K[t^2].$$

Therefore

$$k \neq 0 \quad \text{and} \quad q(t) \not\equiv 0 \bmod t^2K[t^2]. \tag{3}$$

Repeating the argument for $y = tF$, we obtain

$$r(t) \not\equiv 0 \bmod t^2K[t^2] \tag{4}$$

and we get a contradiction:

$$(tE)_{21} = 0,$$

$$(tE)_{21} \equiv k \cdot r(t) \bmod t^2K[t^2] \not\equiv 0. \qquad \square$$

Let $proj_{ij}: M_2(K[t]) \to K[t]$ denote the projection onto the $(i,j)$ entry, i.e. $proj_{ij}(A) = A_{ij},\ A \in M_2(K[t]),\ 1 \leqslant i,j \leqslant 2$.

**Lemma 8.** *Suppose $\mathscr{I}$ is a non-zero ideal of $\mathfrak{g}(x,y)$. Let $V_{ij} = proj_{ij}(\mathscr{I})$. Then there exist polynomials $p_{12}(t), p_{21}(t) \in tK[t^2]$; $p_{11}(t), p_{22}(t) \in t^2K[t^2]$ such that*

$$V_{ij} = K[t^2] \cdot p_{ij}(t).$$

*Moreover,*
(i) $p_{ij} \neq 0$ *and* $p_{ij}(t)$ *is unique up to a nonzero multiplicative constant,*
(ii) $p_{11}(t) = cp_{22}(t)$, *for some* $c \in K^*$,
(iii) $p_{11}(t) = dtp_{12}(t)$ *or* $p_{12}(t) = dtp_{11}(t)$, *for some* $d \in K^*$,
(iv) $p_{11}(t) = etp_{21}(t)$ *or* $p_{21}(t) = etp_{11}(t)$, *for some* $e \in K^*$.

**Proof.** Since $\mathscr{I} \neq 0$, one has $V_{ij} \neq 0$. Let $p_{ij}(t)$ be a minimum degree polynomial in $V_{ij}$. Since $\mathscr{I}$ is invariant by the action of

$$(\operatorname{ad} tE \operatorname{ad} tF)/2 \quad \text{and} \quad (\operatorname{ad} tF \operatorname{ad} tE)/2$$

one has

$$t^2 \cdot p_{ij}(t) \subset V_{ij}$$

and hence

$$K[t^2] \cdot p_{ij}(t) \subset V_{ij}.$$

Given $n(t) \in V_{ij}$, let $q(t)$ and $r(t)$ be the quotient and remainder of the division of $n(t)$ by $p_{ij}(t)$, respectively. Since $\deg r(t) < \deg p_{ij}(t)$ or $r(t) = 0$, it follows that

$$q(t) \in K[t^2]$$

and therefore

$$r(t) = n(t) - q(t)p_{ij}(t) \in V_{ij}.$$

By the minimality of the degree of $p_{ij}(t)$ in $V_{ij}$, it follows $r(t) = 0$ and

$$n(t) \in K[t^2] \cdot p_{ij}(t).$$

Therefore,

$$V_{ij} = K[t^2] \cdot p_{ij}(t).$$

(i) Now suppose $p_{ij}(t)$ satisfies that equality. Clearly $p_{ij}(t)$ is nonzero and unique up to a nonzero multiplicative constant for each pair $(i,j)$.

(ii) The second assertion follows from $V_{11} = V_{22}$.

(iii) Since $\mathscr{I}$ is invariant by the action of

$$\operatorname{ad} tF, \quad \operatorname{ad} tE,$$

one has

$$tp_{12}(t) \in V_{11}, \quad tp_{11}(t) \in V_{12}$$

and hence

$$\deg p_{11}(t) \leqslant \deg p_{12}(t) + 1, \quad \deg p_{12}(t) \leqslant \deg p_{11}(t) + 1$$

or

$$\|\deg p_{11}(t) - \deg p_{12}(t)\| \leqslant 1.$$

Since $\deg p_{11}(t)$ is even and $\deg p_{12}(t)$ is odd, one must have

$$\|\deg p_{11}(t) - \deg p_{12}(t)\| = 1.$$

If $\deg p_{11}(t) = \deg p_{12}(t) + 1$ then $p_{11}(t) = dtp_{12}(t)$.

