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Uniqueness of analytic solutions for stationary plate oscillations in an annulus

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The equations governing the harmonic oscillations of a plate with transverse shear deformation are considered in an annular domain. It is shown that under nonstandard boundary conditions where both the displacements and tractions are zero on the internal boundary curve, the corresponding analytic solution is zero in the entire domain. This property is then used to prove that a boundary value problem with Dirichlet or Neumann conditions on the external boundary and Robin conditions on the internal boundary has at most one analytic solution.

1. Introduction

The interior and exterior Dirichlet and Neumann problems for high frequency harmonic oscillations in thin elastic plates were considered in [1] by means of a classical indirect boundary integral equation formulation. It was found that the exterior Dirichlet (Neumann) problem does not have unique solutions at values of the oscillation frequency for which the homogeneous interior Neumann (Dirichlet) problem admits nonzero solutions, although the exterior problems themselves have at most one solution under certain far-field conditions. In [2] a direct method was used to derive a uniquely solvable pair of integral equations for each exterior problem. The case of Robin boundary conditions was investigated in [3,4], with similar conclusions. Below we lay the foundations of a theory of modified integral equations in which solutions of the exterior problems may be constructed from a single uniquely solvable integral equation.

In what follows, a superscript T denotes matrix transposition and $x = (x_1, x_2)^{\mathrm{T}}$. Let h_0 be the thickness of the plate, λ and μ the elastic constants of the material, ρ the density, and ω the oscillation frequency. The stationary oscillations of the plate when transverse shear effects are taken into account are governed by the system [5]

$$A^{\omega}(\partial_x)u(x) = H(x), \tag{1}$$

where $u = (u_1, u_2, u_3)^{\mathrm{T}}$ is a vector characterizing the displacements, H is related to the averaged body forces and moments, and the matrix operator $A^{\omega}(\partial_x) = A^{\omega}(\partial/\partial x_1, \partial/\partial x_2)$ is defined by

$$A^{\omega}(\xi_1, \xi_2) = \begin{pmatrix} h^2\mu(\Delta + k_3^2) + h^2(\lambda+\mu)\xi_1^2 & h^2(\lambda+\mu)\xi_1\xi_2 & -\mu\xi_1 \\ h^2(\lambda+\mu)\xi_1\xi_2 & h^2\mu(\Delta + k_3^2) + h^2(\lambda+\mu)\xi_2^2 & -\mu\xi_2 \\ \mu\xi_1 & \mu\xi_2 & \mu(\Delta + k^2) \end{pmatrix};$$

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here,

$$h^2 = h_0^2/12, \qquad \Delta = \xi_1^2 + \xi_2^2, \qquad k^2 = \rho\omega^2/\mu, \qquad k_3^2 = k^2 - 1/h^2.$$

Without loss of generality we consider the corresponding homogeneous system

$$A^\omega(\partial_x)u(x) = 0, \tag{2}$$

since a particular solution of (1) may be constructed in terms of a Newtonian potential [6].

The boundary moment–stress operator $T(\partial_x) = T(\partial/\partial x_1, \partial/\partial x_2)$ is defined by [7]

$$T(\xi_1, \xi_2) = \begin{pmatrix} h^2((\lambda+2\mu)\nu_1\xi_1 + \mu\nu_2\xi_2) & h^2(\mu\nu_2\xi_1 + \lambda\nu_1\xi_2) & 0 \\ h^2(\lambda\nu_2\xi_1 + \mu\nu_1\xi_2) & h^2(\mu\nu_1\xi_1 + (\lambda+2\mu)\nu_2\xi_2) & 0 \\ \mu\nu_1 & \mu\nu_2 & \mu(\nu_1\xi_1 + \nu_2\xi_2) \end{pmatrix}, \tag{3}$$

where $\nu = (\nu_1, \nu_2)^{\mathrm{T}}$ is the unit outward normal to some smooth boundary.

