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Theoretical Computer Science 339 (2005) 344–371
Theoretical Computer Science
www.elsevier.com/locate/tcs

A context-free and a 1-counter geodesic language for a Baumslag–Solitar group

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Received 23 November 2003; received in revised form 9 November 2004; accepted 2 March 2005

Communicated by Z. Esik

Abstract

We give a language of unique geodesic normal forms for the Baumslag–Solitar group BS(1,2) that is context-free and 1-counter. We discuss the classes of context-free, 1-counter and counter languages, and explain how they are inter-related.

Keywords: Regular; Context-free; *G*-automaton; Counter; 1-counter; Baumslag–Solitar group; Language of geodesics

1. Introduction

In this article we give a simple combinatorial description of a language of normal forms for the solvable Baumslag–Solitar group $BS(1, 2)$ with the standard generating set, such that each normal form word is geodesic, each group element has a unique normal form representative, and the language is accepted by a (partially blind) 1-counter automaton. It follows that the language is context-free.

Several authors have studied geodesic languages for the (solvable) Baumslag–Solitar groups, including Brazil [1], Collins et al. [2], Freden and McCann [6], Groves [8], Miller

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[1] Supported by EPSRC Grant GR/S53503/01.

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doi:10.1016/j.tcs.2005.03.026

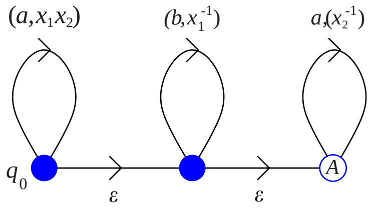

Fig. 1. A counter automaton accepting $a^n b^n a^n$.

[12], and the author and Hermiller [3]. It is well known that Baumslag–Solitar groups are asynchronously automatic but not automatic [5], and the asynchronous language is not geodesic. Groves proved that no geodesic language of normal forms for a solvable Baumslag–Solitar group with standard generating set can be regular [8], so we could say that context-free or 1-counter is the next best thing.

Collins, Edjvet and Gill proved that the growth function (the formal power series where the nth coefficient is the number of elements having a geodesic representative of length n) of a solvable Baumslag–Solitar group is rational [2], and Freden and McCann have studied growth functions for the non-solvable case [6].

If G is a group with generating set $\mathcal{G}$, we say two words u, v are equal in the group, or $u =_G v$, if they represent the same group element. We say u and v are identical if they are equal in the free monoid, that is, they are equal in $\mathcal{G}^*$.

Definition 1 (*G-automaton*). Let G be a group and Σ a finite set. A (non-deterministic) *G-automaton* A_G over Σ is a finite directed graph with a distinguished start *vertex* q_0, some distinguished *accept vertices*, and with edges labeled by $(\Sigma^{\pm 1} \cup \{\varepsilon\}) \times G$. If p is a path in A_G, the element of $(\Sigma^{\pm 1})$ which is the first component of the label of p is denoted by $w(p)$, and the element of G which is the second component of the label of p is denoted $g(p)$. If p is the empty path, $g(p)$ is the identity element of G and $w(p)$ is the empty word. A_G is said to *accept* a word $w \in (\Sigma^{\pm 1})$ if there is a path p from the start vertex to some accept vertex such that $w(p) = w$ and $g(p) =_G 1$.

Definition 2 (*Finite state automaton; Regular*). If G is the trivial group, then A_G is a (non-deterministic) *finite state automaton*. A language is *regular* if it is the set of strings accepted by a finite state automaton.

Definition 3 (*Counter; 1-counter*). A language is *k-counter* if it is accepted by some $\mathbb{Z}^k$-automaton. We call the (standard) generators of $\mathbb{Z}^k$ *counters*. A language is *counter* if it is k-counter for some $k \geqslant 1$.

For example, the language $\{a^n b^n a^n \mid n \in \mathbb{N}\}$ is accepted by the $\mathbb{Z}^2$-automaton in Fig. 1, with alphabet a, b and counters x_1, x_2.

In the case of $\mathbb{Z}$-automata, we assume that the generator is 1 and the binary operation is addition, and we may insist without loss of generality each transition changes the counter by either $0, 1$ or -1. We can do this by adding states and transitions to the automaton

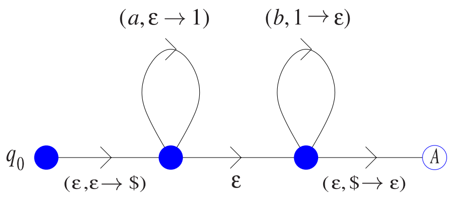

Fig. 2. Pushdown automaton accepting $a^n b^n$.

appropriately. That is, if some edge changes the counter by $k \neq 0, \pm 1$ then divide the edge into $|k|$ edges using more states. The symbols $+, -$ indicate a change of $1, -1$, respectively, on a transition.

Definition 4 (*Pushdown automaton; Context-free*). A *pushdown automaton* is a 6-tuple $(Q, \Sigma, \Gamma, \tau, q_0, A)$ where Q, Σ, Γ and A are all finite sets, and
(1) Q is the set of states,
(2) Σ is the input alphabet together with the empty word ε,
(3) Γ is the stack alphabet together with ε (the empty symbol),
(4) τ is the transition function,
(5) q_0 is the start state,
(6) $A \subseteq Q$ is the set of accept states.
The transition function takes as input a state and an input letter, and outputs a state and a stack instruction of the form $\gamma \to \beta$, which means pop γ from the top of the stack then push β on the top of the stack. Note that $\varepsilon \to \gamma$ means push γ onto the stack, $\gamma \to \varepsilon$ means pop γ off the stack, and $\varepsilon \to \varepsilon$ means do nothing (and in this case will be omitted).

A word is accepted by the automaton if there is a sequence of transitions starting from the state q_0 with an empty stack, pushing and popping stack symbols, to an accept state. Note that you can always push new symbols onto the stack, but you can only pop if the correct symbol is on top of the stack.

A language is *context-free* if it is the language of some pushdown automaton.

As an example, the language $\{a^n b^n \mid n \in \mathbb{N}\}$ is accepted by the pushdown automaton in Fig. 2 with alphabet a, b and stack symbols \$, 1, and this language is not regular [9,15]. Note that our definition of counter automata is not equivalent to a pushdown automata with a stack (with one type of token) for each counter, since in our definition, we cannot test the value of the counter until we are done reading the input. For this reason, these automata are sometimes referred to as “partially blind” or vision-impaired counter automata, since they cannot “see” whether the counter is non-zero except at the end.

Definition 5 (*Baumslag–Solitar group*). The group with presentation $\langle a, t \mid tat^{-1} = a^p \rangle$ is the *solvable Baumslag–Solitar group* $\mathrm{BS}(1, p)$, for $p \in \mathbb{Z}$, $p \geqslant 2$.

In this article we will consider the group $\mathrm{BS}(1, 2)$. Let $\mathcal{G} = \{a, a^{-1}, t, t^{-1}\}$ be the inverse closed generating set for $\mathrm{BS}(1, 2)$. We give a picture of part of the Fig. 3. From the side the

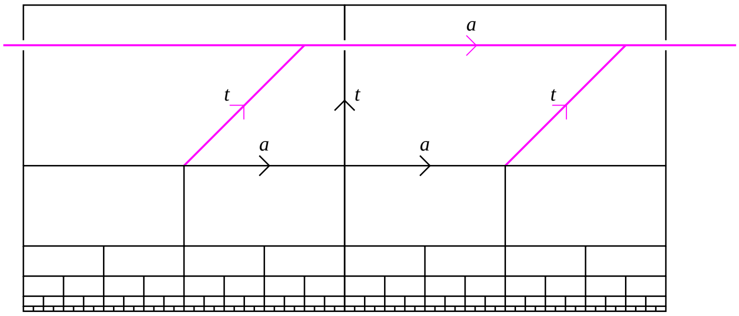

Fig. 3. Part of the Cayley graph for BS(1, 2).

Cayley graph for BS(1, 2) in Cayley graph looks like a binary tree. See [5] for a detailed description of the Cayley graph.

The paper is organised as follows. In Sections 2 and 3 we examine the various definitions of formal languages presented above, and establish their relative intersections and inclusions, which we illustrate in Fig. 5. In particular, we prove that 1-counter languages as defined are context-free. In Section 4, we define a normal form language for BS(1, 2) and prove that each normal form word is geodesic, and the language of normal form words bijects to the set of group elements. In Section 5, we prove that this normal form language is 1-counter, which implies it is context-free. Then in the last section we show that the language of all geodesics for BS(1, 2) is not counter.

2. 1-counter languages

Lemma 6. *Every* 1-*counter language is context-free.*

Proof. Let L be a 1-counter language accepted by a 1-counter machine M. We will construct a (non-deterministic) pushdown automaton N that accepts the language L, with stack symbols $\$_+, \$_-$ and 1. Let M_+ be a copy of M obtained by replacing transitions $(a, +)$ by $(a, \varepsilon \to 1)$ and $(a, -)$ by $(a, 1 \to \varepsilon)$, and let M_- be a copy of M obtained by replacing transitions $(a', -)$ by $(a', \varepsilon \to 1)$ and $(a', +)$ by $(a', 1 \to \varepsilon)$.

N is constructed from these two automata M_+ and M_- as follows. The states of N consist of two distinct states q_+, q_- for each state q of M, plus a new start state s_0 and a new single accept state p. There is a transition labelled $(\varepsilon, \varepsilon \to \$_+)$ from s_0 to the former start state $(q_0)_+$ in M_+. For each q_+ in M_+ there is a transition labelled $(\varepsilon, \$_+ \to \$_-)$ from q_+ to the corresponding state q_- in M_-, and a transition labelled $(\varepsilon, \$_- \to \$_+)$ from q_- to q_+ in M_+.

Finally, for every accept state q in M there is a transition labelled $(\varepsilon, \$_+ \to \varepsilon)$ from q_+ in M_+ to the single accept state p, and $(\varepsilon, \$_- \to \varepsilon)$ from q_- in M_- to the single accept state p.

This new machine works by starting with an empty stack and pushing $\$_+$ on the bottom. Then if the old machine increments the counter, the new machine adds 1 to the stack. From

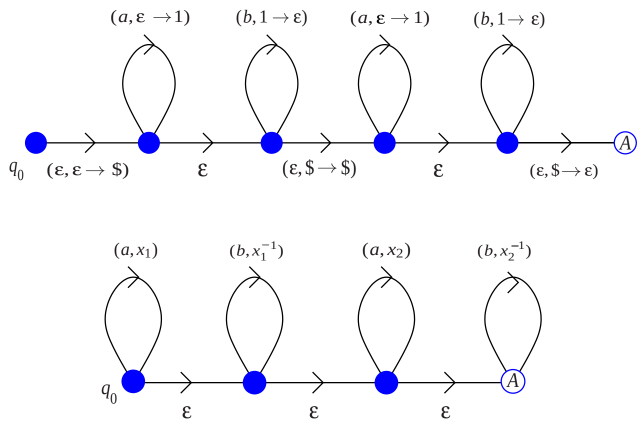

Fig. 4. Pushdown automaton and 2-counter machine accepting $a^m b^m a^n b^n$.

then on if the counter value never dips below zero, the new machine will stay in the M_+ states. However, if there is ever a "pop 1" but the symbol on the stack is $\$_+$, pass over to M_-. Then the height of the stack now represents the negative value of the counter, you stay in this side until the value of the counter comes back to zero, in which case you can switch.

It follows that the language of N is precisely the language of the 1-counter machine L. $\square$

Lemma 7. *The language of strings of the form $a^m b^m a^n b^n$ is both counter and context-free but not* 1*-counter.*

Proof. The pushdown automaton and the $\mathbb{Z}^2$-automaton in Fig. 4 both accept this language, so it is context-free and counter.

Suppose by way of contradiction that the language is 1-counter, and let M be a 1-counter machine for it with p states. Assume without loss of generality that each transition changes the counter by either 0, -1 or 1.

