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Stochastic Processes and their Applications 121 (2011) 2231–2242
stochastic processes and their applications
www.elsevier.com/locate/spa

Convergence to type I distribution of the extremes of sequences defined by random difference equation

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Received 12 March 2011; received in revised form 29 May 2011; accepted 17 June 2011
Available online 25 June 2011

Abstract

We study the extremes of a sequence of random variables (R_n) defined by the recurrence $R_n = M_n R_{n-1} + q$, $n \geq 1$, where R_0 is arbitrary, (M_n) are iid copies of a non-degenerate random variable M, $0 \leq M \leq 1$, and $q > 0$ is a constant. We show that under mild and natural conditions on M the suitably normalized extremes of (R_n) converge in distribution to a double-exponential random variable. This partially complements a result of de Haan, Resnick, Rootzén, and de Vries who considered extremes of the sequence (R_n) under the assumption that $\mathbb{P}(M > 1) > 0$.
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MSC: primary 60G70; secondary 60F05

Keywords: Random difference equation; Convergence in distribution; Extreme value

1. Introduction

We consider a special case of the following random difference equation:

$$R_n = Q_n + M_n R_{n-1}, \quad n \geq 1 \tag{1.1}$$

where R_0 is arbitrary and (Q_n, M_n), $n \geq 1$, are iid copies of a two-dimensional random vector (Q, M), and (Q_n, M_n) is independent of R_{n-1}. Later on we specialize our discussion to a non-degenerate M, and $Q \equiv q$, a positive constant. Much of the impetus for studying equations like (1.1) stems from numerous applications of such schemes in mathematics and other disciplines

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doi:10.1016/j.spa.2011.06.007

of science. We refer the reader to [7,20] for examples of fields in which Eq. (1.1) has been of interest. Further examples of more recent applications are mentioned in [12], and for examples of statistical issues arising in studying solutions of (1.1) see [2].

A fundamental theoretical result that goes back to Kesten [14] asserts that if

$$E \ln |M| < 0 \quad \text{and} \quad E \ln |Q| < \infty \tag{1.2}$$

then the sequence (R_n) converges in distribution to a random variable R, which necessarily satisfies the distributional identity

$$R \overset{d}{=} MR + Q \tag{1.3}$$

(see also [20] for a detailed discussion of the convergence properties of (R_n)). In the same paper, Kesten showed that if $P(|M| > 1) > 0$ and (1.2) holds then, under some mild additional conditions on M and Q, the limiting distribution is always heavy-tailed, that is, $\mathbb{P}(|R| > t) \sim Ct^{-\kappa}$ for a suitably chosen $\kappa > 0$. A different proof of this result was given by Goldie in [9]. By contrast, it was shown in [10] that in the complementary case $|M| \le 1$ if $|Q| \le q$ then the tail of R has decay no slower than exponential.

Interestingly, much more work has been done on the heavy-tailed situation. This is perhaps at least partially a result of the fact that many of the processes appearing in applications (for example GARCH processes in financial mathematics) are in fact heavy-tailed. Nonetheless, the case $|M| \le 1$ and $Q \equiv q$ contains a number of interesting situations, including the class of Vervaat perpetuities; see e.g. [20]. Vervaat perpetuities correspond to M being a Beta$(\alpha, 1)$ random variable for some $\alpha > 0$ and $Q = 1$ in which case one gets

$$R \overset{d}{=} 1 + M_1 + M_1M_2 + M_1M_2M_3 + \cdots \tag{1.4}$$

(some authors prefer not to have a 1 at the beginning, which corresponds to taking $Q = M$). Particular cases of Vervaat perpetuities include the Dickman distribution appearing in number theory (see [6]), in the analysis of the limiting distribution of the `Quickselect` algorithm (see [16]), and in the limit theory of functionals of success epochs in iid sequences of random variables [19, Section 4.7]. Further connections are referenced in [13] and we refer the reader there for more information. For recent work on perfect simulation of Vervaat perpetuities see [8] or [5].

In this note we will be interested in the extremal behavior of the sequence (R_n). For any sequence of random variables (Y_n) we let (Y_n^*) be the sequence of partial maxima, i.e. $Y_n^* = \max_{k \le n} Y_k, n \ge 1$. With this notation, we will seek constants (a_n) and (b_n) such that for all x,

$$P(a_n(R_n^* - b_n) \le x) \longrightarrow G(x), \quad n \to \infty, \tag{1.5}$$

where G is a non-degenerate distribution function.