We assume that [7]

$$\lambda + \mu > 0, \quad \mu > 0. \tag{4}$$

2. Analytic solutions

Let $\mathcal{D}$ be an annular region bounded externally and internally by simple closed C^2-curves ∂S_1 and ∂S_2, respectively. In classical boundary value problems it is shown that if, say, $u = 0$ or $Tu = 0$ on $\partial S_1 \cup \partial S_2$, or if one of these conditions is satisfied on ∂S_1 and the other on ∂S_2, then $u = 0$ in $\mathcal{D}$. The next assertion shows that the same result can be obtained from a nonstandard set of conditions that involves only the internal boundary curve.

Lemma. If u is an analytic solution of (2) in $\mathcal{D} \cup \partial S_2$ and

$$u = Tu = 0 \quad \text{on } \partial S_2,$$

then $u = 0$ in $\mathcal{D}$.

Proof. Since u is analytic in $\mathcal{D}$, it can be represented as a power series with a nonzero radius of convergence, whose coefficients are expressed in terms of the derivatives of the components of u. We claim that all these derivatives, of any order, vanish on ∂S_2.

We verify our claim for the first-order derivatives. By (3) and our assumption, on ∂S_2 we have

$$(\lambda+2\mu)\nu_1\frac{\partial u_1}{\partial x_1} + \mu\nu_2\frac{\partial u_1}{\partial x_2} + \mu\nu_2\frac{\partial u_2}{\partial x_1} + \lambda\nu_1\frac{\partial u_2}{\partial x_2} = 0, \tag{5}$$

$$\lambda\nu_2\frac{\partial u_1}{\partial x_1} + \mu\nu_1\frac{\partial u_1}{\partial x_2} + \mu\nu_1\frac{\partial u_2}{\partial x_1} + (\lambda+2\mu)\nu_2\frac{\partial u_2}{\partial x_2} = 0, \tag{6}$$

$$\nu_1 u_1 + \nu_2 u_2 + \nu_1\frac{\partial u_3}{\partial x_1} + \nu_2\frac{\partial u_3}{\partial x_2} = 0, \tag{7}$$

where $\nu = (\nu_1, \nu_2)^{\mathrm{T}}$ is the unit normal to ∂S_2 pointing into $\mathcal{D}$. Let $s = (s_1, s_2)^{\mathrm{T}} = (-\nu_2, \nu_1)^{\mathrm{T}}$ be the unit tangent to ∂S_2. Since $u = 0$ on ∂S_2, it follows that on this boundary curve

$$\frac{\partial u_1}{\partial s} = -\nu_2\frac{\partial u_1}{\partial x_1} + \nu_1\frac{\partial u_1}{\partial x_2} = 0, \tag{8}$$

$$\frac{\partial u_2}{\partial s} = -\nu_2\frac{\partial u_2}{\partial x_1} + \nu_1\frac{\partial u_2}{\partial x_2} = 0, \tag{9}$$

$$\frac{\partial u_3}{\partial s} = -\nu_2\frac{\partial u_3}{\partial x_1} + \nu_1\frac{\partial u_3}{\partial x_2} = 0, \tag{10}$$

so, by (5), (6), (8) and (9),

$$L^{(1)}\alpha^{(1)} = 0 \quad \text{on } \partial S_2, \tag{11}$$

where

$$L^{(1)} = \begin{pmatrix} (\lambda+2\mu)\nu_1 & \mu\nu_2 & \mu\nu_2 & \lambda\nu_1 \\ \lambda\nu_2 & \mu\nu_1 & \mu\nu_1 & (\lambda+2\mu)\nu_2 \\ -\nu_2 & \nu_1 & 0 & 0 \\ 0 & 0 & -\nu_2 & \nu_1 \end{pmatrix}, \qquad \alpha^{(1)} = \left(\frac{\partial u_1}{\partial x_1}, \frac{\partial u_1}{\partial x_2}, \frac{\partial u_2}{\partial x_1}, \frac{\partial u_2}{\partial x_2}\right)^{\mathrm{T}}.$$