Define $a_1 = a^{p^2}, b_1 = b^{p^2}, a_2 = a^{p^2}, b_2 = b^{p^2}$, and consider the word $s = a_1 b_1 a_2 b_2$ which belongs to the language.

Consider the prefix $a_1 = a^{p^2}$. Since this prefix is longer than the number of states, it must visit some state twice, so $a_1 = x_0 y_0 z_0$ where y_0 represents a loop of length at most p.

If going around y_0 causes a net change of zero in the value of the counter, then going around it twice would give a new word that is accepted by M, but not of the form $a^m b^m a^n b^n$. So assume the net change is k_0 with $|k_0| \geqslant 1$.

Let $s_1 = x_0 z_0$ which has length at least $p^2 - p$, so must go around a loop in M. So $s_1 = x_1 y_1 z_1$ with y_1 a loop of length at most p. Again, if the net change in the counter going around y is zero then we can go around y_1 twice and have a word accepted by M that is not in the language.

If the net change is k_1 of the opposite sign to k_0 then there is a word that goes $|k_1|$ times around the loop y_0 then $|k_0|$ times around y_1, which keeps the final value of the counter at zero, so is accepted by M, but since we are pumping the a^{p^2} prefix of s we have a word that is not in the language.

Thus y_1 changes the counter by k_1 with $|k_1| \geqslant 1$ and having the same sign of k_0. Let $s_2 = x_1 z_1$ with length at least $p^2 - 2p$.

Iteratively we can write $s_i = x_i y_i z_i$ with y_i a loop which changes the value of the counter by an amount k_i of the same sign as k_0, until there are no loops left in $x_i z_i$, which does not happen until at least p iterations (since s_i has length at least $p^2 - ip$).

Since $x_i z_i$ has no loops, it has length at most $p - 1$. So it changes the value of the counter by at most l_1 where $|l_1| < p$. Whereas, the sum of the $|y_i|$ changes the value of the counter by at least p since each one contributes at least 1 to the sum.

Now repeat this analysis for the subwords b_1, a_2 and b_2.

If all the loops in each subword change the counter by the same sign, then we have a contradiction, since the net change of all the loops is greater than $4p$ whereas the net change of the four remaining $x_i z_i$ segments is less than $4p$, so they cannot cancel each other.

Thus at least two subwords have loops of opposite signs. If the loops in a_1 have the same sign as the loops in a_2 and b_2, then the loops in b_1 must have the opposite sign. So suppose that some loop in b_1 changes the counter by k, and some loop in a_2 changes the counter by l of the opposite sign to k. Then pumping the first loop by $|l|$ and the second by $|k|$ gives a word that is accepted by M and not in the language.

Otherwise if the loops in a_1 have the opposite sign to the loops in either a_2 or b_2, then take a loop in a_1 which changes the counter by k and a loop in a_2 or b_2 that changes the counter by l of the opposite sign to k. Then pumping the first loop by $|l|$ and the second by $|k|$ gives a word that is accepted by M and not in the language. □

Corollary 8. 1-*counter languages are not closed under concatenation or intersection.*

Proof. The language $C = \{a^n b^n \mid n \in \mathbb{N}\}$ is 1-counter but CC is not 1-counter by the previous lemma (Lemma 7).

The languages $D = \{a^n b^n c^m \mid m, n \in \mathbb{N}\}$ and $E = \{a^m b^n c^n \mid m, n \in \mathbb{N}\}$ are 1-counter, but $D \cap E = \{a^n b^n c^n \mid n \in \mathbb{N}\}$ is not context-free [9,15] so by Lemma 6 is not 1-counter. □

However, we have

Lemma 9 (*Closure properties of k-counter languages*). *If C, C' are k-counter for $k \geqslant 1$ and L is regular, then $C \cup C'$, $C \cap L$, CL and LC are all k-counter.*

Proof. Let M, M' be k-counter automata for C, C', with start states q_0, q_0', states S, S', and accept states A, A', respectively. Then construct a k-counter automaton accepting $C \cup C'$ with a new start state p_0 joined to q_0, q_0' by two epsilon transitions.

Let N be a finite state automaton for L with states T, start state p_0 and accept states B. Construct a k-counter automaton accepting $C \cap L$ having states $S \times T$, start state (q_0, p_0), such that (q, p) is an accept state if $q \in A$, $p \in B$ (they are both accept states), and if there are transitions from q to r in M labelled by (a, g) and p to s in N labelled by a where $g \in \mathbb{Z}^k$, then there is a transition from (q, p) to (r, s) labelled (a, g).

Construct a k-counter automaton accepting CL with start state q_0 and accept states B by adding an epsilon transition from each accept state of M to p_0.

Construct a k-counter automaton accepting LC with start state p_0 and accept states A by adding an epsilon transition from each accept state of N to q_0. □

Iterating the union operation a finite number of times gives

Corollary 10. *The union of a finite number of k-counter languages is k-counter.*

3. Context-free and not counter

The language $\{a^n b^n a^n \mid n \in \mathbb{N}\}$ accepted by the $\mathbb{Z}^2$-automaton in Fig. 1 is not context-free by standard results [9,15]. In this section we show that conversely, there is a language that is context-free but not counter.

Consider a string of letters a, b, c. We say a string contains a *square* if it has a subword of the form ww. An interesting result from combinatorics is that one can write out a square-free word in a, b, c of arbitrary length. This is due to Thue and Morse and described in [10, Chapter 2]. In particular we have

Proposition 11 (*Thue–Morse*). *Define a homomorphism f on* $\{a, b, c\}$ *by* $f(a) = abc$, $f(b) = ac$ *and* $f(c) = b$. *Then for any* $i \in \mathbb{N}$, $f^i(a)$ *is square-free.*

For example, to compute $f^3(a)$ we have
$a \to abc \to abcacb \to abcacbabcbac$.

In order to show that a language is not counter we make use of the following lemma.

Lemma 12 (*Swapping Lemma*). *If L is counter then there is a constant* $s > 0$, *the "swapping length", such that if* $w \in L$ *with length at least* $2s + 1$ *then w can be divided into four pieces* $w = uxyz$ *such that* $|uxy| \leqslant 2s + 1$, $|x|, |y| > 0$ *and* $uyxz \in L$.

Proof. Let s be the number of states in the counter automaton, and let p be a path in the Z^k-automaton such that $w(p) = w$. If p visits each state at most twice then it cannot have length more than $2s$, so p visits some state at least three times. Let u be the first part of $w(p)$ until it hits this state, then x a non-trivial loop back to this state the second time, y a loop back a third time, and z the rest of w. So $w(p) = uxyz$ ends at an accept state, and the second component of p equals $g(u)g(x)g(y)g(z) =_{\mathbb{Z}^k} 1$. Switching the orders of x and y, the path $uyxz$ still takes you to the same accept state, and $g(uyxz) =_{\mathbb{Z}^k} 1$ since all elements of $\mathbb{Z}^k$ commute, so $uyxz \in L$. □

Note its similarity to the pumping lemmas for regular and context-free languages [9,15]. This lemma is only of any use if your word w has no squares, otherwise you can just swap the square and get the same word (that is $x = y$).

Theorem 13. *There is a language that is context-free but not counter.*

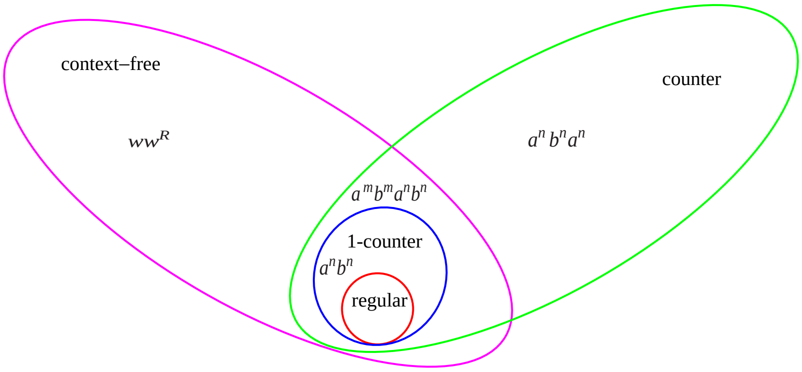

Fig. 5. Intersections of the formal languages.

Proof. Consider the language of all strings in a, b, c of the form ww^R, where w^R is word obtained by reversing w. It is well known that this is a context-free language [9,15], since it is accepted by a pushdown automaton which uses the stack to store the first half of the word, then checks the last half of the word matches.

Suppose by way of contradiction that this language is counter, with swapping length p as in Lemma 12. Let w be a square-free word from Proposition 11 of length at least $2p+1$. Then ww^R can be split into four subwords u, x, y, z such that uxy falls in the first w prefix. Since w has no squares and x, y are adjacent words then it must be that $x \neq y$. But $uyxz$ will fail to be in the language because the second part will not be the reverse of the first part. $\square$

In Fig. 5 we have a diagram of sets of regular, 1-counter, context-free and counter languages, and by the above results we have shown the given inclusions.

The fact that there are counter languages that are not context-free and vice versa can be observed by considering word problems for various groups. The word problem for a group G with generating set $\mathcal{G}$ is the set $WP(G) = \{w \in \mathcal{G}^* : \overline{w} = 1\}$ of all words in the generating set that evaluate to the identity element. By work of Muller and Schupp [14], the word problem for the group $\mathbb{Z}^2$ is not a context-free language, whereas the word problem of the free group on two (or more) generators is context-free. Elston and Ostheimer [4] proved that a group has a deterministic counter word problem (with a so-called inverse property) if and only if it is virtually abelian, so the word problem for Z^2 is counter. To see why $WP(F_2)$ is not counter, consider a Thue–Morse word made up of an arbitrary number of subwords (aaa), (aba), $(ab^{-1}a)$ encoded by the letters x, y, z, followed by its “reverse” in the subwords $(a^{-1}a^{-1}a^{-1})$, $(a^{-1}b^{-1}a^{-1})$, $(a^{-1}ba^{-1})$ encoded by the letters x', y', z'. This word is in the word problem, but applying the Swapping Lemma (Lemma 12) to the encoded word gives a word that does not encode a word in the word problem.

The first examples of languages that are counter but not context-free were given by Mitrana and Stiebe [13]. Mitrana and Stiebe give the following lemma, called the “Interchange Lemma”, which they use to show that the language of palindromes and the language

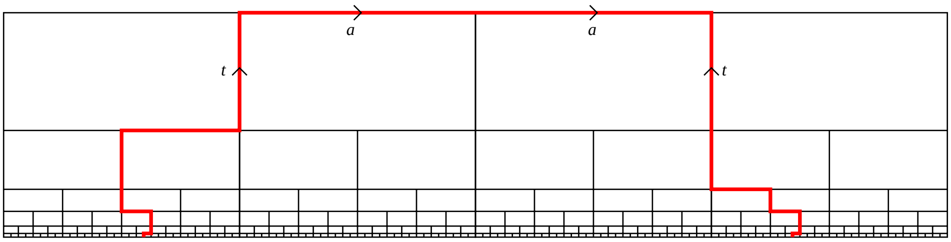

Fig. 6. An X word.

$\{a^i b^i \ |i \geqslant 0\}^*$ are not counter. We include it here for completeness, and to show how it differs from the Swapping Lemma above.

Lemma 14 (*Interchange Lemma [13]*). *If L is the language of a G-automaton where G is an abelian group, then there is a constant p such that for any word $x \in L$ of length at least p, and for any given subdivision of x into subwords $v_1 w_1 v_2 w_2 \dots w_p v_{p+1}$ with $|w_i| \geqslant 1$, there are some r, s such that the word obtained from x by interchanging w_r and w_s is in L.*

4. The normal form language

Recall that $\mathrm{BS}(1, 2) = \langle a, t \,|\, tat^{-1} = a^2 \rangle$ with the (standard) inverse closed generating set $\mathcal{G} = \{a, a^{-1}, t, t^{-1}\}$. We wish to describe geodesic words with respect to this generating set.