Under the assumption that $\mathbb{P}(M > 1) > 0$, the extremes of the sequence (R_n) when both M and Q are non-negative were studied in [4] and were shown to converge (after suitable normalization) to Fréchet (i.e. Type II) distribution with parameter κ. Here, we consider the complementary case, namely that of a light-tailed limiting distribution R. Of course, in this situation one expects convergence in (1.5) to a Gumbel (i.e. a double-exponential or Type I) distribution, provided that there is convergence at all. The latter need not be the case, however. Indeed, if $Q = 1$ and M has a two-point distribution $\mathbb{P}(M = 1) = p = 1 - \mathbb{P}(M = 0)$ then as is seen from (1.4), R has a geometric distribution with parameter $1 - p$ and thus no constants (a_n), (b_n) exist for which (1.5) holds for a non-degenerate distribution G (see [15, Example 1.7.15]). Our main aim here is to show that under fairly general and natural conditions on M (and for a degenerate Q), (1.5) does hold for suitable constants (a_n), (b_n) and a double-exponential distribution $G(x) = \exp(-\mathrm{e}^{-x})$, $-\infty < x < \infty$.

2. Extremal behavior

Following the authors of [4] we assume that both M and Q are non-negative. As we mentioned earlier, we assume that $Q = q > 0$ is a constant. So, we consider

$$R_n = M_n R_{n-1} + q, \quad n \geq 1, \ R_0\text{-given}, \tag{2.6}$$

where M_n and R_{n-1} on the right-hand side are independent and where (M_n) is a sequence of iid copies of a random variable M satisfying

$$0 \leq M \leq 1, \quad M\text{-non-degenerate}. \tag{2.7}$$

(The non-degeneracy assumption is to eliminate the possibility that R itself is degenerate.) Clearly, this is more than (1.2) and thus implies the convergence in distribution of (R_n). Furthermore, it has been known since [10] that in that case the tail of the limiting variable R is no heavier than exponential. Note that if M is bounded away from 1 then R is actually a bounded random variable. To exclude this situation we assume that the right endpoint of M is 1, that is that

$$\sup\{x : \mathbb{P}(M > x) > 0\} = 1. \tag{2.8}$$

Finally, we need to eliminate the possibility that R is a geometric variable. To this end it is enough to assume that

$$\mathbb{P}(M = 0) = 0, \tag{2.9}$$

since this guarantees that the distribution of R is continuous (see e.g. [1, Theorem 1.3]).

We will prove the following theorem:

Theorem 1. *Let (R_n) satisfy (2.6) with M satisfying (2.7)–(2.9). Then there exist sequences (a_n), (b_n) such that for every real x*

$$\mathbb{P}(a_n(R_n^* - b_n) \leq x) \to \exp(-\mathrm{e}^{-x}), \quad \text{as } n \to \infty.$$

3. Proof of Theorem 1

We first outline our proof which generally follows the approach of [4] (see also references therein for earlier developments). Writing out (2.6) explicitly we see that

$$R_n = q + qM_n + qM_nM_{n-1} + \cdots + qM_n \cdots M_2 + M_n \cdots M_1 R_0. \tag{3.10}$$

Under our assumption (2.7) (as a matter of fact, under the first part of (1.2) as well) the product $\prod_{k=1}^{n} M_k$ goes to 0 a.s. Consequently, the extremal behavior of (R_n) is the same regardless of the choice of the initial variable R_0. It is particularly convenient to choose R_0 such that it satisfies (1.3) as then so does every R_k, $k \geq 1$, making the sequence (R_n) stationary. Extremal behavior of stationary sequences is quite well understood (see e.g. [15, Chapter 3]) and we will take advantage of that. To find the extremal behavior of (R_n) one has to do three things:

(i) analyze the extremal behavior of the associated independent sequence $(\hat{R}_n)$ consisting of iid random variables equidistributed with R,
(ii) verify that the sequence (R_n) satisfies the $D(u_n)$ condition for sequences (u_n) of the form $u_n = b_n + x/a_n$, for any x and suitably chosen sequences (a_n), (b_n), and
(iii) show that the sequence (R_n) has the extremal index and find its value.