Now

$$\begin{aligned}\det L^{(1)} &= (\lambda+2\mu)\nu_1[-\mu\nu_1\nu_1^2+(\lambda+2\mu)\nu_2\nu_1(-\nu_2)]-\mu\nu_2[-\mu\nu_1(-\nu_2)\nu_1+(\lambda+2\mu)\nu_2(-\nu_2)^2]\\ &\quad+\mu\nu_2[\lambda\nu_2\nu_1^2-\mu\nu_1(-\nu_2)\nu_1]-\lambda\nu_1[\lambda\nu_2\nu_1(-\nu_2)-\mu\nu_1(-\nu_2)^2]=-\mu(\lambda+2\mu),\end{aligned}$$

which, in view of (4), is nonzero. Hence, from (11) we see that $\alpha^{(1)}=0$ on ∂S_2; that is,

$$\frac{\partial u_1}{\partial x_1}=\frac{\partial u_1}{\partial x_2}=\frac{\partial u_2}{\partial x_1}=\frac{\partial u_2}{\partial x_2}=0 \quad \text{on } \partial S_2.$$

Similarly, by (7) and (10),

$$L_3^{(1)}\alpha_3^{(1)}=0 \quad \text{on } \partial S_2, \tag{12}$$

where

$$L_3^{(1)}=\begin{pmatrix}\nu_1 & \nu_2\\ -\nu_2 & \nu_1\end{pmatrix}, \qquad \alpha_3^{(1)}=\left(\frac{\partial u_3}{\partial x_1},\frac{\partial u_3}{\partial x_2}\right)^{\mathrm{T}}.$$

Since $\det L_3^{(1)}=\nu_1^2+\nu_2^2=1$, from (12) we deduce that $\alpha_3^{(1)}=0$ on ∂S_2; consequently,

$$\frac{\partial u_3}{\partial x_1}=\frac{\partial u_3}{\partial x_2}=0 \quad \text{on } \partial S_2.$$

Suppose now that the derivatives of all orders up to and including $n=2l-1$ of $u_1,\ u_2$, and u_3 are zero on ∂S_2, and let $n=2l$. Applying Δ^{l-1} to the first two equations in (2) and using the analyticity of u, we find that on ∂S_2,

$$\begin{aligned}&\mu\left[\binom{l}{0}\frac{\partial^{2l}u_1}{\partial x_1^{2l}}+\binom{l}{1}\frac{\partial^{2l}u_1}{\partial x_1^{2l-2}\partial x_2^2}+\cdots+\binom{l}{l-1}\frac{\partial^{2l}u_1}{\partial x_1^2\partial x_2^{2l-2}}+\binom{l}{l}\frac{\partial^{2l}u_1}{\partial x_2^{2l}}\right]\\ &\quad+(\lambda+\mu)\left[\binom{l-1}{0}\frac{\partial^{2l}u_1}{\partial x_1^{2l}}+\binom{l-1}{1}\frac{\partial^{2l}u_1}{\partial x_1^{2l-2}\partial x_2^2}+\cdots+\binom{l-1}{l-1}\frac{\partial^{2l}u_1}{\partial x_1^2\partial x_2^{2l-2}}\right]\\ &\quad+(\lambda+\mu)\left[\binom{l-1}{0}\frac{\partial^{2l}u_2}{\partial x_1^{2l-1}\partial x_2}+\binom{l-1}{1}\frac{\partial^{2l}u_2}{\partial x_1^{2l-3}\partial x_2^3}+\cdots+\binom{l-1}{l-1}\frac{\partial^{2l}u_2}{\partial x_1\partial x_2^{2l-1}}\right]=0,\\ &(\lambda+\mu)\left[\binom{l-1}{0}\frac{\partial^{2l}u_1}{\partial x_1^{2l-1}\partial x_2}+\binom{l-1}{1}\frac{\partial^{2l}u_1}{\partial x_1^{2l-3}\partial x_2^3}+\cdots+\binom{l-1}{l-1}\frac{\partial^{2l}u_1}{\partial x_1\partial x_2^{2l-1}}\right]\\ &\quad+\mu\left[\binom{l}{0}\frac{\partial^{2l}u_2}{\partial x_1^{2l}}+\binom{l}{1}\frac{\partial^{2l}u_2}{\partial x_1^{2l-2}\partial x_2^2}+\cdots+\binom{l}{l-1}\frac{\partial^{2l}u_2}{\partial x_1^2\partial x_2^{2l-2}}+\binom{l}{l}\frac{\partial^{2l}u_2}{\partial x_2^{2l}}\right]\\ &\quad+(\lambda+\mu)\left[\binom{l-1}{0}\frac{\partial^{2l}u_2}{\partial x_1^{2l-2}\partial x_2^2}+\binom{l-1}{1}\frac{\partial^{2l}u_2}{\partial x_1^{2l-4}\partial x_2^4}+\cdots+\binom{l-1}{l-1}\frac{\partial^{2l}u_2}{\partial x_2^{2l}}\right]=0.\end{aligned}$$

