Symmetries and Ramsey properties of trees

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Abstract

In this paper we show the extent to which a finite tree of fixed height is a Ramsey object in the class of trees of the same height can be measured by its symmetry group. © 1999 Elsevier Science B.V. All rights reserved

1. Introduction

In the sequel, all the combinatorial structures referred to will be finite. For a natural number r, we write [r] for the set \{1, \ldots, r\}. In this paper, a tree will be a poset T with a minimum element (the root) such that, for each \(x \in T\), the set \(\{y \in T: y < x\}\) is a linear order. The cardinality of this set is called the height of x and denoted by \(ht(x)\). The height of a tree is the maximum height of any element of T. For \(h \geq 0\), we denote the class of trees of height h by \(\mathcal{F}_h\). If \(T_1, T_2 \in \mathcal{F}_h\), an embedding \(\mu\) of \(T_1\) into \(T_2\) is a poset embedding (i.e., \(\mu\) is injective and \(x < y\) iff \(\mu(x) < \mu(y)\) for all \(x, y \in T_1\)) which is, in addition, height-preserving, i.e., \(ht(\mu(x)) = ht(x)\) for all \(x \in T_1\). For a tree \(T\), the number of successors of an element \(x\) is called the arity of \(x\) and denoted by \(ar(x)\). We denote the set of copies \(T_1\) in \(T_2\) by \([T_1, T_2]\). If \(T\) is of height \(h\), a subtree of \(T\) is an image under an embedding of some tree \(S\) of height \(h\).

For \(T \in \mathcal{F}_h\), let \(A(T)\) be the group of height-preserving automorphisms of \(T\). Set

\[
r(T) = \left( \prod_{i=1}^{k} n_i! \right) / |A(T)|,
\]

where \(n_1, \ldots, n_k\) are the arities of the elements of \(T\). For reasons that will soon become clear, we call \(r(T)\) the Ramsey degree of \(T\).

We recall that if \(\mathcal{C}\) is a class of finite structures for which we have a notion of a copy (image under an embedding) of an object \(A\) in an object \(B\) of \(\mathcal{C}\), then \(A\) is said
to be a Ramsey object in \( \mathcal{C} \) if for each \( B \) in \( \mathcal{C} \) and \( r < \omega \), there is some object \( C \) such that for each partition \( \chi: [C,A] \to [r] \), where \([C,A]\) is the set of copies of \( A \) in \( C \), there is a copy \( B' \) of \( B \) in \( C \) such that all the elements of \([B',A]\) are in one block of the partition. The aim of this paper is to show, as was done in [3,4] for posets and bipartite graphs, respectively, the extent to which an object \( T \) is a Ramsey object in \( \mathcal{F}_h \) can be measured. Our main theorem explains how the number \( r(T) \) does it for a tree \( T \).

**Theorem 1.** For integers \( s, h \geq 0 \) and trees \( T, T_1, T_2 \), both of height \( h \), there is a tree \( T_2 \) of the same height such that, for every \( \chi: [T_2, T] \to [s] \), there is a copy, \( T'_1 \), of \( T_1 \) in \( T_2 \) such that \( \chi \) assumes at most \( r(T) \) values on \([T'_1, T]\). Moreover, for a given tree \( T \) of height \( h \), and for \( r \geq r(T) \), there is some \( T_1 \in \mathcal{F}_h \), such that, for any tree \( T_2 \in \mathcal{F}_h \) with \([T_2, T_1] \neq \emptyset \), there is some partition \( \chi: [T_2, T] \to [r] \) with the property that \( \chi \) assumes, on any copy of \( T_1 \) in \( T_2 \), at least \( r(T) \) values.

It follows that \( T \) will be a Ramsey object in \( \mathcal{F}_h \) iff \( r(T) = 1 \). A tree is complete when the arities of any two elements of the same height are the same. We shall soon see that \( r(T) = 1 \) iff \( T \) is complete. It follows that the complete trees are the only Ramsey objects in \( \mathcal{F}_h \). For some further results on the Ramsey properties of trees, the reader is referred to [1,2,6,7]. An analogous result for distributive lattices appears in [8].