Otherwise $\deg p_{12}(t) = \deg p_{11}(t) + 1$ and $p_{12}(t) = dtp_{11}(t)$, for some $d \in K^*$.

(iv) Same as (iii). □

Recall that a nonzero polynomial is said to be *monic* if its leading coefficient (the coefficient of the nonzero term of maximum degree) is 1. We adopt the convention that the polynomials $p_{ij}(t)$ are monic; such polynomials are called *minimal polynomials* of $\mathscr{I}$.

**Corollary 9.** *Let $p_{ij}(t)$ be minimal polynomials of $\mathscr{I}$. Then*
(i) *they are unique*,
(ii) $p_{11}(t) = p_{22}(t)$,
(iii) $p_{11}(t) = tp_{12}(t)$ *or* $p_{12}(t) = tp_{11}(t)$,
(iv) $p_{11}(t) = tp_{21}(t)$ *or* $p_{21}(t) = tp_{11}(t)$.

**Proof.** Immediate from Lemma 8. □

**Example 10.** We list the minimal polynomials corresponding to the ideals described in Example 5.
(1) $\mathfrak{g}(x, y) : p_{11}(t) = p_{22}(t) = t^2,\ p_{12}(t) = p_{21}(t) = t.$
(2) $\mathscr{I}_d : p_{11}(t) = p_{22}(t) = t^2,\ p_{12}(t) = p_{21}(t) = t^3.$
(3) $\mathscr{I}_l : p_{11}(t) = p_{22}(t) = t^2,\ p_{12}(t) = t^3,\ p_{21}(t) = t.$
(4) $\mathscr{I}_u : p_{11}(t) = p_{22}(t) = t^2,\ p_{12}(t) = t,\ p_{21}(t) = t^3.$
(5) $\mathscr{I}_c : p_{11}(t) = p_{22}(t) = t^2,\ p_{12}(t) = p_{21}(t) = t.$

**Proof.** This follows from the explicit description of such ideals in Example 5. □

**Definition 11.** Given a subspace $\mathscr{I}$ of $\mathfrak{g}(x, y)$ and $p(t) \in K[t^2]$ a monic polynomial, we call the subspace of $\mathfrak{g}(x, y)$

$$p(t) \cdot \mathscr{I} = \{p(t)a \mid a \in \mathscr{I}\}$$

the *multiple* of $\mathscr{I}$ with *factor* $p(t) \in K[t^2]$.

If $\mathscr{I}$ is an ideal of $\mathfrak{g}(x, y)$ so is $p(t)\mathscr{I}$ and conversely; in this case $p(t)\mathscr{I}$ is a 3-graded ideal if and only if $\mathscr{I}$ is a 3-graded ideal of $\mathfrak{g}(x, y)$. The multiplication by $p(t)$ induces a one-to-one correspondence between the sets of generators of the two ideals. In particular, if one of them is finitely generated so is the other and they have the same minimum number of generators. Also it is clear that the minimal polynomials of $p(t)\mathscr{I}$ are those corresponding to $\mathscr{I}$ multiplied by $p(t)$.

**Theorem 12.** *A nonzero ideal* $\mathscr{I}$ *of* $\mathfrak{g}(x,y)$ *falls into one of the following classes*:

(I) $\mathscr{I}=p(t)\mathfrak{g}(x,y)=\langle a,b\rangle,\ a=\begin{pmatrix} 0 & tp(t) \\ 0 & 0 \end{pmatrix},\ b=\begin{pmatrix} 0 & 0 \\ tp(t) & 0 \end{pmatrix}$, *monic* $p(t)\in K[t^2]$.