By the inductive assumption, on ∂S_2 we also have

$$\begin{aligned}&\frac{\partial}{\partial s}\left(\frac{\partial^{2l-1}u_1}{\partial x_1^{2l-1}}\right)=-\nu_2\frac{\partial^{2l}u_1}{\partial x_1^{2l}}+\nu_1\frac{\partial^{2l}u_1}{\partial x_1^{2l-1}\partial x_2}=0,\ \ldots,\ \frac{\partial}{\partial s}\left(\frac{\partial^{2l-1}u_1}{\partial x_2^{2l-1}}\right)=-\nu_2\frac{\partial^{2l}u_1}{\partial x_1\partial x_2^{2l-1}}+\nu_1\frac{\partial^{2l}u_1}{\partial x_2^{2l}}=0,\\ &\frac{\partial}{\partial s}\left(\frac{\partial^{2l-1}u_2}{\partial x_1^{2l-1}}\right)=-\nu_2\frac{\partial^{2l}u_2}{\partial x_1^{2l}}+\nu_1\frac{\partial^{2l}u_2}{\partial x_1^{2l-1}\partial x_2}=0,\ \ldots,\ \frac{\partial}{\partial s}\left(\frac{\partial^{2l-1}u_2}{\partial x_2^{2l-1}}\right)=-\nu_2\frac{\partial^{2l}u_2}{\partial x_1\partial x_2^{2l-1}}+\nu_1\frac{\partial^{2l}u_2}{\partial x_2^{2l}}=0.\end{aligned}$$

The above equalities yield a linear algebraic system of $4l+2$ equations in $4l+2$ unknowns, which can be written in the form

$$L^{(2l)}\alpha^{(2l)}=0 \quad \text{on } \partial S_2, \tag{13}$$

where

$$L^{(2l)} = \begin{pmatrix} b_0{}^{(l)} & 0 & b_1{}^{(l)} & \cdots & 0 & b_l{}^{(l)} & 0 & c_0{}^{(l)} & \cdots & 0 & c_{l-1}{}^{(l)} & 0 \\ 0 & c_0{}^{(l)} & 0 & \cdots & c_{l-1}{}^{(l)} & 0 & d_0{}^{(l)} & 0 & \cdots & d_{l-1}{}^{(l)} & 0 & d_l{}^{(l)} \\ -\nu_2 & \nu_1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\nu_2 & \nu_1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & \nu_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -\nu_2 & \nu_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -\nu_2 & \nu_1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -\nu_2 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -\nu_2 & \nu_1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & -\nu_2 & \nu_1 \end{pmatrix},$$

$$\alpha^{(2l)} = \left(\frac{\partial^{2l} u_1}{\partial x_1^{2l}}, \frac{\partial^{2l} u_1}{\partial x_1^{2l-1}\partial x_2}, \dots, \frac{\partial^{2l} u_1}{\partial x_2^{2l}}, \frac{\partial^{2l} u_2}{\partial x_1^{2l}}, \frac{\partial^{2l} u_2}{\partial x_1^{2l-1}\partial x_2}, \dots, \frac{\partial^{2l} u_2}{\partial x_2^{2l}} \right)^{\mathrm{T}},$$

$$b_n{}^{(l)} = \begin{cases} \mu \binom{l}{n} + (\lambda+\mu)\binom{l-1}{n}, & 0 \le n \le l-1, \\ \mu \binom{l}{l}, & n = l, \end{cases} \qquad d_n{}^{(l)} = \begin{cases} \mu \binom{l}{0}, & n = 0, \\ \mu \binom{l}{n} + (\lambda+\mu)\binom{l-1}{n-1}, & 1 \le n \le l, \end{cases}$$