Definition 15 (E, N, P, X). A word is of the form E if it is a^i. A word is of the form N if it has no t letters and at least one t^{-1} letter. A word is of the form P if it has no t^{-1} letters and at least one t letter.

A word is of the form X if it is the concatenation of a P word of t-exponent k, followed by an N word of t-exponent $(-k)$. That is, an X word is a word of type PN with zero t-exponent.

Benson Farb called words of type X "mesas", since drawing an X word in the Cayley graph resembles this land formation. See Fig. 6.

While the following fact is well known, we include an elementary proof of it here for completeness.

Lemma 16 (*Commutation*). *If u has zero t-exponent then $au = ua$ and $a^{-1}u = ua^{-1}$.*

Proof. If u is type X then $u =_{BS} a^i$ so $au = a^{i+1} = ua$.

If u is type NP then let $u = vw$ where v is type N with t-exponent $-k$ and w is type P (so has t-exponent k). Each time we push a^i past a t^{-1} it becomes a^{2i} since $at^{-1} = t^{-1}a^2$. Then $au = avw = va^{2^k}w$. Each time we push a^{2^i} past a t it becomes $a^{2^{i-1}}$ since $a^2 t = ta$. So $au = avw = va^{2^k}w = vwa = ua$. Finally if u is any other form, first replace each occurrence of $ta^i t^{-1}$ in u by a^{2i}. Then u becomes a word of type NP with zero t-exponent.

We can pass a through this word as in the previous case, and then put u back in its original form and we are done. $\square$

Lemma 17 (*Miller [12]*). *Every geodesic word in $\mathcal{G}^*$ is a subword of a word of type NPN or PNP.*

See Lemma 1 of [8] for a proof. We can use this lemma to describe a subset of geodesic words that represent every group element.

Define a type $NP_{\leqslant}$ word to be a word of type *NP* with non-positive t-exponent sum, and type $NP_{>}$ to be a word of type *NP* with positive t-exponent sum.

Lemma 18 (*Ten types*). *Every element of $BS(1,2)$ has a geodesic representative in $\mathcal{G}^*$ that is one of ten types*:
$E, X, N, XN, NP_{\leqslant}, XNP$ *having* t-exponent $\leqslant 0$, *or*
$P, PX, NP_{>}, NPX$ *having* t-exponent > 0, *such that no more than three a or a^{-1} letters can occur in succession in the geodesic.*

Hermiller and the author used a similar characterisation in our work on minimal almost convexity [3].

Proof of Lemma 18. Every group element can be represented by some geodesic word in $\mathcal{G}^*$. If a geodesic word has no $t^{\pm 1}$ letters then it is type *E*. Otherwise by Lemma 17 it is a word of type N, P, NP, PN, NPN or *PNP*.

If the geodesic is type *NP* then it either has non-positive t-exponent sum, so is type $NP_{\leqslant}$, or positive t-exponent sum, so is type $NP_{>}$.

If the geodesic is type *PN* then it either has zero t-exponent sum, so is type *X*, negative t-exponent sum, so is type *XN*, or positive t-exponent sum, so is type *PX*.

Suppose the geodesic is a word w of type *NPN*. If w has positive t-exponent sum it is type *NPX*. If w has zero t-exponent sum, then write it as ux where u is type *NP* with zero t-exponent sum and x is type *X*. Since $x =_{BS} a^p$ for some integer p then applying Lemma 16 we get $w = ux_{BS}ua^p =_{BS} a^p u =_{BS} xu$ which has the same length and is type *XNP*. If w has negative t-exponent sum, then $w = a^{\varepsilon_1}t^{-1}uta^{\varepsilon_2}txt^{-1}a^{\varepsilon_2}t^{-1}v$ where u is type *E* or *NP* with zero t-exponent sum, x is type *E* or *X*, v is type *E* or *N*, and $\varepsilon_i \in \mathbb{Z}$. Then by Lemma 16

$$
\begin{aligned}
w &=_{BS} a^{\varepsilon_1+\varepsilon_2+\varepsilon_3}(t^{-1}ut)(txt^{-1})t^{-1}v \\
&=_{BS} a^{\varepsilon_1+\varepsilon_2+\varepsilon_3}(txt^{-1})(t^{-1}ut)t^{-1}v
\end{aligned}
$$

which is not geodesic since we can cancel tt^{-1} at the end.

Finally, suppose the geodesic is a word w of type *PNP*. If w has negative or zero t-exponent sum it is type *XNP*. If w has positive t-exponent sum, then $w = a^{\varepsilon_1}txt^{-1}a^{\varepsilon_2}t^{-1}uta^{\varepsilon_3}tv$ where x is type *E* or *X*, u is type *E* or *NP* with zero t-exponent sum, v is type *E* or *P*, and $\varepsilon_i \in \mathbb{Z}$. Then by Lemma 16

$$
\begin{aligned}
w &=_{BS} a^{\varepsilon_1+\varepsilon_2+\varepsilon_3}(txt^{-1})(t^{-1}ut)tv \\
&=_{BS} a^{\varepsilon_1+\varepsilon_2+\varepsilon_3}(t^{-1}ut)(txt^{-1})tv
\end{aligned}
$$

which is not geodesic since we can cancel $t^{-1}t$ at the end.

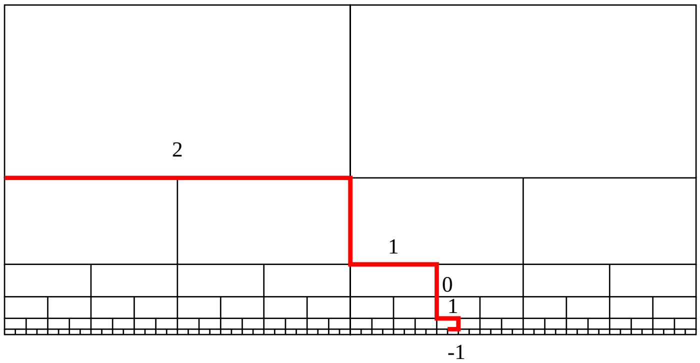

Fig. 7. The N-run $2101(-1)$.

The additional condition that no more than three a's are allowed in succession is obtained by observing that $a^6 =_{BS} ta^3t^{-1}$ so any power of a greater than five is not geodesic, and since $a^4 =_{BS} ta^2t^{-1}$ and $a^5 =_{BS} ta^2t^{-1}a =_{BS} ata^2t^{-1}$ we choose to replace a-exponents of 4 or 5 by subwords of the same length. An identical argument eliminates powers of a^{-1} greater than three. $\square$

Definition 19 (*Run*). An *N-run* is a word of the form

$$a^{\varepsilon_k}t^{-1}a^{\varepsilon_{k-1}}t^{-1}\dots t^{-1}a^{\varepsilon_1}t^{-1}a^{\varepsilon_0}.$$

A *P-run* is a word of the form

$$a^{\varepsilon_0}ta^{\varepsilon_1}t\dots ta^{\varepsilon_{k-1}}ta^{\varepsilon_k}.$$

We can write a run in shorthand by just writing the a-exponents. For example, $a^2t^{-1}at^{-1}a^0 t^{-1}at^{-1}a^{-1}$ can be written as $2101(-1)$.

We call the a-exponents *entries* of the run. A run is *non-trivial* if it has at least one non-zero entry. Note that a run that has at least one t or t^{-1} letter will have at least two entries, since by definition a run starts and ends with a power of a (possibly a^0).

We say a geodesic has at most one non-trivial run if it can be expressed as the concatenation of geodesic N- or P-runs such that at most one factor is non-trivial. For example, the word $t^2a^2t^{-1}at^{-2}$ can be written as $(t^2)(a^2t^{-1}at^{-2})$, so has at most one run.

Drawing the N-run represented by $2101(-1)$ in the Cayley graph we start to see what behaviour is allowed in a geodesic. For instance, the sub-runs $1(-1)$ and $(-1)1$ are not allowed since

$$at^{-1}a^{-1} \to t^{-1}a, \quad a^{-1}t^{-1}a \to t^{-1}a^{-1}.$$

Also, if the N-run $2101(-1)$ were preceded by a t^{-1} then we would have $t^{-1}a^2$ which can be written as at^{-1}. In fact, the only time you could ever see an entry that is not 0, 1 or -1 is at the *start* of an N-run, or the *end* of a P-run (Fig. 7).

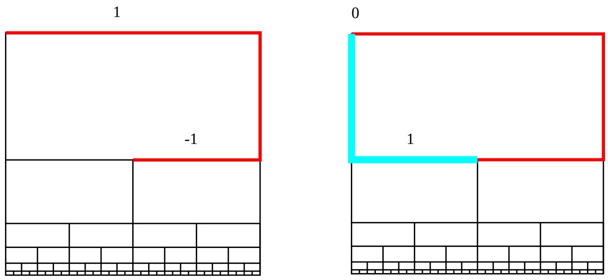

Fig. 8. No $1(-1)$, $(-1)1$ in a run.

Lemma 20 (*No* $|i| > 6$). *If a run represents a geodesic word and has an entry i that is not one of* $1, 0$ *and* (-1), *then i must be one of* $2, 3, 4, 5, (-2)(-3), (-4), (-5)$ *and occurs at the start of an N-run or the end of a P-run.*

Proof. If $i \geqslant 6$ occurs at any point in a run then $a^6 \to ta^3t^{-1}$ so the run is not geodesic.

For N-runs, if $i \geqslant 2$ occurs after the start of the run then $t^{-1}a^2 \to at^{-1}$ so the run is not geodesic. If $i \leqslant -2$ occurs after the start of the run then $t^{-1}a^{-2} \to a^{-1}t^{-1}$ so the run is not geodesic.

For P-runs, if $i \geqslant 2$ occurs before the end of the run then $a^2t \to ta$ so the run is not geodesic. If $i \leqslant -2$ occurs before the end of the run then $a^{-2}t \to ta^{-1}$ so the run is not geodesic. □

Lemma 21 (*No consecutive* $1(-1), (-1)1$). *A geodesic run cannot contain* $1(-1)$ *or* $(-1)1$.

Proof. For an N-run:

$$\begin{array}{llll} 1(-1) \to 01, & at^{-1}a^{-1} & \to & t^{-1}a, \\ (-1)1 \to 0(-1), & a^{-1}t^{-1}a & \to & t^{-1}a^{-1}. \end{array}$$

For a P-run:

$$\begin{array}{ll} 1(-1) \to (-1)0, & ata^{-1} \to a^{-1}t, \\ (-1)1 \to 10, & a^{-1}ta \to at. \end{array}$$

See Fig. 8. □

Lemma 22 (*No consecutive* $11, (-1)(-1)$). *There exist rewrite rules which do not increase length which can be applied to a geodesic run to eliminate all occurrences of consecutive* 11 *or* $(-1)(-1)$ *after the first two entries of an N-run and before the last two entries of a P-run.*

Proof. Let $i \in \mathbb{Z}$.

For an N-run:

$$\begin{array}{rlrl} i11 \to & (i+1)0(-1), & a^it^{-1}at^{-1}a & \to a^{i+1}t^{-2}a^{-1}, \\ i(-1)(-1) \to & (i-1)01, & a^it^{-1}a^{-1}t^{-1}a^{-1} & \to a^{i-1}t^{-2}a. \end{array}$$

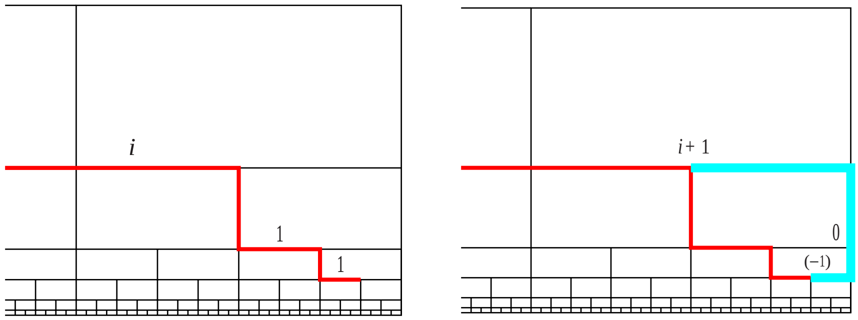

Fig. 9. No $i11$, $i(-1)(-1)$ in an N-run.