(II) $\mathscr{I}=p(t)\mathscr{I}_d=\langle a\rangle,\ a=\begin{pmatrix} t^2p(t) & 0 \\ 0 & -t^2p(t) \end{pmatrix}$, *monic* $p(t)\in K[t^2]$.

(III) $\mathscr{I}=p(t)\mathscr{I}_l=\langle a\rangle,\ a=\begin{pmatrix} 0 & 0 \\ tp(t) & 0 \end{pmatrix}$, *monic* $p(t)\in K[t^2]$.

(IV) $\mathscr{I}=p(t)\mathscr{I}_u=\langle a\rangle,\ a=\begin{pmatrix} 0 & tp(t) \\ 0 & 0 \end{pmatrix}$, *monic* $p(t)\in K[t^2]$.

(V) $\mathscr{I}=p(t)\mathscr{I}_c=\langle a\rangle,\ a=\begin{pmatrix} 0 & tp(t) \\ ctp(t) & 0 \end{pmatrix}$, $c\in K^*$, *monic* $p(t)\in K[t^2]$.

*These classes are disjoint and their descriptions contain no repetition.*

**Proof.** We analyze the possibilities stated in Corollary 9 for the minimal polynomials $p_{ij}$.

(a) First suppose

$$p_{11}(t)=tp_{12}(t),\quad p_{21}(t)=tp_{11}(t).$$

We have

$$p_{12}(t)=tp(t),$$

for some $p(t)\in K[t^2]$. Then

$$p_{11}(t)=t^2p(t),\quad p_{21}(t)=t^3p(t).$$

It follows $\mathscr{I}=p(t)\mathscr{I}'$, where $\mathscr{I}'$ is an ideal with minimal polynomials

$$q_{11}(t)=t^2,\quad q_{12}(t)=t,\quad q_{21}(t)=t^3.$$

We show that $\mathscr{I}'=\mathscr{I}_u$, where $\mathscr{I}_u=\langle a\rangle,\ a=tE$.

Indeed, choose

$$\begin{pmatrix} * & t \\ * & * \end{pmatrix}\in\mathscr{I}'.$$

(The symbol '$*$' at a position means that we are not interested in the actual entry of the matrix at that position.)

Since

$$\begin{pmatrix} t^2 & 0 \\ * & -t^2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ t^3 & 0 \end{pmatrix} \in \mathscr{I}' \tag{5}$$

we have

$$\begin{pmatrix} t^2 & 0 \\ 0 & -t^2 \end{pmatrix}, \begin{pmatrix} * & t \\ 0 & * \end{pmatrix} \in \mathscr{I}'. \tag{6}$$

Finally from (6), it follows:

$$a = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \in \mathscr{I}'. \tag{7}$$

Now it is immediate that $\mathscr{I}' = \langle a \rangle$.

(b) Suppose now

$$p_{11}(t) = tp_{21}(t), \quad p_{12}(t) = tp_{11}(t)$$

for the ideal $\mathscr{I}$.

Then $\theta(\mathscr{I})$ is an ideal described in (a) and therefore $\mathscr{I}$ is a multiple of

$$\mathscr{I}_l = \langle a \rangle, \quad a = tF.$$

In fact, the involution $\theta$ permutes the classes of ideals (III) and (IV).

(c) Let

$$p_{12}(t) = tp_{11}(t), \quad p_{21}(t) = tp_{11}(t).$$

We can write

$$p_{11}(t) = t^2 p(t)$$

for some $p(t) \in K[t^2]$.