Some of the difficulties with carrying out this program are caused by the fact that, contrary to the heavy-tailed situation, little is known about the tail asymptotics in the case of light tails. A notable exception is the case of Vervaat perpetuities (see [19, Section 4.7] for a discussion). General results on the light-tail case are scarce (see [10,12,11]) and less precise than Kesten's result in the heavy-tailed situation. As a consequence, less precise information about the norming constants (a_n), (b_n) will be available. Our substitute for Kesten's result will be two-sided bounds obtained recently in [11].

We will treat the three items above in separate subsections.

3.1. The associated independent sequence

We appeal to the general theory of extremes as described in e.g. [15, Chapter 1]. First, we know from [1, Theorem 1.3] that (2.9) and the non-degeneracy assumption on M imply that R has continuous distribution function F_R. Therefore, the condition (1.7.3) of Theorem 1.7.13 of [15] is satisfied and thus, for every $x > 0$ there exist $u_n = u_n(x)$ such that

$$\lim_{n\to\infty} n\mathbb{P}(R > u_n) = \mathrm{e}^{-x}. \tag{3.11}$$

In fact, since R is continuous, u_n may be taken to be

$$u_n(x) = F_R^{-1}\left(1 - \frac{\mathrm{e}^{-x}}{n}\right),$$

where F_R is the probability distribution function of R. The question now is whether u_n's may be chosen to be linear functions of x, i.e. whether there exist constants a_n and b_n, $n \geq 1$, such that for $x > 0$ we have

$$u_n(x) = \frac{x}{a_n} + b_n, \quad n \geq 1. \tag{3.12}$$

To address that question we will utilize a recent result of [11] which states that there exist absolute constants c_i, $i = 0, 1, 2, 3$, such that for sufficiently large $y > 0$,

$$\exp\left\{c_0 y \ln p_{\frac{c_1}{y}}\right\} \leq \mathbb{P}(R > y) \leq \exp\left\{c_2 y \ln p_{\frac{c_3}{y}}\right\},$$

where, following [10], for $0 < \delta < 1$ we set

$$p_\delta = \mathbb{P}(1 - \delta < M \leq 1) = 1 - F_M(1 - \delta) \quad \text{and} \quad p_0 = \lim_{\delta\to 0} p_\delta = \mathbb{P}(M = 1). \tag{3.13}$$

Notice that by (2.8) p_δ is strictly positive for $\delta \in (0, 1)$. Now, if

$$\mathbb{P}(R > u_n) = \frac{\mathrm{e}^{-x}}{n},$$

then

$$\exp\left\{c_0 u_n \ln p_{\frac{c_1}{u_n}}\right\} \leq \frac{\mathrm{e}^{-x}}{n}.$$

Therefore, if the w_n's are chosen such that

$$\exp\left\{c_0 w_n \ln p_{\frac{c_1}{w_n}}\right\} = \frac{e^{-x}}{n},$$

then $u_n \geq w_n$. By the same argument, if the v_n's are such that

$$\exp\left\{c_2 v_n \ln p_{\frac{c_3}{v_n}}\right\} = \frac{e^{-x}}{n},$$

then $\mathbb{P}(R > v_n) \leq \frac{e^{-x}}{n}$, so $u_n \leq v_n$. Hence for every $x > 0$,

$$w_n(x) \leq u_n(x) \leq v_n(x)$$

and thus for every $n \geq 1$ there would exist $0 \leq \alpha_n \leq 1$ such that

$$u_n = \alpha_n w_n + (1 - \alpha_n) v_n.$$

If both (v_n) and (w_n) were linear, say,

$$w_n(x) = \frac{x}{a'_n} + b'_n, \qquad v_n(x) = \frac{x}{a''_n} + b''_n,$$

for some (a'_n), (b'_n), (a''_n), and (b''_n), then (3.12) would hold with

$$a_n = \left(\frac{\alpha_n}{a'_n} + \frac{1 - \alpha_n}{a''_n}\right)^{-1} \quad \text{and} \quad b_n = \alpha_n b'_n + (1 - \alpha_n) b''_n.$$

It therefore suffices to show the existence of linear norming for partial maxima of iid random variables (W_n) whose common distribution F_W satisfies

$$1 - F_W(y) = \exp\{c_0 y \ln p_{c_1/y}\}, \quad \text{for } y \geq y_0,$$

where $p_{c_1/y}$ is given by (3.13) for some fixed random variable M satisfying (2.7)–(2.9).