These moves are illustrated in Fig. 9. We can always perform these rewrites to get a word of the same length or shorter. That is, suppose you have an N-run, which is geodesic so we assume has no $1(-1)$ or $(-1)1$. Starting at the right end of the N-run, if there is an $i11$, we know that $i \geqslant 0$. Replacing this by $(i+1)0(-1)$ gives a word that is not geodesic if $i > 0$, otherwise gives $10(-1)$. Now if the preceding entry is (-1) the word is not geodesic, so is $0, 1$ or we are at the start of the run. A similar argument holds when we see $i(-1)(-1)$.

So iterate this procedure until the start of the run is reached. This eliminates all occurrences of adjacent non-zero entries after the first two entries. That is, if the N-run starts with 110 for example, the rules do not apply.

For a P-run:

$$\begin{aligned} 11i &\to (-1)0(i+1), & atata^i &\to a^{-1}t^2a^{i+1}, \\ (-1)(-1)i &\to 10(i-1), & a^{-1}ta^{-1}ta^i &\to at^2a^{i-1}. \end{aligned}$$

Similarly we can always perform these rewrites to get a word of the same length or shorter, this time starting at the left end of the word and moving right, so we can eliminate all adjacent non-zero entries except in the last two positions. $\square$

Next we will show that every geodesic of one of the ten types can be "pushed" into a geodesic word for the same group element that have at most one non-trivial run. As an example, if $w = a^{\varepsilon_0}ta^{\varepsilon_1}t\ldots a^{\varepsilon_k}ta^nt^{-1}a^{\eta_k}t^{-1}\ldots t^{-1}a^{\eta_1}t^{-1}a^{\eta_0}$ is a geodesic X word, then we can push the inner subword $a^{\varepsilon_k}ta^nt^{-1}a^{\eta_k}$ to $ta^nt^{-1}a^{\varepsilon_k+\eta_k}$, and iteratively push at each level to get $t^ka^nt^{-1}a^{\varepsilon_k+\eta_k}t^{-1}\ldots t^{-1}a^{\varepsilon_1+\eta_1}t^{-1}a^{\varepsilon_0+\eta_0}$. We show this in Fig. 10.

Lemma 23 (*At most one run*). *Every group element is represented by some geodesic of one of the ten types having at most one non-trivial run.*

Proof. By Lemma 18 each group element is represented by some geodesic of one of the ten types. If the word is type E, N, P then there is at most one non-trivial run. If it is X, XN or PX then by Lemma 16 we can push a letters to one side of the X word to get at most one non-trivial run, as we did in the example above. For $NP_{\leqslant}$ words we have $w = w_N w_{NP}$ where w_{NP} has zero t-exponent, so by Lemma 16 we can push a letters to the left of the

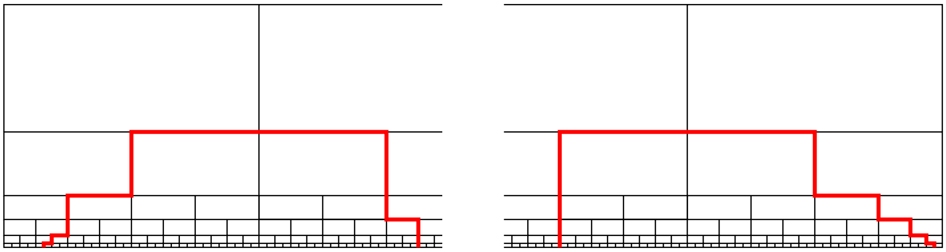

Fig. 10. Pushing an *X* word to have one non-trivial *N*-run.

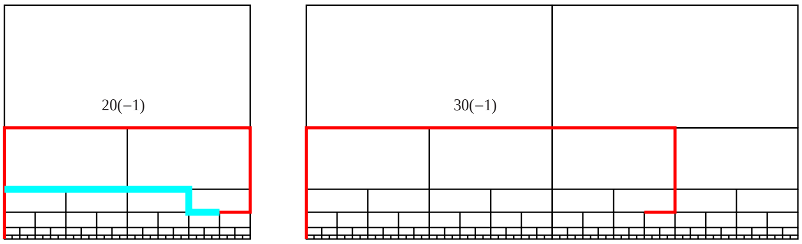

Fig. 11. Prefixes for *N*-runs of an *X* word.

NP word to get at most one run non-trivial run. For *XNP* words we have $w = w_X w_N w_{NP}$ where w_{NP} has zero t-exponent, so by Lemma 16 we can push a letters to one side of the *X* and *NP* words to get at most one non-trivial run. For $NP_>$ words we have $w = w_{NP} w_P$ where w_{NP} has zero t-exponent, so by Lemma 16 we can push a letters to the right of the *NP* word to get at most one non-trivial run. For *NPX* words we have $w = w_{NP} w_P w_X$ where w_{NP} has zero t-exponent, so by Lemma 16 we can push a letters to one side of the *X* and *NP* words to get at most one non-trivial run. $\square$

Given that every word can be pushed into a word having at most one non-trivial run, and we can choose which patterns are not allowed in a run, we are ready to define the normal form language.

The only issue that remains is the prefix of each run. For example, a geodesic of type *X* can be pushed into a word with exactly one *N*-run. The start of this run can be chosen to be either $a^2t^{-1}, a^3t^{-1}, a^{-2}t^{-1}$ or $a^{-3}t^{-1}$, for if the run starts with 1 then $tat^{-1} \to a^2$ so is not geodesic. If it starts with 4 or 5 then by Lemma 18 $a^4 \to ta^2t^{-1}$ and $a^5 \to ta^2t^{-1}a$ so we elect to write it starting with a 2 instead, and if the run starts with $i \geqslant 6$ then it is not geodesic.

The next few entries could be any one of the following:
$200, 201, 210, 300, 301, 30(-1), 310$ or the negatives of these.

Note that the prefix $20(-1)$ is not allowed since $t^2a^2t^{-2}a^{-1}$ is not geodesic, whereas $30(-1)$ is allowed since $t^2a^3t^{-2}a^{-1}$ is geodesic. See Fig. 11.

Each case is treated separately in the following lemma. Then after these prefixes (suffixes for *P*-runs) the run has only 0, 1, (-1) with no consecutive non-zero entries.

Lemma 24 (*Prefixes/suffixes of runs*). *In this lemma we assume that each word has been pushed into a word with at most one non-trivial run, and that each run has at least three* $t^{\pm 1}$ *letters*.

- *The N-run in a geodesic word of type* X, XN, XNP *with non-positive t-exponent sum must start with one of*
  200, 201, 210, 300, 301, 30(-1), 310 *or the negatives of these.*
- *The N-run in a geodesic word of type* N, $NP_{\leqslant}$ *with non-positive t-exponent sum must start with one of*
  000, 001, 010, 100, 101, 10(-1), 110, 200, 201, 20(-1), 210, 300, 301, 30(-1), 310 *or the negatives of these.*
- *The P-run in a geodesic word of type* P, $NP_{>}$ *with positive t-exponent sum must end with one of*
  000, 100, 010, 001, 101, (-1)01, 011, 002, 102, (-1)02, 012, 003, 103, (-1)03, 013 *or the negatives of these*.
- *The P-run in a geodesic word of type* PX, NPX *with positive t-exponent sum must end with one of*
  002, 102, 012, 003, 103, (-1)03, 013 *or the negatives of these.*

Proof. If an *N*-run starts with $i11$ or $i(-1)(-1)$ then by Lemma 22 we can replace $i11$ by $(i+1)0(-1)$ and $i(-1)(-1)$ by $(i-1)01$ without increasing length. Thus the first three entries of an *N*-run will include a 0.

If an *N*-run in a word of type X, XN or *XNP* starts with *i* with $|i|\geqslant 4$ then we can replace $ta^{4+j}t^{-1}$ by $t^2a^2t^{-1}a^jt^{-1}$ to get a word of the same type and preserving length. If an *N*-run in a word of type X, XN or *XNP* starts with *i* with $|i|\leqslant 1$ then we can replace ta^it^{-1} by a^{2i}, reducing length, contradicting the fact that the word is geodesic. Thus an *N*-run in a word of type X, XN or *XNP* starts with 2, 3, (-2) or (-3).

This gives the following possibilities for the first three entries:
200, 201, 20(-1), 210, 2(-1)0, 300, 301, 30(-1), 310, 3(-1)0 or the negatives of these. We can eliminate 2(-1)0 and 3(-1)0 since they encode $a^it^{-1}a^{-1}t^{-1} = a^{i-1}t^{-1}at^{-1}$ for $i = 2, 3$ so are not geodesic. We also observe that 20(-1) encodes $t^2a^2t^{-2}a^{-1}$ which is not geodesic (as seen in Fig. 11).

This leaves 200, 201, 210, 300, 301, 30(-1), 310 (or their negatives) as the possible prefixes to the *N*-run in a geodesic of type X, XN or *XNP*. It is easy to check that each of these prefixes is geodesic.

If the *N*-run in a word of type N, $NP_{\leqslant}$ starts with *i* with $|i|\geqslant 4$ then we can replace $ta^{4+j}t^{-1}$ by $t^2a^2t^{-1}a^jt^{-1}$ preserving length. Note that they become words of type *XN* or *XNP*. If the *N*-run in a word of type N, $NP_{\leqslant}$ starts with *i* with $|i|\leqslant 3$ then we can have prefixes of the form $i0$, $i10$ when $i > 0$ and $i(-1)0$ when $i < 0$.

Explicitly, this gives
000, 001, 010, 100, 101, 10(-1), 110, 200, 201, 20(-1), 210, 300, 301, 30(-1), 310
or their negatives. It is easy to check that each of these prefixes is geodesic. Note that in this case we cannot eliminate 20(-1) since there are no preceding *t*'s.

The proof for P-runs follows a similar argument, and is omitted. □

Lemma 25 (*Short runs*). *In this lemma we assume that each word has been pushed into a word with at most one non-trivial run, and that each run has no more than two $t^{\pm 1}$ letters.*

- *The geodesics of type X, XN and XNP are the set L_1 of words of the form*

$$\begin{array}{lrl}
ta^it^{-1}a^j, & ij = & (\pm 2)0, (\pm 3)0, 21, 31, (-2)(-1), (-3)(-1); \\
ta^it^{-1}a^jt^{-1}a^k, & ijk = & (\pm 2)00, (\pm 3)00, 201, (\pm 3)01, (-2)0(-1), \\
& & (\pm 3)0(-1), 210, 310, (-2)(-1)0, (-3)(-1)0; \\
t^2a^it^{-1}a^jt^{-1}a^k, & ijk = & (\pm 2)00, (\pm 3)00, 201, (\pm 3)01, (-2)0(-1), \\
& & (\pm 3)0(-1), 210, 310, (-2)(-1)0, (-3)(-1)0; \\
ta^it^{-1}a^jt^{-1}a^kt, & ijk = & 201, (\pm 3)01, (-2)0(-1), (\pm 3)0(-1).
\end{array}$$