Then

$$p_{12}(t) = t^3 p(t), \quad p_{21}(t) = t^3 p(t).$$

$\mathscr{I} = p(t)\mathscr{I}'$, where $\mathscr{I}'$ is an ideal with minimal polynomials

$$q_{11}(t) = t^2, \quad q_{12}(t) = t^3, \quad q_{21}(t) = t^3.$$

We show that

$$\mathscr{I}' = \mathscr{I}_d, \quad \text{where } \mathscr{I}_d = \langle a \rangle, \ a = \begin{pmatrix} t^2 & 0 \\ 0 & -t^2 \end{pmatrix}.$$

Indeed, choose

$$\begin{pmatrix} t^2 & * \\ * & -t^2 \end{pmatrix} \in \mathscr{I}'.$$

Since

$$\begin{pmatrix} 0 & t^5 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ t^5 & 0 \end{pmatrix} \in \mathscr{I}' \tag{8}$$

one has

$$\begin{pmatrix} 0 & t^n \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ t^n & 0 \end{pmatrix} \in \mathscr{I}' \quad \text{for } n = 5, 7, 9 \ldots$$

and we can start over choosing this time an element of the form

$$\begin{pmatrix} t^2 & c_1 t^3 \\ c_2 t^3 & -t^2 \end{pmatrix} \in \mathscr{I}', \quad c_1, c_2 \in K. \tag{9}$$

Then there exist elements

$$\begin{pmatrix} c_3 t^4 & t^3 \\ 0 & -c_3 t^4 \end{pmatrix}, \begin{pmatrix} c_4 t^4 & 0 \\ t^3 & -c_4 t^4 \end{pmatrix} \in \mathscr{I}' \quad \text{for certain } c_3, c_4 \in K \tag{10}$$

and

$$\begin{pmatrix} t^4 & 0 \\ c_5 t^5 & -t^4 \end{pmatrix}, \begin{pmatrix} t^4 & c_6 t^5 \\ 0 & -t^4 \end{pmatrix} \in \mathscr{I}' \quad \text{for certain } c_5, c_6 \in K. \tag{11}$$

From relations (9) and (10), we conclude

$$\begin{pmatrix} t^2 + c_7 t^4 & 0 \\ 0 & -t^2 - c_7 t^4 \end{pmatrix} \in \mathscr{I}' \quad \text{for some } c_7 \in K. \tag{12}$$

From relations (8) and (11), we have

$$\begin{pmatrix} t^4 & 0 \\ 0 & -t^4 \end{pmatrix} \in \mathscr{I}'. \tag{13}$$

Finally from (12) and (13), it follows:

$$a = \begin{pmatrix} t^2 & 0 \\ 0 & -t^2 \end{pmatrix} \in \mathscr{I}'. \tag{14}$$

From its minimal polynomials, it is clear that $\mathscr{I}' = \langle a \rangle$.

(d) Finally, suppose

$$p_{11}(t) = tp_{12}(t), \quad p_{11}(t) = tp_{21}(t).$$

One has

$$p_{12}(t) = tp(t)$$

for some $p(t) \in K[t^2]$. Then

$$p_{11}(t) = t^2 p(t), \quad p_{21}(t) = tp(t).$$

$\mathscr{I} = p(t)\mathscr{I}'$, where $\mathscr{I}'$ is an ideal with minimal polynomials

$$q_{11}(t) = t^2, \quad q_{12}(t) = t, \quad q_{21}(t) = t.$$

We want to check that either

$$\mathscr{I}' = \mathscr{I}_c = \langle a \rangle, \quad a = \begin{pmatrix} 0 & t \\ ct & 0 \end{pmatrix} \quad \text{for some } c \in K^*$$

or

$$\mathscr{I}' = \mathfrak{g}(x, y).$$

We start with an element of the form

$$\begin{pmatrix} * & t \\ * & * \end{pmatrix} \in \mathscr{I}'.$$

(As before, the symbol '$*$' means that the actual entry of the matrix is not relevant here.)