In accordance with [15, Theorem 1.5.1], to show that

$$\mathbb{P}(a'_n(W_n - b'_n) \leq x) \to \exp(-e^{-x}),$$

holds for every real x, the constants (a'_n) and (b'_n) must be constructed such that for every such x,

$$n(1 - F_W(b'_n + x/a'_n)) \to e^{-x}, \quad \text{as } n \to \infty,$$

i.e. that

$$n \exp\left\{c_0 \left(b'_n + \frac{x}{a'_n}\right) \ln p_{\frac{c_1}{b'_n + x/a'_n}}\right\} \to e^{-x}, \quad \text{as } n \to \infty. \tag{3.14}$$

Choose b'_n such that

$$c_0 b'_n \ln p_{c_1/b'_n} = -\ln n. \tag{3.15}$$

Then the left-hand side of (3.14) is

$$\exp\left\{c_0 b'_n \left(\ln p_{\frac{c_1}{b'_n + x/a'_n}} - \ln p_{c_1/b'_n}\right)\left(1 + \frac{x}{a'_n b'_n}\right) + c_0 \frac{x}{a'_n} \ln p_{c_1/b'_n}\right\}.$$

To choose (a'_n), first note that the difference of logarithms in the first summand is negative. Hence, if for any n, $a'_n \le -K \ln p_{c_1/b'_n}$ for some $K < c_0$ then the exponent is no more than $-xc_0/K < -x$. Therefore, for any admissible choice of (a'_n) we must have $\liminf_n a'_n / \ln p_{c_1/b'_n} \le -c_0$ which implies in particular that $a'_n b'_n \to \infty$. Thus, the exponent in the above formula is asymptotic to

$$c_0 b'_n \left(\ln p_{\frac{c_1}{b'_n + x/a'_n}} - \ln p_{c_1/b'_n} \right) + c_0 \frac{x}{a'_n} \ln p_{c_1/b'_n}.$$

We can further assume that for each n $1 - c/b'_n$ is a differentiability point of F_M and that the derivative, f_M, is finite at $1 - c_1/b'_n$. It then follows that the exponent is asymptotic to

$$-c_0 \frac{c_1 x}{a'_n b'_n p_{c_1/b'_n}} f_M \left(1 - \frac{c_1}{b'_n} \right) + c_0 \frac{x}{a'_n} \ln p_{c_1/b'_n}$$

and thus we may choose

$$a'_n = c_0 \left(\frac{c_1}{b'_n p_{c_1/b'_n}} f_M \left(1 - \frac{c_1}{b'_n} \right) - \ln p_{c_1/b'_n} \right). \tag{3.16}$$

3.2. The $D(u_n)$ condition

To check that the $D(u_n)$ condition holds for sequences of the form $b_n + x/a_n$ we proceed in the same fashion as [4, Proof of Theorem 2.1]; the argument there was, in turn, based on [17, Proof of Lemma 3.1]. Recall that, according to [15, Lemma 3.2.1(ii)], it suffices to show that if $1 \le i_1 < \cdots < i_r < j_1 < \cdots < j_s \le n$ are such that $j_1 - i_r \ge \lambda n$ for $\lambda > 0$ then

$$\mathbb{P}\left(\bigcap_{k=1}^{r} \{R_{i_k} \le u_n\} \cap \bigcap_{m=1}^{s} \{R_{j_m} \le u_n\} \right) - \mathbb{P}\left(\bigcap_{k=1}^{r} \{R_{i_k} \le u_n\} \right) \mathbb{P}\left(\bigcap_{k=1}^{r} \{R_{i_k} \le u_n\} \right) \to 0,$$

as $n \to \infty$. Set $I = \{i_1, \ldots, i_r\}$ and $J = \{j_1, \ldots, j_s\}$ and for any set A of positive integers let $R^*_A = \max_{a \in A} R_a$.

It follows from (2.6) that for $j > i$ we have

$$\begin{aligned} R_j &= q + qM_j + \cdots + qM_j \cdots M_{i+2} + M_j \cdots M_{i+1} R_i \\ &=: S_{j,i} + M_j \cdots M_{i+1} R_i, \end{aligned}$$

where for $j > i$ we have set

$$S_{j,i} := q + qM_j + \cdots + qM_j \cdots M_{i+2}.$$

Hence, for any $\epsilon_n > 0$ we obtain

$$\begin{aligned} \{R^*_J \le u_n\} &= \bigcap_{j \in J} \{S_{j,i_r} + M_j \cdots M_{i_r+1} R_{i_r} \le u_n\} \\ &\supset \bigcap_{j \in J} \{S_{j,i_r} \le u_n - \epsilon_n\} \cap \{M_j \cdots M_{i_r+1} R_{i_r} \le \epsilon_n\} \\ &= \bigcap_{j \in J} \left\{S_{j,i_r} \le u_n - \epsilon_n\right\} \setminus \bigcup_{j \in J} \left\{M_j \cdots M_{i_r+1} R_{i_r} > \epsilon_n\right\}. \end{aligned}$$