- *The geodesics of type N and $NP_{\leqslant}$ are the set L_2 of words of the form*

$$\begin{array}{lrl}
a^it^{-1}a^j, & ij = & 00, (\pm 1)0, (\pm 2)0, (\pm 3)0, 0(\pm 1), 0(\pm 2), 0(\pm 3), \\
& & 11, 21, 31, (-1)(-1), (-2)(-1), (-3)(-1); \\
a^it^{-1}a^jt, & ij = & 0(\pm 1), 0(\pm 2), 0(\pm 3), 11, 21, 31, \\
& & (-1)(-1), (-2)(-1), (-3)(-1); \\
a^it^{-1}a^jt^{-1}a^k, & ijk = & 000, (\pm 1)00, (\pm 2)00, (\pm 3)00, 0(\pm 1)0, 0(\pm 2)0, 0(\pm 3)0, \\
& & 00(\pm 1), 00(\pm 2), 00(\pm 3), (\pm 1)0(\pm 1), (\pm 2)0(\pm 1), (\pm 3)0 \\
& & (\pm 1), 110, 210, 310, (-1)(-1)0, (-2)(-1)0, (-3)(-1)0; \\
a^it^{-1}a^jt^{-1}a^kt, & ijk = & 00(\pm 1), 00(\pm 2), 00(\pm 3), (\pm 1)0(\pm 1), (\pm 2)0(\pm 1), (\pm 3)0 \\
& & (\pm 1); \\
a^it^{-1}a^jt^{-1}a^kt^2, & ijk = & 00(\pm 1), 00(\pm 2), 00(\pm 3), (\pm 1)0(\pm 1), (\pm 2)0(\pm 1), (\pm 3) \\
& & 0(\pm 1).
\end{array}$$

- *The geodesics of type P and $NP_{>}$ are the set L_3 of words of the form*

$$\begin{array}{lrl}
a^ita^j, & ij = & 00, 0(\pm 1), 0(\pm 2), 0(\pm 3), (\pm 1)0, 11, 12, 13, \\
& & (-1)(-1), (-1)(-2), (-1)(-3); \\
t^{-1}a^ita^j & ij = & (\pm 1)0, 11, 12, 13, (-1)(-1), (-1)(-2), (-1)(-3); \\
a^ita^jta^k, & ijk = & 000, 00(\pm 1), 00(\pm 2), 00(\pm 3), (\pm 1)0(\pm 1), (\pm 1)0(\pm 2), (\pm 1) \\
& & 0(\pm 3), 0(\pm 1)0, 011, 012, 013, 0(-1)(-1), 0(-1)(-2), \\
& & 0(-1)(-3); \\
t^{-1}a^ita^jta^k, & ijk = & (\pm 1)0(\pm 1), (\pm 1)0(\pm 2), (\pm 1)0(\pm 3); \\
t^{-2}a^ita^jta^k, & ijk = & (\pm 1)0(\pm 1), (\pm 1)0(\pm 2), (\pm 1)0(\pm 3).
\end{array}$$

- *The geodesics of type PX and NPX* (*must have positive t-exponent*) *are the set L_4 of words of the form*

$$\begin{array}{lrl}
a^ita^jta^kt^{-1}, & ijk = & 00(\pm 2), 00(\pm 3), 012, 013, 0(-1)(-2), 0(-1)(-3), \\
& & 102, 10(\pm 3), (-1)0(-2), (-1)0(\pm 3).
\end{array}$$

Proof. The proof is by exhaustive search. For the first two cases we have either one or two t^{-1} letters, so we consider $t^pa^it^{-1}a^jt^q$ and $t^pa^it^{-1}a^jt^{-1}a^kt^q$. The t-exponent must be non-positive, so $p + q \leqslant 1$ in the first case and $p + q \leqslant 2$ in the second case. For the a-exponents, $|i| \leqslant 3$ and $|j|, |k| \leqslant 1$. This gives a finite set of possibilities, so we run through each and check if it gives a geodesic. Note that the pattern $20(-1)$ is not a geodesic if it appears in an N-run preceded by a t, yet it is geodesic if it is in a N or $NP_{\geqslant}$ geodesic.

By Lemma 22 we choose to reject runs of the form $(i, 1, 1)$ and $(i, -1, -1)$ in favour of $(i + 1, 0, -1)$ and $(i - 1, 0, 1)$, respectively, so that we never see three non-zero entries in a row, even at the start of a run. The details of the exhaustive check are omitted.

For the third and fourth cases we have either one or two t letters, so we consider $t^{-p}a^i ta^j t^{-q}$ and $t^{-p}a^i ta^j ta^k t^{-q}$. The t-exponent must be positive, so $p, q = 0$ in the third and $p+q \leqslant 1$ in the fourth cases. For the a-exponents, $|k| \leqslant 3$ and $|j|, |k| \leqslant 1$. This gives a finite set of possibilities, so we run through each and check if it gives a geodesic.

By Lemma 22 we choose to reject runs of the form $(1, 1, i)$ and $(-1, -1, i)$ in favour of $(-1, 0, i + 1)$ and $(1, 0, i - 1)$, respectively, so that we never see three non-zero entries in a row, even at the end of a run. The details of the exhaustive check are omitted. $\square$

Definition 26 (*Normal form*). There are ten distinct types of normal form words.

- Type $\mathcal{NF}_E$ words are precisely $\varepsilon, a^{\pm 1}, a^{\pm 2}, a^{\pm 3}$.
- Type $\mathcal{NF}_X, \mathcal{NF}_{XN}$ and $\mathcal{NF}_{XNP}$, all with zero or negative t-exponent, are the words: $t^k a^{\varepsilon_l} t^{-1} a^{\varepsilon_{l-1}} t^{-1} \ldots a^{\varepsilon_1} t^{-1} a^{\varepsilon_0} t^m$ such that $k > 0$ and $l \geqslant k + m$, $\varepsilon_0 \neq 0$ if $m > 0$, the N-run starts with one of 200, 201, 210, 300, 301, 30(−1), 310 or the negatives of these, and after this has only 0, 1, (−1) with no consecutive non-zero entries (that is, no 1(−1), (−1)1, 11 or (−1)(−1) in the run).

  If there are less than three t^{-1} letters in the run, then the word is in the set L_1 of Lemma 25.
- Type $\mathcal{NF}_N$ and $\mathcal{NF}_{NP_{\leqslant}}$, all with negative t-exponent, are the words: $a^{\varepsilon_l} t^{-1} a^{\varepsilon_{l-1}} t^{-1} \ldots a^{\varepsilon_1} t^{-1} a^{\varepsilon_0} t^k$ such that $0 \leqslant k \leqslant l$, $\varepsilon_0 \neq 0$ if $k > 0$, the N-run starts with one of 000, 001, 010, 100, 101, 10(−1), 110, 200, 201, 20(−1), 210, 300, 301, 30(−1), 310 or the negatives of these, and after this has only 0, 1, (−1) with no consecutive non-zero entries.

  If there are less than three t^{-1} letters in the run, then the word is in the set L_2 of Lemma 25.
- Type $\mathcal{NF}_P$ *and* $\mathcal{NF}_{NP_{>}}$, all with positive t-exponent, are the words: $t^{-k} a^{\varepsilon_0} ta^{\varepsilon_1} t \ldots a^{\varepsilon_{l-1}} ta^{\varepsilon_l}$ such that $0 \leqslant k < l$, $\varepsilon_0 \neq 0$ if $k > 0$, the P-run ends with one of 000, 100, 010, 001, 101, (−1)01, 011, 002, 102, (−1)02, 012, 003, 103, (−1)03, 013 or the negatives of these, and before this has only 0, 1, (−1) with no consecutive non-zero entries.

  If there are less than three t letters in the run, then the word is in the set L_3 of Lemma 25.
- Type $\mathcal{NF}_{PX}$ and $\mathcal{NF}_{NPX}$, all with positive t-exponent, are the words: $t^{-k} a^{\varepsilon_0} ta^{\varepsilon_1} t \ldots a^{\varepsilon_{l-1}} ta^{\varepsilon_l} t^{-m}$ such that $k > 0$, $m \geqslant 0$ and $k + m < l$, $\varepsilon_0 \neq 0$ if $k > 0$, the P-run ends with one of 002, 102, 012, 003, 103, (−1)03, 013 or the negatives of these, and before this has only 0, 1, (−1) with no consecutive non-zero entries.

  The P-run must have at least two t letters since the t-exponent of the word is positive. If there are less than three t letters in the run, then the word is in the set L_4 of Lemma 25.

Lemma 27 (*The language of normal forms surjects to the group*). *Every group element is represented by a normal form word.*

Proof. By Lemma 23 every group element is represented by a geodesic having at most one run. Then by Lemma 22 we can remove any occurrences of 11 and $(-1)(-1)$ in the run (except possibly at the start of N and $NP_{\leqslant}$ words and the end of P and $NP_{>}$ words) without lengthening the word. Then if the resulting run does not start (or end) with one of the number patterns given in Lemma 24 relative to its type, it is not geodesic, and if it does, the word is in normal form. □

Definition 28 (*HNN-extension*). If G is a group with presentation $\langle \mathcal{G} \mid \mathcal{R} \rangle$ and $\phi : A \to B$ is an isomorphism of subgroups $A, B \subseteq G$, define the *HNN-extension* G_ϕ of G by ϕ to be the group with presentation $\langle \mathcal{G}, t \mid \mathcal{R}, \{tat^{-1} = \phi(a) : a \in A\}\rangle$. The generator t is called the *stable letter* and A, B are called *associated subgroups*.

The group $\mathrm{BS}(1,2)$ is an HNN-extension of $\langle a \rangle$ with the isomorphism $\phi(a) = a^2$ between associated subgroups $\langle a \rangle$ and $\langle a^2 \rangle$. The following fact about HNN-extensions can be read in [11].

Lemma 29 (*Britton's Lemma*). *If w is a word containing a $t^{\pm 1}$ letter in an* HNN-extension *of G_ϕ with associated subgroups A, B and if $w =_{G_\phi} 1$ then w must contain a subword* (*called a pinch*) *of the form tat^{-1} or $t^{-1}\phi(a)t$ for some element $a \in A$.*

Corollary 30 (*t-exponent*). *For each element $g \in \mathrm{BS}(1,2)$ there is an integer k such that every word for g has t-exponent k.*

Proof. If w represents the identity and has no $t^{\pm 1}$ letters then its t-exponent sum is zero. If w represents the identity and has $t^{\pm 1}$ letters then by Britton's Lemma it contains a pinch. Removing a pinch leaves the t-exponent of w unchanged, so either you can remove all $t^{\pm 1}$ letters, in which case the t-exponent sum was zero, or you cannot remove all $t^{\pm 1}$ letters, in which case the word did not represent the identity.

If w and u are two words for the same group element with t-exponents k and l, respectively, then $wu^{-1} =_{BS} 1$ and has t-exponent $k - l = 0$, so w and u have the same t-exponent. □

Lemma 31 (*a-exponents*). *The X word $w = t^k a^j t^{-1} a^{\varepsilon_{k-1}} t^{-1} \dots t^{-1} a^{\varepsilon_0}$ represents the element a^N where*

$$N = 2^k j + \sum_{i=0}^{k-1} 2^i \varepsilon_i.$$

Moreover if each $|\varepsilon_i| \leqslant 1$ for all $i \leqslant k-1$, $|j| \geqslant 2$ and ε_{k-1} is zero or the same sign as j, then $|N| \geqslant 4$.

Also, the X word $w = a^{\varepsilon_0} t a^{\varepsilon_1} t \dots t a^{\varepsilon_{k-1}} t a^j t^{-k}$ represents the element a^N where

$$N = 2^k j + \sum_{i=0}^{k-1} 2^i \varepsilon_i$$

and moreover if each $|\varepsilon_i| \leqslant 1$ *for all* $i \leqslant k-1$, $|j| \geqslant 2$ *and* ε_{k-1} *is zero or the same sign as* j*, then* $|N| \geqslant 4$.

Proof. To prove the first assertion we will use induction on k. If $k = 1$ we have $w = ta^j t^{-1} a^{\varepsilon_0} = a^{2j+\varepsilon_0}$.

Assuming the statement holds for k, then

$$\begin{aligned} w &= t^{k+1} a^j t^{-1} a^{\varepsilon_k} t^{-1} a^{\varepsilon_{k-1}} t^{-1} \ldots t^{-1} a^{\varepsilon_0} \\ &= t^k a^{2j+\varepsilon_k} t^{-1} a^{\varepsilon_{k-1}} t^{-1} \ldots t^{-1} a^{\varepsilon_0} = a^N, \end{aligned}$$

where $N = 2^k(2j + \varepsilon_k) + \sum_{i=0}^{k-1} 2^i \varepsilon_i$.