$$c_n{}^{(l)} = (\lambda+\mu)\binom{l-1}{n}, \quad 0 \le n \le l-1.$$

After a lengthy calculation (see [8] for full details), we find that the binomial coefficients satisfy the equalities

$$\sum_{s=0}^{r} \binom{l-1}{l-1-s}\binom{l}{l-(r-s)} + \sum_{s=0}^{r-1} \binom{l}{l-s}\binom{l-1}{l-(r-s)} = \sum_{s=0}^{r} \binom{l}{l-s}\binom{l}{l-(r-s)},$$

$$\sum_{s=0}^{l-1} \left[\binom{l-1}{s}\binom{l}{l-s} + \binom{l}{s}\binom{l-1}{l-1-s} \right] = \sum_{s=0}^{l} \binom{l}{s}\binom{l}{l-s},$$

which, in turn, help us show that

$$\det L^{(2l)} = \mu(\lambda + 2\mu).$$

As already remarked, this is nonzero, so from (13) it follows that

$$\frac{\partial^{2l} u_1}{\partial x_1^{2l}} = \frac{\partial^{2l} u_1}{\partial x_1^{2l-1}\partial x_2} = \dots = \frac{\partial^{2l} u_2}{\partial x_1 \partial x_2^{2l-1}} = \frac{\partial^{2l} u_2}{\partial x_2^{2l}} = 0 \quad \text{on } \partial S_2.$$

In the case of u_3, for $n = 2l$ we similarly find that

$$\Delta^l u_3 = 0 \quad \text{on } \partial S_2,$$

which means that

$$\binom{l}{0} \frac{\partial^{2l} u_3}{\partial x_1^{2l}} + \binom{l}{1} \frac{\partial^{2l} u_3}{\partial x_1^{2l-2}\partial x_2^2} + \dots + \binom{l}{l-1} \frac{\partial^{2l} u_3}{\partial x_1^2 \partial x_2^{2l-2}} + \binom{l}{l} \frac{\partial^{2l} u_3}{\partial x_2^{2l}} = 0 \quad \text{on } \partial S_2.$$

As above, on ∂S_2 we also have

$$\frac{\partial}{\partial s}\left(\frac{\partial^{2l-1} u_3}{\partial x_1^{2l-1}} \right) = -\nu_2 \frac{\partial^{2l} u_3}{\partial x_1^{2l}} + \nu_1 \frac{\partial^{2l} u_3}{\partial x_1^{2l-1}\partial x_2} = 0, \ \dots, \ \frac{\partial}{\partial s}\left(\frac{\partial^{2l-1} u_3}{\partial x_2^{2l-1}} \right) = -\nu_2 \frac{\partial^{2l} u_3}{\partial x_1 \partial x_2^{2l-1}} + \nu_1 \frac{\partial^{2l} u_3}{\partial x_2^{2l}} = 0.$$

This leads to a linear algebraic system of $2l + 1$ equations in $2l + 1$ unknowns, of the form

$$L_3^{(2l)} \alpha_3^{(2l)} = 0 \quad \text{on } \partial S_2, \qquad (14)$$

where

$$L_3^{(2l)} = \begin{pmatrix} \binom{l}{0} & 0 & \binom{l}{1} & 0 & \cdots & 0 & \binom{l}{l-1} & 0 & \binom{l}{l} \\ -\nu_2 & \nu_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -\nu_2 & \nu_1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -\nu_2 & \nu_1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \ddots & \nu_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\nu_2 & \nu_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\nu_2 & \nu_1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\nu_2 & \nu_1 \end{pmatrix},$$

$$\alpha_3^{(2l)} = \left(\frac{\partial^{2l} u_3}{\partial x_1^{2l}}, \frac{\partial^{2l} u_3}{\partial x_1^{2l-1} \partial x_2}, \ldots, \frac{\partial^{2l} u_3}{\partial x_1 \partial x_2^{2l-1}}, \frac{\partial^{2l} u_3}{\partial x_2^{2l}} \right)^{\mathrm{T}}.$$