Note that R_k and $S_{n,m}$ are independent whenever $m \geq k$, so $\{R_i : i \in I\}$ and $\{S_{j,i_r} : j \in J\}$ are independent, and hence we get

$$P(R_I^* \leq u_n, R_J^* \leq u_n) \geq P(R_I^* \leq u_n)P(S_{J,i_r}^* \leq u_n - \epsilon_n) - P\left(\bigcup_{j\in J} M_j \cdots M_{i_r+1}R_{i_r} > \epsilon_n\right).$$

Also,

$$\{S_{J,i_r}^* \leq u_n - \epsilon_n\} \supset \{R_J^* \leq u_n - 2\epsilon_n\} \cap \bigcap_{j\in J}\{M_j \cdots M_{i_r+1}R_{i_r} \leq \epsilon_n\},$$

which further leads to

$$P(R_I^* \leq u_n, R_J^* \leq u_n) \geq P(R_I^* \leq u_n)P(R_J^* \leq u_n - 2\epsilon_n) - 2P\left(\bigcup_{j\in J} M_j \cdots M_{i_r+1}R_{i_r} > \epsilon_n\right).$$

By essentially the same argument we also get

$$P(R_I^* \leq u_n, R_J^* \leq u_n) \leq P(R_I^* \leq u_n)P(R_J^* \leq u_n + 2\epsilon_n) + 2P\left(\bigcup_{j\in J} M_j \cdots M_{i_r+1}R_{i_r} > \epsilon_n\right).$$

Combining these two estimates we obtain

$$\begin{aligned}&|P(R_I^* \leq u_n, R_J^* \leq u_n) - P(R_I^* \leq u_n)P(R_J^* \leq u_n)|\\ &\quad\leq \max\{P(R_J^* \leq u_n) - P(R_J^* \leq u_n - 2\epsilon_n), P(R_J^* \leq u_n + 2\epsilon_n) - P(R_J^* \leq u_n)\}\\ &\qquad + 2P\left(\bigcup_{j\in J} M_j \cdots M_{i_r+1}R_{i_r} > \epsilon_n\right).\end{aligned}$$

Thus condition $D(u_n)$ will be verified once we show that both terms in the sum on the right-hand side vanish as $n \to \infty$. To handle the first term we use stationarity and the fact that $j_s \leq n$ to find that the maximum above is bounded by

$$\sum_{j\in J} P(u_n - 2\epsilon_n \leq R_j \leq u_n + 2\epsilon_n) \leq nP(u_n - 2\epsilon_n \leq R \leq u_n + 2\epsilon_n).$$

Recall that the (u_n) satisfy (3.11) and (3.12). Thus, setting $\epsilon_n = \epsilon/a_n$ with $\epsilon > 0$ sufficiently small we get

$$nP(u_n - 2\epsilon_n \leq R \leq u_n + 2\epsilon_n) \to \mathrm{e}^{-(x-2\epsilon)} - \mathrm{e}^{-(x+2\epsilon)} = O(\epsilon).$$

Turning attention to the second term, using $M_k \leq 1$ we see that

$$\begin{aligned}\mathbb{P}\left(\bigcup_{j\in J} M_j \cdots M_{i_r+1}R_{i_r} > \epsilon/a_n\right) &\leq \sum_{j\in J} \mathbb{P}(M_j \cdots M_{i_r+1}R_{i_r} > \epsilon/a_n)\\ &\leq nP(M_{j_1-i_r} \cdots M_1 R_0 > \epsilon/a_n).\end{aligned}$$

Intersect the event underneath this probability with $\{R > 2b_n\}$ and its complement to see that this term is bounded by

$$n\mathbb{P}(R > 2b_n) + n\mathbb{P}(M_{j_1-i_r}\cdots M_1 > \epsilon/(2a_nb_n)). \tag{3.17}$$