The smallest possible value for $|N|$ occurs when $|j| = 2$, $\varepsilon_{k-1} = 0$ and each $\varepsilon_i = -\frac{j}{|j|}$. In this case

$$\begin{aligned} |N| &\geqslant 2^k(2) + 0 + \sum_{i=0}^{k-2} 2^i(-1) \\ &= 2^k(2) - \sum_{i=0}^{k-2} 2^i \\ &= 2^k(2) - (2^{k-1} - 1) \\ &\geqslant 2(2) - (1 - 1) = 4 \text{ since } k \geqslant 1. \end{aligned}$$

To prove the second assertion we will again use induction on k. If $k = 1$ we have

$$w = a^{\varepsilon_0} t a^j t^{-1} = a^{2j+\varepsilon_0}.$$

Assuming the statement holds for k, then

$$\begin{aligned} w &= a^{\varepsilon_0} t \ldots t a^{\varepsilon_{k-1}} t a^{\varepsilon_k} t a^j t^{-k-1} \\ &= a^{\varepsilon_0} t \ldots t a^{\varepsilon_{k-1}} t a^{2^j+\varepsilon_k} t^{-k} = a^N, \end{aligned}$$

where $N = 2^k(2j + \varepsilon_k) + \sum_{i=0}^{k-1} 2_i \varepsilon_i$.

The smallest possible value for $|N|$ occurs when $|j| = 2$, $\varepsilon_{k-1} = 0$ and each $\varepsilon_i = -\frac{j}{|j|}$. In this case

$$\begin{aligned} |N| &\geqslant 2^k(2) + 0 + \sum_{i=0}^{k-2} 2^i(-1) \\ &= 2^k(2) - \sum_{i=0}^{k-2} 2^i \\ &= 2^k(2) - (2^{k-1} - 1) \\ &\geqslant 2(2) - (1 - 1) = 4 \text{ since } k \geqslant 1. \qquad \square \end{aligned}$$

Lemma 32 (*Uniqueness for* $\mathcal{NF}_E \cup \mathcal{NF}_X$). *If* $w, u \in \mathcal{NF}_E \cup \mathcal{NF}_X$ *and* $w =_{BS} u$ *then* w *and* u *are identical.*

Proof. If $w, u \in \mathcal{NF}_E$ then $w = a^i$ and $u = a^j$ and $a^i =_{BS} a^j$ means $a^{i-j} = 1$, so $i = j$ and w and u are identical.

If $w \in \mathcal{NF}_X$ then we can write $w = t^k a^{\varepsilon_k} t^{-1} a^{\varepsilon_{k-1}} t^{-1} \ldots a^{\varepsilon_1} t^{-1} a^{\varepsilon_0}$ with $k > 0$, which evaluates to the power N with $|N| \geqslant 4$ by Lemma 31, so w cannot be equal to a word in $\mathcal{NF}_E$.

If $u \in \mathcal{NF}_X$ and $w =_{BS} u$ then we can write $u = t^l a^{\eta_l} t^{-1} a^{\eta_{l-1}} t^{-1} \ldots t^{-1} a^{\eta_k} t^{-1} \ldots a^{\eta_1} t^{-1} a^{\eta_0}$, where without loss of generality we are assuming that $k \leqslant l$. Since both words evaluate to the same power of a we have

$$\begin{aligned} &\varepsilon_k 2^k + \varepsilon_{k-1} 2^{k-1} + \cdots + \varepsilon_1 2 + \varepsilon_0 \\ &\quad = \eta_l 2^l + \eta_{l-1} 2^{l-1} + \cdots + \eta_k 2^k + \cdots + \eta_1 2 + \eta_0. \end{aligned}$$

Let $i \in \mathbb{N}$ such that $\varepsilon_j = \eta_j$ for all $j < i$ and $\varepsilon_i \neq \eta_i$. Then cancelling and dividing through by 2^i we have

$$\varepsilon_k 2^{k-i} + \varepsilon_{k-1} 2^{k-1-i} + \cdots + \varepsilon_i = \eta_l 2^{l-i} + \eta_{l-1} 2^{l-1-i} + \cdots + \eta_i. \tag{1}$$

If $i = k$ then $|\varepsilon_k| = 2$ or 3 and we have $\varepsilon_k = \eta_l 2^{l-k} + \eta_{l-1} 2^{l-1-k} + \ldots + \eta_k$. If $l = k$ then $\varepsilon_k = \eta_k$ so w and u are identical. If $l \geqslant k+1$ then $|\varepsilon_k| = |\eta_l 2^{l-k} + \eta_{l-1} 2^{l-k-1} + \ldots \eta_k| \geqslant 4$ since $|\eta_l| \geqslant 2$ and η_{l-1} is either 0 or the same sign as η_l, but $|\varepsilon_k| \leqslant 3$ so this is a contradiction.

If $i < k$ then ε_i, η_i are either $0, \pm 1$ since they occur in the middle of a run. By Eq. (1) they must be of the same parity, and they cannot both be zero so one is 1 and one is (-1). If $i + 1 < k$ then $\varepsilon_{i+1} = \eta_{i+1} = 0$ and we contradict the equation since one side is equal to $1 \bmod 4$ and the other is $(-1) \bmod 4$.

So $i + 1 = k$, so the run in w starts with 210 or 310 (or their negatives). Then $w = t^k a^s t^{-1} a w''$ and $u = t^k u' t^{-1} a^{-1} w''$ with $s = 2, 3$ so $u' =_{BS} a^{s+1}$ so is a^3 or a^4, which by Lemma 18 is written as $t a^2 t^{-1}$ if it occurs in a normal form word. Then the run in u must start with either $3(-1)$ or $20(-1)$, neither of which is allowed in a normal form word, so w and u are identical. □

Lemma 33 (*Uniqueness for $\mathcal{NF}_N \cup \mathcal{NF}_{XN}$*). *If $w, u \in \mathcal{NF}_N \cup \mathcal{NF}_{XN}$ and $w =_{BS} u$ then w and u are identical.*

Proof. If w and u are two normal form words representing the same group element, then they have the same t-exponent by Lemma 30. If $w, u \in \mathcal{NF}_{XN}$ with t-exponent $(-k)$ then $t^k w, t^k u$ are in $\mathcal{NF}_X$ so by Lemma 32 they are identical. Note that $\mathcal{NF}_{XN}$ and $\mathcal{NF}_X$ words have the same N-run structure, the only difference is the length of the t^l prefix.

If $w \in \mathcal{NF}_N$ then let $w = a^{\varepsilon_k} t^{-1} \ldots t^{-1} a^{\varepsilon_0}$ and let $u = u' t^{-1} a^{\eta_{k-1}} t^{-1} \ldots t^{-1} a^{\eta_0}$ where u' evaluates to a^n and is type X or E. The words $t^k w$ and $t^k u$ evaluate to the same power of a, which is $\varepsilon_k 2^k + \cdots + \varepsilon_0 = n 2^k + \eta_{k-1} 2^{k-1} + \cdots + \eta_0$. Let $i \in \mathbb{N}$ such that $\varepsilon_j = \eta_j$ for all $j < i$ and $\varepsilon_i \neq \eta_i$. Then cancelling and dividing through by 2^i we get

$$\varepsilon_k 2^{k-i} + \cdots + \varepsilon_i = n 2^{k-i} + \eta_{k-1} 2^{k-i-1} + \cdots + \eta_i. \tag{2}$$

If $i = k$ then $\varepsilon_k = n$. Now $|\varepsilon_k| \leqslant 3$ and u' is an E or X word with the same a-exponent. By Lemma 31 if u' is type X then it evaluates to a^N with $|N| \geqslant 4$, so u' is type E, indeed it is exactly a^{ε_k}, so w and u are identical.

If $i < k$ then $\varepsilon_i, \eta_i = \pm 1$ since they are in the middle of a run, and have the same parity by Eq. (2). If $i < k+1$ then we have a contradiction since $\varepsilon_{i+1} = \eta_{i+1} = 0$ and the equation has $4x+1$ on one side and $4y-1$ on the other for integers x, y. So $i = k+1$ and $\varepsilon_k 2 + \varepsilon_{k-1} = n2 + \eta_{k-1}$ so $n = \varepsilon_k \pm 1$ since $\varepsilon_{k-1} - \eta_{k-1} = \pm 2$, and ε_{k-1} has the same sign as ε_k.

If u' is type X then $|n| \geqslant 4$ by Lemma 31 but $|\varepsilon_k| \leqslant 3$, so the only chance for equality is when the run in *w* starts with 31 and $\eta_{k-1} = -1$. Then $u' =_{BS} a^4$ which is written as ta^2t^{-1} in a normal form word, but then the run in *u* starts with $20(-1)$ which is not allowed. Thus *u* is also in $\mathcal{NF}_N$. Without loss of generality assume $\varepsilon_k > 0$ so $\varepsilon_{k-1} = 1$ and $\eta_{k-1} = -1$. Then *n* must be negative since the run in *u* starts with $n(-1)$, and we have a contradiction. □

Lemma 34 (*Uniqueness for* $\mathcal{NF}_P \cup \mathcal{NF}_{PX}$). *If* $w, u \in \mathcal{NF}_P \cup \mathcal{NF}_{PX}$ *and* $w =_{BS} u$ *then w and u are identical.*

Proof. If $w, u \in \mathcal{NF}_P \cup \mathcal{NF}_{PX}$ then w^{-1} and u^{-1} are in $\mathcal{NF}_N \cup \mathcal{NF}_{XN}$, so by Lemma 33 since $w^{-1} =_{BS} u^{-1}$ then w^{-1} and u^{-1} are identical, and so *w* and *u* are identical. □

Lemma 35 (*Uniqueness*). *Every group element is represented by a unique normal form word.*

Proof. If *w* and *u* are two normal form words representing the same group element, then they have the same *t*-exponent by Lemma 30.

If *w* and *u* have zero *t*-exponent then they are of the form E, X, NP or *XNP*. If neither is *NP* or *XNP* then they are identical by Lemma 32. If one is *NP* or *XNP* then let $w = w't^{-1}a^{\varepsilon_{k-1}}t^{-1} \dots t^{-1}a^{\varepsilon_0}t^k$ and $u = u't^{-1}a^{\eta_{l-1}}t^{-1} \dots t^{-1}a^{\eta_0}t^l$ where w', u' evaluate to powers of *a* and assume without loss of generality that $k > 0$ and $k \geqslant l$. Then $wu^{-1} = w't^{-1}a^{\varepsilon_{k-1}}t^{-1} \dots t^{-1}a^{\varepsilon_0}t^{k-l}a^{-\eta_0}t \dots ta^{-\eta_{l-1}}(u')^{-1} =_{BS} 1$. Since $k > 0$ then $\varepsilon_0 = \pm 1$ so if we replace w' and u' by the corresponding powers of *a* (by pinching ta^st^{-1} subwords) we have a word that does not admit any pinches, contradicting Britton's Lemma. Thus $k = l$. Then the words wt^{-k} and ut^{-k} are equal and in $\mathcal{NF}_N \cup \mathcal{NF}_{XN}$ so by Lemma 33 must be identical, so *w* and *u* are identical.

If *w* and *u* have negative *t*-exponent then they are of the form N, XN, NP or *XNP*. If neither is *NP* or *XNP* then they are identical by Lemma 33. If one is *NP* or *XNP* then let $w = w't^{-1}a^{\varepsilon_{k-1}}t^{-1} \dots t^{-1}a^{\varepsilon_0}t^l$ and let $u = u't^{-1}a^{\eta_{p-1}}t^{-1} \dots t^{-1}a^{\eta_0}t^q$ where $k > l$, $p > q$, and w', u' evaluate to powers of *a*. Assume without loss of generality that $l > 0$ and $l \geqslant q$. Then $wu^{-1} = w't^{-1}a^{\varepsilon_{k-1}}t^{-1} \dots t^{-1}a^{\varepsilon_0}t^{l-q}a^{-\eta_0}t \dots ta^{-\eta_{p-1}}(u')^{-1} =_{BS} 1$. Since $l > 0$ then $\varepsilon_0 = \pm 1$ so after replacing w' and u' by the corresponding powers of *a*, we have a word that does not admit any more pinches, contradicting Britton's Lemma. Thus $l = q$. Then the words wt^{-l} and ut^{-l} are equal and in $\mathcal{NF}_N \cup \mathcal{NF}_{XN}$ so by Lemma 33 must be identical, so *w* and *u* are identical.