2. Proof of main theorem

For each \( i \geq 0 \), the set of elements of \( T \) of height \( i \) is called the \( i \)th level of \( T \) and is denoted by \( L_i(T) \). For \( T \in \mathcal{F}_h \), let \( T^c \) be the complete tree such that, for every \( i \leq h \) the arities of all \( x \in L_i(T^c) \) will be \( \max \{ \text{ar}(x): x \in L_i(T) \} \). We call \( T^c \) the completion of \( T \). An orientation of the tree \( T \) is a sequence \( O \) of the form \((O_1, \ldots, O_h)\), where, for some embedding \( \mu \) of \( T \) into \( T^c \), each \( O_i \) is the sequence of arities of \( \mu(T) \), the image of \( T \) under \( \mu \), as they occur at level \( i \) of \( T^c \), when we scan the latter from the left to the right. We call the pair \((T, O)\) an oriented tree. We say in this case that the orientation \( O \) is induced by \( \mu \). (Note that a given orientation can be induced by more than one embedding of \( T \) into \( T^c \).) We can visualise \((T, O)\) as the tree \( T \) drawn in the plane in such a manner that the arities of the \( i \)th level of \( T \) appear as in the sequence \( O_i \) when \( L_i(T) \) is scanned from the left to the right.

It is clear that each complete tree has exactly one orientation. Moreover, if a tree has a unique orientation, then it must be complete. For if \( T \) were not complete, consider the first level where there are elements \( x \) and \( y \) with \( \text{ar}(x) \neq \text{ar}(y) \). Take any embedding \( \mu \) of \( T \) into \( T^c \). Let \( \sigma \) be an automorphism of \( T^c \) that interchanges \( x \) and \( y \). Clearly \( \sigma \mu \) is another embedding of \( T \) into \( T^c \). It is clear that the orientations induced by \( \mu \) and \( \sigma \mu \) are distinct.

If \( T' \) is a subtree of \( T \), then \( T' \) inherits a unique orientation from any given orientation \( O \) of \( T \), namely, if \( O \) is induced by the embedding \( \mu: T \to T^c \), say, the inherited
orientation of \( T' \) is induced by the restriction of \( \mu \) to \( T' \). Here it is important to note that the inherited orientation of \( T' \) depends on \( O \) only. If \((T,O^1)\) and \((S,O^2)\) are oriented trees, an embedding \( \mu:(T,O^1)\rightarrow(S,O^2) \) is a tree embedding such that \( \mu(T) \) inherits the orientation \( O^1 \) from the orientation \( O^2 \) of \( S \).

**Lemma 1.** For a given tree \( T \), the Ramsey degree, \( r(T) \), is the number of distinct orientations of \( T \). Hence \( r(T) = 1 \) iff \( T \) is complete.

**Proof.** Let \( h \) be the height of \( T \). The proof proceeds by induction on \( h \), the case \( h = 0 \) being trivial. Let \( T \) be a tree of height \( h \). We can represent, for a given orientation \( O \) of \( T \), the oriented tree \((T,O)\) as a sequence \((T_1,O^{(1)}),\ldots,(T_r,O^{(r)})\) of (oriented) trees of height \(<h\). Here \( T_1,\ldots,T_r \) are the components of the forest, such that, when viewed from the left to the right, \((T,O)\) is exactly a root connected with the sequence \((T_1,O^{(1)}),\ldots,(T_r,O^{(r)})\) of trees. Suppose that there are exactly \( \tau \) isomorphism types of trees among the \( T_i \) (now with their orientations ignored) with \( \alpha_i \) of them being of the \( i \)-th isomorphism type, for \( i = 1,\ldots,\tau \). Let \( M \) be the multiset \( \{\alpha_1,\ldots,\alpha_\tau\} \).