From

$$\begin{pmatrix} t^2 & 0 \\ * & -t^2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ t^3 & 0 \end{pmatrix} \in \mathscr{I}', \tag{15}$$

we have

$$\begin{pmatrix} 0 & t \\ * & 0 \end{pmatrix} \in \mathscr{I}' \text{ and then } \begin{pmatrix} 0 & t \\ ct & 0 \end{pmatrix} \in \mathscr{I}' \quad \text{for some } c \in K. \tag{16}$$

Therefore,

$$\begin{pmatrix} t^2 & 0 \\ 0 & -t^2 \end{pmatrix}, \begin{pmatrix} 0 & t^3 \\ 0 & 0 \end{pmatrix} \in \mathscr{I}'. \tag{17}$$

From (15) and (17), we conclude that

$$\mathscr{I}' = \mathscr{V}_1 \oplus \mathscr{V}_2, \tag{18}$$

where

$$\mathscr{V}_1 = \mathscr{I}' \cap \left\{ \begin{pmatrix} 0 & c_1 t \\ c_2 t & 0 \end{pmatrix} \middle| c_1, c_2, \in K \right\}$$

and

$$\mathscr{V}_2 = \left\{ \begin{pmatrix} p(t) & q(t) \\ r(t) & -p(t) \end{pmatrix} \middle| p(t) \in t^2 K[t^2],\ r(t), q(t) \in t^3 K[t^2] \right\}.$$

From (16), we know that $\dim \mathscr{V}_1 \geqslant 1$.

If $\dim \mathscr{V}_1 = 2$ (over $K$) then $\mathscr{I}' = \mathfrak{g}(x, y)$. Otherwise, $\dim \mathscr{V}_1 = 1$ and

$$a = \begin{pmatrix} 0 & t \\ ct & 0 \end{pmatrix}$$

forms a basis for $\mathscr{V}_1$. In this case, it follows that $\mathscr{I}' = \langle a \rangle$. Notice that $c \neq 0$ since otherwise $q_{21}(t) = t^3$.

Concluding the proof, we notice that two ideals corresponding to distinct choices of $p(t)$ in the same class are different, since they have different minimal polynomials. If two ideals lie in distinct classes they are different as well. The reason is the same, except for classes (I) and (V), but, in that case, an ideal of type (V) will be generated by one element whereas one in (I) will not. □

**Corollary 13.** *If $\mathscr{I}$ is an ideal of $\mathfrak{g}(x, y)$ then $t^2 \mathscr{I} \subset \mathscr{I}$.*

**Proof.** It is enough to verify this statement for the ideals of Example 5. □

**Proposition 14.** *If $\mathscr{I} \neq 0$ is a 3-graded ideal of $\mathfrak{g}(x, y)$ and $p_{11}(t), p_{12}(t), p_{21}(t)$ its corresponding minimal polynomials then*

$$\mathscr{I} = K[t^2](p_{11}(t)H) + K[t^2](p_{12}(t)E) + K[t^2](p_{21}(t)F)$$

(*In other words*, $\{p_{11}(t)H, p_{12}(t)E, p_{21}(t)F\}$ *forms a basis of* $\mathscr{I}$ *over* $K[t^2]$.)

**Proof.** It follows from Lemma 8. Alternatively, since $\mathscr{I}$ is a multiple of an ideal listed in Example 5, it suffices to check the statement for the 3-graded ones in the latter case. □

## 3. Embedding theorems

**Theorem 15.** *Let $\mathscr{I}$ be an ideal of $\mathfrak{g}(x, y)$ and $\tilde{\mathscr{I}}$ the ideal inside $\mathfrak{sl}_2(tK[t])$ generated by $\mathscr{I}$. Then the homomorphism*

$$\mathfrak{i}_{\mathscr{I}} : \mathfrak{g}(x, y)/\mathscr{I} \to \mathfrak{sl}_2(tK[t])/\tilde{\mathscr{I}},$$

$$\mathfrak{i}_{\mathscr{I}}(a+\mathscr{I}) = a+\tilde{\mathscr{I}}, \quad a \in \mathfrak{g}(x,y),$$