If *w* and *u* have positive *t*-exponent then they are of the form P, PX, NP or *NPX*. If neither is *NP* or *NPX* then they are identical by Lemma 34. If one is *NP* or *NPX* then assume that *w* is, so let $w = t^{-k}a^{\varepsilon_0}t \dots a^{\varepsilon_{l-1}}tw'$ with $k > 0, k < l, \varepsilon_0 = \pm 1$ and w' evaluates to a power of *a*, and let $u = t^{-p}a^{\eta_0}t \dots a^{\eta_{q-1}}tu'$ where $p < q$ and u' evaluates to a power of

a. Assume without loss of generality that $k \geqslant p$ (if $k < p$ then reverse the roles of w and u). Then $u^{-1}w = (u')^{-1}t^{-1}a^{-\eta_{q-1}} \dots t^{-1}a^{-\eta_0}t^{p-k}a^{\varepsilon_0}t \dots ta^{\varepsilon_{l-1}}w' =_{BS} 1$. If $k \neq p$ then since $\varepsilon_0 = \pm 1$ then after replacing w' and u' by their corresponding powers of a we have a word that cannot be pinched, contradicting Britton's Lemma. Thus $k = p$. Then the words $t^k w$ and $t^k u$ are equal and in $\mathcal{NF}_P \cup \mathcal{NF}_{PX}$ so by Lemma 34 must be identical, so w and u are identical. $\square$

Lemma 36 (*Normal forms are geodesic*). *Each normal form word is a geodesic.*

Proof. Suppose that a word $w \in \mathcal{NF}$ is not geodesic. Choose a geodesic word $u =_{BS} w$ that is one of the ten types in Lemma 18. By Lemma 23 we can move u into a word u' of the same length having one run.

If u' is in normal form then since w and u' are both normal form words that equate to the same group element then w, u' must be identical by Lemma 35.

If u' is not in normal form, it either violates the prefix rules (as in Lemma 24) or has an adjacent pair of non-zero digits in its run.

If the run in u' has an occurrence of $1(-1)$ or $(-1)1$ then u' is not geodesic. If the run in u' has an occurrence of 11 or $(-1)(-1)$ that is not at the start of an N-run or the end of a P-run, then by Lemma 22 we can perform a length preserving rewrite to eliminate it. If this causes u' to have a $1(-1)$ then u was not geodesic, and it causes u' to have a 11 or $(-1)(-1)$ then repeatedly applying Lemma 22 from right to left in an N-run, or left to right in a P-run, we can eliminate all occurrences of pairs of non-zero digits.

Finally, if the start or end is not one of the prefixes in Lemma 24 then either u' is not geodesic (if the prefix is $20(-1)$ for example), or is equal to a normal form word of the same length, which means that the original word w is geodesic. $\square$

5. The main theorem

Theorem 37. *The language $\mathcal{NF}$ is a 1-counter language.*

Proof. The ten types of normal-form geodesics listed in Definition 26 break up into five cases. The set $\mathcal{NF}_E$ is a 1-counter language since it is finite. We can describe a $\mathbb{Z}$-automaton for each of the remaining four cases to accept the remaining nine types.

Consider the set of normal forms words of type X, XN and XNP. The language L_1 of Lemma 25 describes the set of normal form words of these types with at most two t^{-1} letters in the N-run, and since L_1 is finite, it is a regular language.

Let L'_1 be the set of words of the form $\{t^k a^i t^{-2} a t^{-1}, t^k a^j t^{-2} a^{-1} t^{-1} \mid k = 1, 2, 3, i = 2, \pm 3, j = -2, \pm 3\}$. This is a finite set so is regular, and is the set of X (and XN) normal form words with three t^{-1}'s in the N-run, that corresponds to the prefix $201, 301, 30(-1)$ and their negatives.

The remaining X, XN and XNP normal form words (with an N-run of 3 or more t^{-1} letters) are accepted by the automaton on the left of Fig. 12. The edge labeled κ stands for

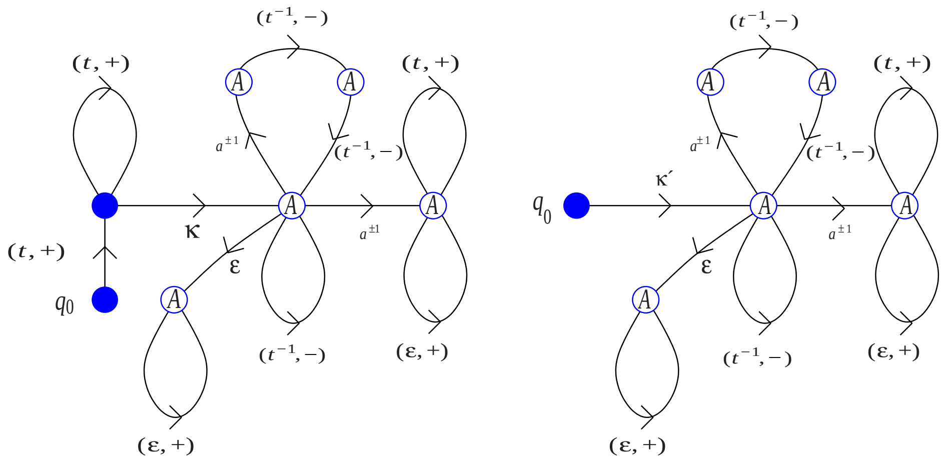

Fig. 12. Counter automata for normal form X, XN, XNP words and N, $NP_{\leqslant}$ words with N-run length at least 3.

a collection of paths labeled by

$$\begin{array}{ll}
a^i(t^{-1},-)(t^{-1},-)(t^{-1},-), & i=\pm 2,\pm 3;\\
a^i(t^{-1},-)(t^{-1},-)a(t^{-1},-)(t^{-1},-), & i=2,\pm 3;\\
a^i(t^{-1},-)(t^{-1},-)a^{-1}(t^{-1},-)(t^{-1},-), & i=-2,\pm 3;\\
a^i(t^{-1},-)a(t^{-1},-)(t^{-1},-), & i=2,3;\\
a^i(t^{-1},-)a^{-1}(t^{-1},-)(t^{-1},-), & i=-2,-3.
\end{array}$$

The union of these three (regular and 1-counter) languages is 1-counter.

Next, consider the set of normal forms words of type N and $NP_{\leqslant}$. The language L_2 of Lemma 25 describes the set of normal form words of these types with at most two t^{-1} letters in the N-run, and since L_2 is finite, it is a regular language.

Let L_2' be the set of words of the form $\{a^i t^{-2} a^{\pm 1} t^{-1} \mid i=0,\pm 1,\pm 2,\pm 3\}$. This is a finite set so is regular, and is the set of N (and $NP_{\leqslant}$) normal form words with three t^{-1}'s in the N-run, that corresponds to the prefix $00(\pm 1)$, $10(\pm 1)$, $20(\pm 1)$, $30(\pm 1)$ and their negatives.

The remaining N and $NP_{\leqslant}$ normal form words (with an N-run of 3 or more t^{-1} letters) are accepted by the automaton on the right of Fig. 12. The edge labeled κ' stands for a collection of paths labeled by

$$\begin{array}{ll}
a^i(t^{-1},-)(t^{-1},-)(t^{-1},-), & i=0,\pm 1,\pm 2,\pm 3;\\
a^i(t^{-1},-)(t^{-1},-)at^{-1}(t^{-1},-), & i=0,\pm 1,\pm 2,\pm 3;\\
a^i(t^{-1},-)(t^{-1},-)a^{-1}(t^{-1},-)(t^{-1},-), & i=0,\pm 1,\pm 2,\pm 3;\\
a^i(t^{-1},-)a(t^{-1},-)(t^{-1},-), & i=0,1,2,3;\\
a^i(t^{-1},-)a^{-1}(t^{-1},-)(t^{-1},-), & i=0,-1,-2,-3.
\end{array}$$

Next, consider the set of normal forms words of type P and $NP_{>}$. The language L_3 of Lemma 25 describes the set of normal form words of these types with at most two t letters in the P-run, and since L_3 is finite, it is a regular language.

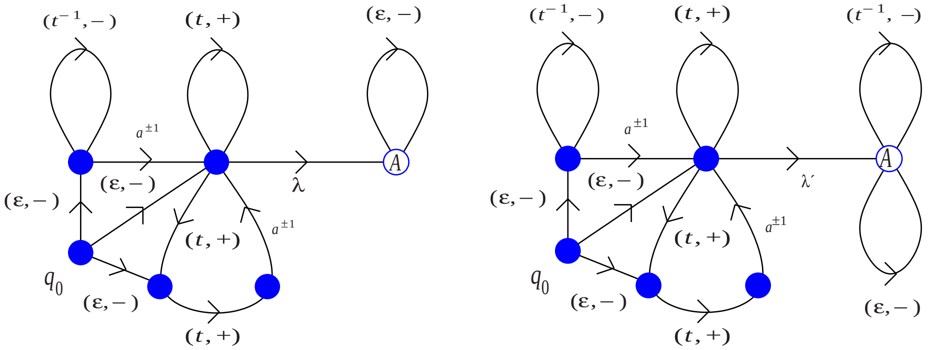

Fig. 13. Counter automata for normal form P, $NP_>$ words and PX, NPX words with P-run length at least 3.

Let L'_3 be the set of words of the form $\{ta^{\pm1}t^2a^i \mid i = 0, \pm1, \pm2, \pm3\}$. This is a finite set so is regular, and is the set of P (and $NP_>$) normal form words with three t's in the P-run, that corresponds to the suffix $(\pm1)00$, $(\pm1)01$, $(\pm1)02$, $(\pm1)03$ and their negatives.

The remaining P and $NP_>$ normal form words (with a P-run of 3 or more t letters) are accepted by the automaton on the left of Fig. 13. The edge labeled λ stands for a collection of paths labeled by

$$\begin{array}{ll}
(t,+)(t,+)(t,+)a^i, & i = 0, \pm1, \pm2, \pm3;\\
(t,+)(t,+)a(t,+)(t,+)a^i, & i = 0, \pm1, \pm2, \pm3;\\
(t,+)(t,+)a^{-1}(t,+)(t,+)a^i, & i = 0, \pm1, \pm2, \pm3;\\
(t,+)(t,+)a(t,+)a^i, & i = 0, 1, 2, 3;\\
(t,+)(t,+)a^{-1}(t,+)a^i, & i = 0, -1, -2, -3.
\end{array}$$

Lastly, consider the set of normal forms words of type *PX* and *NPX*. The language L_4 of Lemma 25 describes the set of normal form words of these types with (at most) two t letters in the P-run, and since L_4 is finite, it is a regular language.

Let L'_4 be the set of words of the form $\{tat^2a^it^{-k}, ta^{-1}t^2a^jt^{-k}, \mid k = 1, 2, 3, i = 2, \pm3, j = -2, \pm3\}$. This is a finite set so is regular, and is the set of *PX* (and *NPX*) normal form words with three t's in the P-run, that corresponds to the suffix 102, 103, $(-1)03$ and their negatives.