Furthermore, since for any $T > 0$ and sufficiently large n, $2b_n = b_n + \frac{a_nb_n}{a_n} > b_n + T/a_n$, the first term (3.17) is bounded by

$$n\mathbb{P}(R > 2b_n) \le n\mathbb{P}(R > b_n + T/a_n) \to \mathrm{e}^{-T},$$

and thus goes to 0 upon letting $T \to \infty$. Turning to the second term in (3.17) we see that by Markov's inequality and independence of the M_k's it is bounded by

$$\frac{2na_nb_n}{\epsilon}(EM)^{j_1-i_r}. \tag{3.18}$$

We need to see that this vanishes as $n \to \infty$. Recall that $EM < 1$ and $j_1 - i_r \ge \lambda n$ where $\lambda > 0$, so $(\mathbb{E}M)^{j_1-i_r}$ decays exponentially fast in n. Furthermore,

$$a_nb_n \le K\max\{a_n', a_n''\}\cdot\max\{b_n', b_n''\}.$$

Recall that b_n' and b_n'' satisfy (3.15) (with different constants). Thus they both are $O(\ln n)$ as are $\ln p_{c_1/b_n'}$ and $\ln p_{c_3/b_n''}$. Hence,

$$a_n' \le K\left(\frac{c_1}{b_n'}f_M\left(1-\frac{c_1}{b_n'}\right)\frac{1}{p_{c_1/b_n'}} + \ln n\right).$$

Since f_M is an integrable function, we may assume that $\frac{c_1}{b_n'}f_M(1 - c_1/b_n') = O(1)$ as $n \to \infty$. Finally, recall that (b_n') satisfies (3.15). Therefore,

$$p_{c_1/b_n'} = \exp\left(-\frac{\ln n}{c_0b_n'}\right) = n^{-1/c_0b_n'} \ge n^{-\alpha}, \quad \alpha > 0,$$

where the last inequality follows from the fact that $b_n' \to \infty$ as $n \to \infty$ which is evident from (3.15). It follows that na_nb_n has a polynomial growth in n and thus that for every $\epsilon > 0$ (3.18) goes to 0 as $n \to \infty$.

3.3. The extremal index

We establish the following fact about the extremal index of (R_n). It implies, in particular, that if M does not have an atom at 1, then the extremal behavior of (R_n) is exactly the same as it would be for independent R_n's.

Proposition 2. *Let (R_n) be a stationary sequence satisfying the recurrence (2.6). Then (R_n) has the extremal index θ whose value is*

$$\theta = \limsup_n \mathbb{P}(MR + q \le u_n | R > u_n) = 1 - p_0 = 1 - \mathbb{P}(M = 1).$$

Proof. Again following the authors of [4] we rely on Theorem 4.1 of Rootzén [18]. Since we have shown that $D(u_n)$ holds for every sequence u_n of the form $b_n + x/a_n$, $x > 0$, it remains to verify condition (4.3) of that theorem, i.e. to show that

$$\limsup_{n\to\infty} |\mathbb{P}(R_{\lceil n\epsilon\rceil} \le u_n | R_0 > u_n) - \theta| \to 0, \quad \text{as } \epsilon \searrow 0.$$

To this end, for given $\epsilon > 0$, let $m := m_\epsilon := \lceil n\epsilon \rceil$. Then

$$\mathbb{P}(R_m^* \le u_n | R_0 > u_n) = \mathbb{P}(R_m \le u_n | R_{m-1}^* \le u_n, R_0 > u_n)\mathbb{P}(R_{m-1}^* \le u_n | R_0 > u_n).$$

By the Markov property, for $m \ge 2$ the first probability on the right-hand side is

$$\begin{aligned}&\mathbb{P}(R_m \le u_n | R_{m-1} \le u_n) = \mathbb{P}(M_m R_{m-1} + q \le u_n | R_{m-1} \le u_n)\\ &\quad = \mathbb{P}(MR + q \le u_n | R \le u_n).\end{aligned}$$

Continuing in the same fashion we find that

$$\begin{aligned}\mathbb{P}(R_m^* \le u_n | R_0 > u_n) &= \mathbb{P}^{m-1}(MR + q \le u_n | R \le u_n)\mathbb{P}(R_1 \le u_n | R_0 > u_n)\\ &= (1 - \mathbb{P}(MR + q > u_n | R \le u_n))^{m-1}\, \mathbb{P}(MR + q \le u_n | R > u_n).\end{aligned}$$

So, clearly,

$$\limsup_n \mathbb{P}(R_m^* \le u_n | R_0 > u_n) \le \limsup_n \mathbb{P}(MR + q \le u_n | R > u_n).$$