*is a monomorphism of Lie algebras.*

**Proof.** It is obvious that $\mathfrak{i}_{\mathscr{I}}$ is a well-defined homomorphism; we need now to prove that $\mathfrak{i}_{\mathscr{I}}$ is injective. Since

$$\mathfrak{g}(x,y)/(\mathfrak{g}(x,y)\cap\tilde{\mathscr{I}}) \cong (\mathfrak{g}(x,y)+\tilde{\mathscr{I}})/\tilde{\mathscr{I}}$$

it is enough to show that

$$\mathfrak{g}(x,y)\cap\tilde{\mathscr{I}} = \mathscr{I}.$$

Clearly

$$\mathscr{I} \subset \mathfrak{g}(x,y)\cap\tilde{\mathscr{I}}.$$

On the other hand, any element of $\tilde{\mathscr{I}}$ is a linear combination of monomials of the form

$$t^m(\operatorname{ad} tE)^{n_1}(\operatorname{ad} tF)^{n_2}(\operatorname{ad} tH)^{n_3}X,$$

where $m, n_1, n_2, n_3 \geqslant 0$ and $X$ is one of the (one or two) generators of $\mathscr{I}$, listed in Theorem 12.

We now consider matrices defined over $K(t)$, the field of rational functions in $t$. Hence one can write

$$t^m(\operatorname{ad} tE)^{n_1}(\operatorname{ad} tF)^{n_2}(\operatorname{ad} tH)^{n_3}X = t^n(\operatorname{ad} tE)^{n_1}(\operatorname{ad} tF)^{n_2}(\operatorname{ad} t^2H)^{n_3}X,$$

where $n = m - n_3 \in \mathbb{Z}$. In order that an element belongs to $\mathfrak{g}(x,y)\cap\tilde{\mathscr{I}}$ it is necessary that each monomial in a linearly independent expansion as above satisfies

$$n = 2k \in \mathbb{Z}.$$

Now if $n = 2k \geqslant 0$, such a monomial belongs to $\mathscr{I}$ by Corollary 13. Otherwise $n = -2k$, $k > 0$, and we can write the monomial as

$$(\operatorname{ad} tE)^{n_1}(\operatorname{ad} tF)^{n_2}(\operatorname{ad} t^2H)^{n_3-2k}(\operatorname{ad} tH)^{2k}X.$$

But $(\operatorname{ad} tH)^{2k}$ acts on $X$ as the multiplication by $ct^{2k}$, where $c \in K$, for any generator $X$ listed in Theorem 12. Again, by Corollary 13, the monomial lies in $\mathscr{I}$ and so does the element of $\mathfrak{g}(x,y)\cap\tilde{\mathscr{I}}$. $\square$

We now study the 3-graded ideals $\mathscr{I}$ of $\mathfrak{g}(x,y)$, as described in Theorem 12, case by case, to show that the quotient $\mathfrak{g}(x,y)/\mathscr{I}$ can be embedded in $\mathfrak{sl}_2(K[t]/\langle p(t)\rangle)$, $\langle p(t)\rangle = p(t)K[t]$.

(I)

$$\mathscr{I} = \langle a,b\rangle, \quad a = \begin{pmatrix} 0 & p(t) \\ 0 & 0 \end{pmatrix}, \ b = \begin{pmatrix} 0 & 0 \\ p(t) & 0 \end{pmatrix}, \ \text{nonzero } p(t) \in tK[t^2].$$

Let

$$\tilde{\mathfrak{i}} : \mathfrak{g}(x,y) \to \mathfrak{sl}_2(K[t]/\langle p(t)\rangle)$$

such that

$$\tilde{\mathfrak{i}}(x) = (t + \langle p(t) \rangle)E, \quad \tilde{\mathfrak{i}}(y) = (t + \langle p(t) \rangle)F.$$

$$\begin{pmatrix} b(t) & a(t) \\ c(t) & -b(t) \end{pmatrix} \in \ker \tilde{\mathfrak{i}} \quad \text{if and only if}$$

$$a(t) \in p(t)K[t] \cap tK[t^2], \quad b(t) \in p(t)K[t] \cap t^2K[t^2], \quad c(t) \in p(t)K[t] \cap tK[t^2]$$