The remaining *PX* and *NPX* normal form words (with a P-run of 3 or more t letters) are accepted by the automaton on the right-hand side of Fig. 13. The edge labeled λ' stands for a collection of paths labeled by

$$\begin{array}{ll}
(t,+)(t,+)(t,+)a^i, & i = \pm2, \pm3;\\
(t,+)(t,+)a(t,+)(t,+)a^i, & i = 2, \pm3;\\
(t,+)(t,+)a^{-1}(t,+)(t,+)a^i, & i = -2, \pm3;\\
(t,+)(t,+)a(t,+)a^i, & i = 2, 3;\\
(t,+)(t,+)a^{-1}(t,+)a^i, & i = -2, -3.
\end{array}$$

By Lemma 9 the union of a 1-counter and a regular language is 1-counter so each of the ten types is 1-counter, and by Lemma 10 the union of 1-counter languages is 1-counter. $\square$

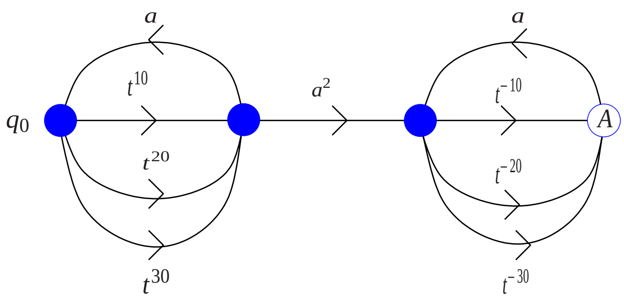

Fig. 14. A finite state automaton accepting the language L in the proof of Theorem 39.

Corollary 38. *The language of normal forms for* $\mathrm{BS}(1,2)$ *with the standard generating set is context-free.*

6. Full language of geodesics

In this section we prove that the language of all geodesic words in the standard generating set is not counter. To prove this we will mimic the proof of Theorem 13. Recall that in that proof we constructed a word ww^R on three symbols whose prefix is square-free and suffix is its reverse, and applied the Swapping Lemma (Lemma 12) to obtain a contradiction.

Let w be a word in $\mathrm{BS}(1,2)$ with no a^{-1} letters. Define the *t-encoding* of w to be a string of integers $n_1n_2\ldots n_k$ such that $w = t^{n_1}at^{n_2}\ldots at^{n_k}$. If w starts (or, respectively, ends) with an a then $n_1 = 0$ (or, respectively, $n_k = 0$).

As an example, the word

$$at^2a^2ta^3t^4at^{-9}at^2at^{-1} = t^0at^2at^0atat^0at^0at^4at^{-9}at^2at^{-1}$$

is encoded as $0201004(-9)2(-1)$. Note that previously our encodings have been of a-exponents, but this new encoding will be useful for the argument to follow.

Theorem 39. *The language of all geodesic words in* $\mathrm{BS}(1,2)$ *with respect to the generating set* $\{a^{\pm 1}, t^{\pm 1}\}$ *is not counter.*

Proof. Suppose that the full language is counter, and call it C. Define L to be the set of words in $\{a, t^{\pm 1}\}$ accepted by the finite state automaton in Fig. 14. That is, L is the set of *PN* words whose t-encodings are words of the form $\{10, 20, 30\}\{10, 20, 30\}^*0\{-10, -20, -30\}\{-10, -20, -30\}^*$.

Since L is regular, the intersection of C and L is counter. Let M be a counter automaton accepting $C \cap L$, with alphabet $a^{\pm 1}, t^{\pm 1}$. We can construct a new counter automaton M' which accepts the set of t-encoded words of $C \cap L$ as follows:

The states, start state, accept states and counters are the same as for M. The new alphabet is $\{0, \pm 10, \pm 20, \pm 30\}$. The transitions are defined as follows:

If there is a path labelled by $t^i a$ in M from p to q, then add an edge in M' from p to q labeled by i, and the counters are changed by the same amount as they were following the path $t^i a$ in M. Thus a word is accepted by M if and only if its encoding is accepted by M'. Since M accepts $C \cap L$, the only subwords of the form $t^i a$ that appear in accepted words are for $i = 0, \pm 10, \pm 20$ or ± 30. Let p be the swapping length for M'.

Next, take a Thue–Morse word in three symbols, which we choose to be 10, 20, 30, of length greater than $2p$. This word encodes a P word u of some t-exponent $10c$. We wish to find some kind of "reverse" of u, as we did in the proof of Theorem 13. We find a word v to act as the "reverse" by the following procedure:

(1) Write u as $t^{10}a^{\varepsilon_1}t^{10}a^{\varepsilon_2}\dots t^{10}a^{\varepsilon_k}t^{10}$ where $\varepsilon_i = 0, 1$.
(2) Reverse this word.
(3) Replace a^0 with a^1 and a^1 with a^0 in this word.
(4) Replace t^{10} with t^{-10} in this word to get v.

For example, the Thue–Morse word 10, 20, 30, 10, 30, 20, 10, 20, 30, 20, 10, 30 encodes the word

$$u = t^{10}at^{20}at^{30}at^{10}at^{30}at^{20}at^{10}at^{20}at^{30}at^{20}at^{10}at^{30}.$$

Step 1: Write u as

$$u = |a^1|a^0|a^1|a^0|a^0|a^1|a^1|a^0|a^0|a^1|a^0|a^1|a^1|a^0|a^1|a^0|a^0|a^1|a^0|a^1|a^1|a^0|a^0|,$$

where the t^{10} terms are replaced by bars |, to make it easier to read.

Step 2: Reversing this word gives

$$u^R = |a^0|a^0|a^1|a^1|a^0|a^1|a^0|a^0|a^1|a^0|a^1|a^1|a^0|a^1|a^0|a^0|a^1|a^1|a^0|a^0|a^1|a^0|a^1|.$$

Step 3: Replacing a^0 by a^1 and vice versa gives

$$|a^1|a^1|a^0|a^0|a^1|a^0|a^1|a^1|a^0|a^1|a^0|a^0|a^1|a^0|a^1|a^1|a^0|a^0|a^1|a^1|a^0|a^1|a^0|.$$

Step 4: Replacing t^{10} by t^{-10} gives

$$\begin{aligned} v = &\dagger a^1 \dagger a^1 \dagger a^0 \dagger a^0 \dagger a^1 \dagger a^0 \dagger a^1 \dagger a^1 \dagger a^0 \dagger a^1 \dagger a^0 \dagger a^0 \\ &\dagger a^1 \dagger a^0 \dagger a^1 \dagger a^1 \dagger a^0 \dagger a^0 \dagger a^1 \dagger a^1 \dagger a^0 \dagger a^1 \dagger a^0 \dagger \\ = &\, t^{-10}at^{-10}at^{-30}at^{-20}at^{-10}at^{-20}at^{-30}at^{-20}at^{-10}at^{-30}at^{-10}at^{-20}at^{-20}, \end{aligned}$$

where † represents t^{-10}.

The t-encoding for v is then

$$(-10)(-10)(-30)(-20)(-10)(-20)(-30)(-20)(-10)(-30)(-10)(-20)(-20).$$

Note that v does not have to be square-free. Note also that the t-exponent of v is $-10c$, where $10c$ is the t-exponent of u.

Now to understand what motivated us to produce this v from u, consider the word $w = ua^2v = uat^0av$. This word is type X. Drawing w in a sheet of the Cayley graph we see that at every tenth level there is an a letter, either on the part going up the sheet (the u part) or the part going down (the v part). See the left-hand side of Fig. 15.

We will now show that w is a geodesic. Consider the word w' obtained from w by commuting all a letters to the right. Since there is exactly one a at every tenth level of w,

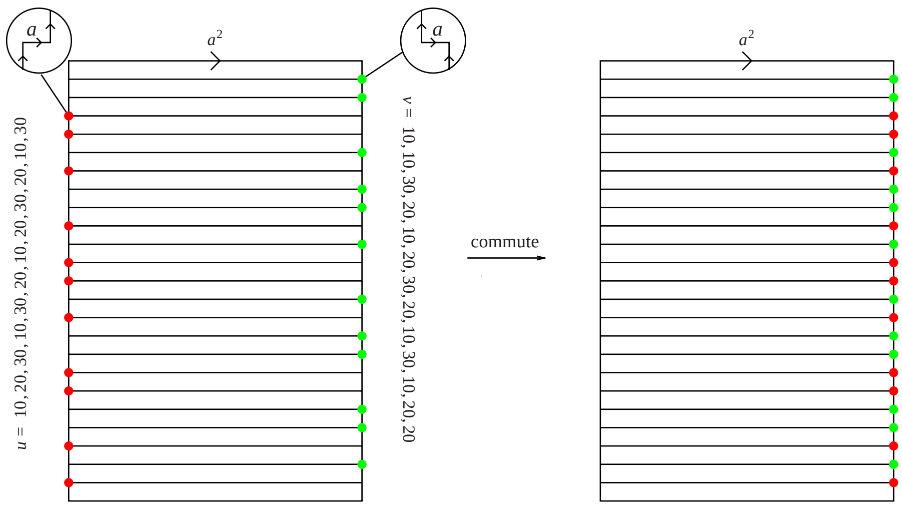

Fig. 15. The word $w = ua^2v$ drawn in a sheet of the Cayley graph.

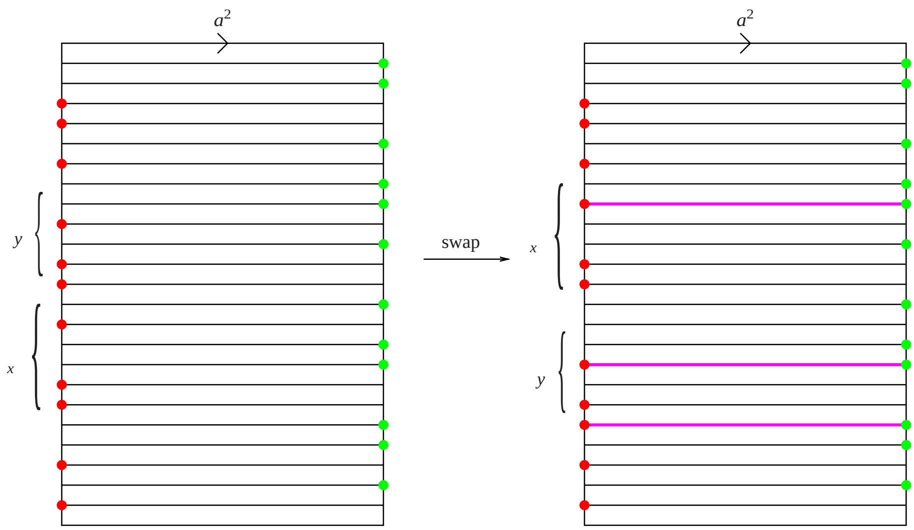

Fig. 16. Swapping two subwords in the P part of w leads to a word with $t^{-1}a^2t^{-1}$.

we have $w' = t^{10c}a^2t^{-10}(at^{-10})^{c-1}$. Then w' is a normal form X word, since its N-run is of the form 200 . . . with no consecutive non-zero entries. Thus by Lemma 36 is geodesic, and since w' has the same length as w then w is geodesic. So w is in $C \cap L$, it is accepted by the counter automaton M, and its t-encoding is accepted by M'.

Applying the Swapping Lemma (Lemma 12) to the encoding of w, we switch two adjacent subwords in the first half of w, that is, in the t-encoding of u, which is square-free.

This new string is a t-encoding of some other word in the group, which is an X word, essentially the same as w except that at some level(s) there is an a step on the left- and right-hand sides of the sheet in the Cayley graph, indicated by the thin lines joining the dots in Fig. 16.

This causes a problem, for when we commute a-letters to the right in this word, we will see $t^{-1}a^2t^{-1}$ at some point(s) in the N-run, and thus the swapped word is not a geodesic, so not in $C \cap L$, and this is a contradiction. □

Acknowledgements

My sincere thanks to Bob Gilman, Ray Cho, Walter Neumann, Jon McCammond, Susan Hermiller, Sarah Rees, Rick Thomas, Nik Ruskuc, Kim Ruane, Mauricio Gutierrez, Sean Cleary, Jennifer Taback and Gretchen Ostheimer for their help and suggestions that have all contributed to this work. I wish to thank the reviewer of this paper for pointing out that the normal form language described here is a 1-counter language, as well as many other very useful suggestions and corrections. The labels for the figures were produced using Andrew Rechnitzer's `equation_edit` program.

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