On the other hand,

$$\begin{aligned}n\mathbb{P}(MR + q > u_n | R \le u_n) &= n\frac{\mathbb{P}(MR + q > u_n, R \le u_n)}{\mathbb{P}(R \le u_n)} \le n\frac{\mathbb{P}(MR + q > u_n)}{1 - \mathbb{P}(R > u_n)}\\ &= n\frac{\mathbb{P}(R > u_n)}{1 - \mathbb{P}(R > u_n)} \to \mathrm{e}^{-x},\end{aligned}$$

as $n \to \infty$ by the very choice of (u_n). Thus

$$\limsup_n n\mathbb{P}(MR + q > u_n | R \le u_n) \le \mathrm{e}^{-x} =: c < \infty$$

and so

$$\liminf_n (1 - \mathbb{P}(MR + q > u_n | R \le u_n))^{m-1} \ge \mathrm{e}^{-c\epsilon}$$

and hence

$$\limsup_n \mathbb{P}(R_m^* \le u_n | R_0 \le u_n) \ge \mathrm{e}^{-c\epsilon} \limsup_n \mathbb{P}(MR + q \le u_n | R > u_n).$$

It follows that

$$\begin{aligned}&\lim_{\epsilon \searrow 0} \limsup_{n\to\infty} \left\{(1 - \mathbb{P}(MR + q > u_n | R \le u_n))^{m-1}\, \mathbb{P}(MR + q \le u_n | R > u_n)\right\}\\ &\quad = \limsup_{n\to\infty} \mathbb{P}(MR + q \le u_n | R > u_n).\end{aligned}$$

We now turn to evaluating

$$\limsup_{n\to\infty} \mathbb{P}(MR + q \le u_n | R > u_n).$$

It is clear that if $p_0 > 0$ then for every n such that $u_n \ge q$ we have $\mathbb{P}(MR + q \le u_n | R > u_n) = 1 - p_0$, so assume that $p_0 = 0$ and write

$$\begin{aligned}\mathbb{P}(MR + q \le u_n | R > u_n) &= 1 - \mathbb{P}(MR + q > u_n | R > u_n)\\ &= 1 - \frac{\mathbb{P}(MR + q > u_n, R > u_n)}{\mathbb{P}(R > u_n)}.\end{aligned}$$

It remains to show that the numerator in the last expression is of lower order than the denominator. To do that, let (t_n) be a sequence converging to infinity but in such a way that $t_n = o(b_n)$. Then

$$\mathbb{P}(MR + q > u_n, R > u_n) = \int_{u_n}^{\infty} \mathbb{P}(Mt + q > u_n)\mathrm{d}F_R(t)$$
$$= \left(\int_{u_n}^{u_n+t_n} + \int_{u_n+t_n}^{\infty}\right)\mathbb{P}(Mt + q > u_n)\mathrm{d}F_R(t).$$

Note that the probability under the integral is an increasing function of t. Bounding it trivially by 1 in the second term we see that this term is bounded by $\mathbb{P}(R > u_n + t_n)$. This can be further bounded by

$$\mathbb{P}\left(R > b_n + \frac{x}{a_n} + t_n\right) = \mathbb{P}\left(R > b_n + \frac{x + a_n t_n}{a_n}\right) \le \mathbb{P}\left(R > b_n + \frac{x + T}{a_n}\right),$$

whenever $a_n t_n \ge T$. It follows by the choice of (u_n) and the $D(u_n)$ condition that

$$\frac{\mathbb{P}(R > u_n + t_n)}{\mathbb{P}(R > u_n)} \le \mathrm{e}^{-T},$$

for arbitrarily large T and sufficiently large n and thus it vanishes as $n \to \infty$. The first integral is bounded by

$$\mathbb{P}(M(u_n + t_n) + q > u_n)\mathbb{P}(u_n < R < u_n + t_n) \le \mathbb{P}\left(M > 1 - \frac{t_n + q}{u_n + t_n}\right)\mathbb{P}(R > u_n).$$

Since the first term goes to $p_0 = 0$ as $n \to \infty$, we see that this term is $o(\mathbb{P}(R > u_n))$ as $n \to \infty$. This shows that the extremal index is 1 when $p_0 = 0$ and completes the proof. □