or, equivalently,

$$a(t) \in p(t)K[t^2], \quad b(t) \in tp(t)K[t^2], \quad c(t) \in p(t)K[t^2].$$

Therefore $\ker \tilde{\mathfrak{i}} = \mathscr{I}$ and

$$\tilde{\mathfrak{i}} : \mathfrak{g}(x, y) \to \mathfrak{s}l_2(K[t]/\langle p(t) \rangle)$$

induces a monomorphism

$$\mathfrak{i} : \mathfrak{g}(x, y)/\mathscr{I} \to \mathfrak{s}l_2(K[t]/\langle p(t) \rangle)$$

such that

$$\mathfrak{i}(x + \mathscr{I}) = (t + \langle p(t) \rangle)E, \quad \mathfrak{i}(y + \mathscr{I}) = (t + \langle p(t) \rangle)F.$$

(IV) To study class (IV), we consider the embedding $\mathfrak{i}_{3,1}$ instead of $\mathfrak{i}_{1,1}$. Recall that

$$\mathfrak{i}_{3,1}(x) = t^3E, \quad \mathfrak{i}_{3,1}(y) = tF.$$

Hence an ideal $\mathscr{I}$ in class (IV) assumes the form

$$\mathscr{I} = \langle a \rangle, \quad a = \begin{pmatrix} 0 & p(t) \\ 0 & 0 \end{pmatrix} \quad \text{for nonzero } p(t) \in t^3K[t^4].$$

Let

$$\tilde{\mathfrak{i}} : \mathfrak{g}(x, y) \to \mathfrak{s}l_2(K[t]/\langle p(t) \rangle)$$

such that

$$\tilde{\mathfrak{i}}(x) = (t^3 + \langle p(t) \rangle)E, \quad \tilde{\mathfrak{i}}(y) = (t + \langle p(t) \rangle)F.$$

$$\begin{pmatrix} b(t) & a(t) \\ c(t) & -b(t) \end{pmatrix} \in \ker \tilde{\mathfrak{i}} \quad \text{if and only if}$$

$$a(t) \in p(t)K[t] \cap t^3K[t^4], \; b(t) \in p(t)K[t] \cap t^4K[t^4], \; c(t) \in p(t)K[t] \cap tK[t^4].$$

They are, respectively, equivalent to

$$a(t) \in p(t)K[t^4], \quad b(t) \in p(t)tK[t^4], \quad c(t) \in p(t)t^2K[t^4].$$

Hence $\ker \tilde{\mathfrak{i}} = \mathscr{I}$ and

$$\tilde{\mathfrak{i}} : \mathfrak{g}(x, y) \to \mathfrak{sl}_2(K[t]/\langle p(t)\rangle)$$

induces a monomorphism

$$\mathfrak{i} : \mathfrak{g}(x, y)/\mathscr{I} \to \mathfrak{sl}_2(K[t]/\langle p(t)\rangle)$$

such that

$$\mathfrak{i}(x + \mathscr{I}) = (t^3 + \langle p(t)\rangle)E, \quad \mathfrak{i}(y + \mathscr{I}) = (t + \langle p(t)\rangle)F.$$

(III) We embed $\mathfrak{g}(x, y)$ into $\mathfrak{sl}_2(K[t])$ by means of

$$\mathfrak{i}_{1,3} : \mathfrak{g}(x, y) \to \mathfrak{sl}_2(K[t])$$

defined as

$$\mathfrak{i}_{1,3}(x) = tE, \quad \mathfrak{i}_{1,3}(y) = t^3F.$$

An ideal $\mathscr{I}$ in class (III) has the form

$$\mathscr{I} = \langle a\rangle, \quad a = \begin{pmatrix} 0 & 0 \\ p(t) & 0 \end{pmatrix} \quad \text{for nonzero } p(t) \in t^3K[t^4].$$

Let

$$\tilde{\mathfrak{i}} : \mathfrak{g}(x, y) \to \mathfrak{sl}_2(K[t]/\langle p(t)\rangle)$$

such that

$$\tilde{\mathfrak{i}}(x) = (t + \langle p(t)\rangle)E, \quad \tilde{\mathfrak{i}}(y) = (t^3 + \langle p(t)\rangle)F.$$