It is easily seen that

$$\det L_3^{(2l)} = \binom{l}{0} \nu_1 \nu_1^{2l-1} + \binom{l}{1} (-\nu_2)(-\nu_2)\nu_1^{2l-2} + \cdots + \binom{l}{l-1} (-\nu_2)(-\nu_2)^{2l-3}\nu_1^2 + \binom{l}{l} (-\nu_2)(-\nu_2)^{2l-1} = 1.$$

Hence, (14) implies that

$$\frac{\partial^{2l} u_3}{\partial x_1^{2l}} = \frac{\partial^{2l} u_3}{\partial x_1^{2l-1} \partial x_2} = \cdots = \frac{\partial^{2l} u_3}{\partial x_1 \partial x_2^{2l-1}} = \frac{\partial^{2l} u_3}{\partial x_2^{2l}} = 0 \quad \text{on } \partial S_2.$$

The derivatives of order $n = 2l + 1$ of the three components of u are shown to be zero on ∂S_2 by the same procedure, with the obvious modifications. Mathematical induction now implies that the derivatives of any order of these functions vanish on ∂S_2. Using power series expansions, we deduce that u is zero in the neighborhood of any point of this part of the boundary and, since u is an analytic solution of (2) in $\mathcal{D}$, we apply the argument of continuity to conclude that $u = 0$ in $\mathcal{D}$. □

3. Uniqueness theorem

We use the above lemma to derive a result that is instrumental in eliminating nonzero solutions of problems with a certain type of homogeneous boundary conditions.

Let K be a (3×3)-matrix whose elements are such that

$$K_{ij} = \bar{K}_{ji} \quad \text{for } i \neq j \tag{15}$$

and either

$$\mathrm{Im}(K_{11}),\ \mathrm{Im}(K_{22}),\ \mathrm{Im}(K_{33}) > 0 \tag{16}$$

or

$$\mathrm{Im}(K_{11}),\ \mathrm{Im}(K_{22}),\ \mathrm{Im}(K_{33}) < 0. \tag{17}$$

Theorem. If u is an analytic solution of (2) in $\mathcal{D} \cup \partial S_2$ such that

$$u = 0 \quad \text{or} \quad Tu = 0 \quad \text{on } \partial S_1 \tag{18}$$

and

$$Tu + Ku = 0 \quad \text{on } \partial S_2, \tag{19}$$

where K satisfies (15) and either (16) or (17), then $u = 0$ in $\mathcal{D}$.

Proof. Applying the reciprocity relation (see [9]) to u and $\bar{u}$ in $\mathcal{D}$ and taking (18), (19) and (15) into account, we arrive at

$$\begin{aligned} 0 &= \int_{\partial S_1} \{u^{\mathrm{T}} T\bar{u} - \bar{u}^{\mathrm{T}} Tu\}\, ds - \int_{\partial S_2} \{u^{\mathrm{T}} T\bar{u} - \bar{u}^{\mathrm{T}} Tu\}\, ds = \int_{\partial S_2} \{u^{\mathrm{T}} \bar{K}\bar{u} - \bar{u}^{\mathrm{T}} Ku\}\, ds \\ &= \int_{\partial S_2} \{u_i \bar{K}_{ij} \bar{u}_j - \bar{u}_j K_{ji} u_i\}\, ds = -2i \int_{\partial S_2} \{\mathrm{Im}(K_{11})|u_1|^2 + \mathrm{Im}(K_{22})|u_2|^2 + \mathrm{Im}(K_{33})|u_3|^2\}\, ds. \end{aligned}$$

From this and (16) or (17) it follows that $u = 0$ on ∂S_2, and (19) yields $Tu = 0$ on ∂S_2. Therefore, by the Lemma, $u = 0$ in $\mathcal{D}$, which proves the assertion. □

Remark. This result is essential in establishing the unique solvability, via the Fredholm Alternative, of modified integral equations arising in the exterior problems for high frequency harmonic oscillations. Such a proof requires to show that a function satisfying a homogeneous dissipative (Robin-type) condition on a curve interior to a scatterer is zero in the interior region of the scatterer bounded by that curve. Work is now in progress to construct analytic solutions of (2) that satisfy condition (19) on some suitable curve.

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