4. Remarks

1. The main drawback of Theorem 1 is that it does not give a good handle on the norming constants (a_n) and (b_n). This is generally caused by a lack of precise information about the tails of the limiting random variable R. However, even in the rare cases in which more precise information about tails of R is available, the formulas seem to be too complicated to make the precise statements about (a_n) and (b_n) practical. For example, for when $q = 1$ and M has a Beta$(\alpha, 1)$ distribution, $\alpha > 0$ (i.e. R is a Vervaat perpetuity), Vervaat [19, Theorem 4.7.7] (on the basis of earlier arguments of de Bruijn [3]) found the expression for the density of R. This, in principle, could be used to get precise enough asymptotics of the tail function of R and thus determine the asymptotic values of (b_n) and (a_n). However, the nature of these formulas makes obtaining explicit asymptotic expressions for (a_n) and (b_n) difficult if not impossible. As far as we know, Vervaat perpetuities provide the only class of examples (within our restrictions on M and Q) for which the asymptotics of the tail function is known. On the other hand, Theorem 1 typically gives the order of the magnitude of (a_n) and (b_n).

2. The expression (3.16) for (a'_n) often simplifies to $a'_n \sim -c_0 \ln p_{c_1/b'_n}$ (with corresponding simplification for (a_n)). This will happen, for example, whenever $p_0 = 0$ and $\delta f_M(1-\delta)/p_\delta$ is bounded as $\delta \to 0$, and in particular, when M is a Beta(α, β) random variable, $\alpha, \beta > 0$. In that case, b'_n may be chosen to be asymptotic to $\frac{\ln n}{c_0 \beta \ln \ln n}$ and then $a'_n \sim c_0 \beta \ln \ln n$. Hence, (a_n) and (b_n) are of order $\ln \ln n$ and $\ln n / \ln \ln n$, respectively. Note that Vervaat perpetuity corresponds to $\beta = 1$ and Dickman distribution to $\alpha = \beta = 1$.

3. There are, however, situations for which the above remark is not true. The following situation was considered in [12, Theorem 6]. Let M have density given by

$$f_M(t) = K \exp\left\{-\frac{1}{(1-t^r)^{1/(r-1)}}\right\}, \quad 1 < r < \infty,\ 0 < t < 1,$$

where $K = K_r$ is a normalizing constant. Then, as $\delta \searrow 0$,

$$\begin{aligned} p_\delta &\sim (1-(1-\delta)^r)^{r/(r-1)} \exp\{-(1-(1-\delta)^r)^{-1/(r-1)}\} \\ &\sim (r\delta)^{r/(r-1)} \exp\{-(r\delta)^{-1/(r-1)}\}, \end{aligned}$$

and so

$$\frac{c_1 f_M(1 - c_1/b'_n)}{b'_n p_{c_1/b'_n}} \sim \left(\frac{b'_n}{c_1 r^r}\right)^{1/(r-1)}.$$

On the other hand,

$$-\ln p_{c_1/b'_n} \sim \left(\frac{b'_n}{c_1}\right)^{1/(r-1)} + \frac{r}{r-1}\ln(b'_n/rc_1) = \left(\frac{b'_n}{c_1}\right)^{1/(r-1)} \left(1 + O\left(\frac{\ln b'_n}{b_n'^{1/(r-1)}}\right)\right),$$

so the two terms appearing in (3.16) are of the same order. Here again, the norming constants (a_n), (b_n) in Theorem 1 may be determined up to absolute multiplicative factors and are of order $(\ln n)^{1/r}$ and $(\ln n)^{(r-1)/r}$, respectively.

4. Consider another example from [12] in which

$$f_M(t) = K \exp\left(-\int_{1-t}^{1} \frac{e^{1/s}}{s} ds\right), \quad 0 < t < 1.$$

Then (see [12, Lemma 8]) $\ln p_\delta \sim -\delta e^{1/\delta}$ as $\delta \to 0$. Similarly, one can check that

$$\frac{\delta f_M(1-\delta)}{p_\delta} \sim \frac{\delta e^{-\delta e^{1/\delta}} e^{\delta e^{1/\delta}}}{\delta e^{-1/\delta}} = e^{1/\delta},$$

so this time the first term in the expression (3.16) is of higher order than the second. It follows from the asymptotics above that $a'_n \sim (\ln n)/c_0 c_1$ and $b'_n \sim c_1 \ln\ln n$ and hence (a_n), (b_n) are of order $\ln n$ and $\ln\ln n$, respectively.

Acknowledgments

The author was supported in part by the NSA grant #H98230-09-1-0062.

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