Similarly to (IV), $\ker \tilde{\mathfrak{i}} = \mathscr{I}$ and

$$\tilde{\mathfrak{i}} : \mathfrak{g}(x, y) \to \mathfrak{sl}_2(K[t]/\langle p(t)\rangle)$$

descends to the monomorphism

$$\mathfrak{i} : \mathfrak{g}(x, y)/\mathscr{I} \to \mathfrak{sl}_2(K[t]/\langle p(t)\rangle)$$

such that

$$\mathfrak{i}(x + \mathscr{I}) = (t + \langle p(t)\rangle)E, \quad \mathfrak{i}(y + \mathscr{I}) = (t^3 + \langle p(t)\rangle)F.$$

(II)

$$\mathscr{I} = \langle a\rangle, \quad a = \begin{pmatrix} p(t) & 0 \\ 0 & -p(t) \end{pmatrix} \quad \text{for nonzero } p(t) \in t^2K[t^2].$$

We define

$$\tilde{\mathfrak{i}} : \mathfrak{g}(x, y) \to \mathfrak{sl}_2(K[t]/\langle p(t)\rangle)$$

such that

$$\tilde{\mathfrak{i}}(x) = (t + \langle p(t) \rangle)E, \quad \tilde{\mathfrak{i}}(y) = (t + \langle p(t) \rangle)F.$$

Then

$$\begin{pmatrix} b(t) & a(t) \\ c(t) & -b(t) \end{pmatrix} \in \ker \tilde{\mathfrak{i}} \quad \text{if and only if}$$

$$a(t) \in p(t)K[t] \cap tK[t^2], \quad b(t) \in p(t)K[t] \cap t^2K[t^2], \quad c(t) \in p(t)K[t] \cap tK[t^2].$$

They are, respectively, equivalent to

$$a(t) \in tp(t)K[t^2], \quad b(t) \in p(t)K[t^2], \quad c(t) \in tp(t)K[t^2].$$

Therefore $\ker \tilde{\mathfrak{i}} = \mathscr{I}$ and

$$\tilde{\mathfrak{i}} : \mathfrak{g}(x, y) \to \mathfrak{s}l_2(K[t]/\langle p(t) \rangle)$$

descends to the monomorphism

$$\mathfrak{i} : \mathfrak{g}(x, y)/\mathscr{I} \to \mathfrak{s}l_2(K[t]/\langle p(t) \rangle)$$

given by

$$\mathfrak{i}(x + \mathscr{I}) = (t + \langle p(t) \rangle)E, \quad \mathfrak{i}(y + \mathscr{I}) = (t + \langle p(t) \rangle)F.$$

We have proved the following result:

**Theorem 16.** *Any* 3*-graded Lie algebra generated by an element of degree* 1 *and another of degree* $-1$ *over a field* $K$ *of characteristic zero can be realized as a* 3*-graded Lie subalgebra of* $\mathfrak{s}l_2(K[t]/\langle p(t) \rangle)$ *for some* $p(t) \in K[t]$. *Moreover*, *it is symmetric if and only if it is isomorphic*, *as a graded Lie algebra*, *to the* 3*-graded Lie subalgebra generated by*

$$(t + \langle p(t) \rangle)E \quad \textit{and} \quad (t + \langle p(t) \rangle)F$$

*inside* $\mathfrak{s}l_2(K[t]/\langle p(t) \rangle)$ *for some* $p(t) \in K[t]$ (*further embeddings are given in the proof*).

It seems somewhat evident to us that Theorem 16 could be used to obtain a classification or similar description of the 3-graded Lie algebras in a pair of generators. Moreover, since the same 3-graded Lie algebra can be embedded into $\mathfrak{s}l_2(K[t]/\langle p(t) \rangle)$ in many different ways, it would be interesting to understand how it depends on the chosen polynomial $p(t) \in K[t]$ and selection of representatives for the generators in $\mathfrak{s}l_2(K[t]/\langle p(t) \rangle)$.

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