Each orientation of \( T \) is obtained by first finding orientations \( O^{(1)},\ldots,O^{(r)} \) for \( T_1,\ldots,T_r \), respectively, and by then permuting the oriented trees \((T_1,O^{(1)}),\ldots,(T_r,O^{(r)})\). Two such permutations yield the same orientation of \( T \) if they induce the same multipermutation of \( M \). Hence, by induction, writing \( s(T) \) for the number of orientations of \( T \), we have

\[
s(T) = \left( \begin{array}{c} \ell \\ \alpha_1,\ldots,\alpha_\tau \end{array} \right) s(T_1)\cdots s(T_r)
\]

\[
= \left( \begin{array}{c} \ell \\ \alpha_1,\ldots,\alpha_\tau \end{array} \right) r(T_1)\cdots r(T_r).
\]

Moreover, it is easily seen that

\[
|A(T)| = \alpha_1!\cdots\alpha_\tau! \quad |A(T_1)|\cdots|A(T_r)|.
\]

Consequently, \( s(T) = r(T) \). Finally, we have already shown that \( s(T) = 1 \) iff \( T \) is complete.

If \( A \) is a set and \( n \geq 0 \) we write \( \binom{A}{n} \) for the set of \( n \)-subsets of \( A \). In particular, \( \binom{\emptyset}{n} \) is the singleton \( \{\emptyset\} \). For integers \( n_1,\ldots,n_r \geq 0 \) and \( m,n,s \geq 1 \), we write

\[
n \Rightarrow [m, (n_1,\ldots,n_r)]
\]

when the following holds:

For sets \( Y_1,\ldots,Y_r \) with \( |Y_i| \geq n \) and an \( s \)-colouring \( \psi: \prod_{i=1}^{r} \binom{Y_i}{n} \rightarrow [s] \), there is, for each \( i \in [r] \), some subset \( Z_i \) of \( Y_i \), each set \( Z_i \) having exactly \( m \) elements, such that \( \psi \) assumes a constant value on the set \( \prod_{i=1}^{r} \binom{Z_i}{n} \).

The existence of \( n \), for given \( n_i,m \) and \( s \) is a well-known result (the so-called product Ramsey theorem, see for example [5]).

In the sequel, we shall think of complete trees as being represented in the plane. If \( T_1,T_2 \) are complete trees of the same height, an embedding \( \mu:T_1 \rightarrow T_2 \) is said to be
If, for each level $L_i$ of $T_1$, if $x, y \in L_i$ and $x$ is to the left of $y$ then $\mu(x)$ will be to the left $\mu(y)$ on the ith level of $T_2$. We have introduced this idea in order to ensure that the induced orientations of subtrees of $T_1$ are preserved under $\mu$.

For trees $T, T_1, T_2 \in \mathcal{T}_h$ where $T_1$ and $T_2$ are complete, for a natural number $s$ and an orientation $O$ of $T$ we write

$$T_2 \Rightarrow [T_1, (T, O)]$$

when the following holds:

For every $s$-colouring $\chi$ of the copies of the oriented $(T, O)$ in $T_2$, there is a strong embedding $\mu : T_1 \rightarrow T_2$ such that $\chi$ assumes a constant value on all the copies of $(T, O)$ in $T_2$ which are contained in the image of $T_1$ under $\mu$.

In order to prove Theorem 1 it suffices to prove

**Lemma 2.** For $h \geq 0$, $s \geq 1$ and $T, T_1 \in \mathcal{T}_h$ with $T_1$ complete, there exists, for every orientation $O$ of $T$, a complete $T_2 \in \mathcal{T}_h$ such that $T_2 \Rightarrow [T_1, (T, O)]$.

We now deduce Theorem 1 from Lemma 2. Let $O^1, \ldots, O^r$ with $r = r(T)$ be the distinct orientations of $T$. Set $T^0 = T_1^c$ and apply Lemma 2 to find complete trees $T^1, \ldots, T^r$ such that

$$T^{i+1} \Rightarrow [T^i, (T, O^i)]$$

for $i = 0, \ldots, r - 1$. Then a simple downward induction shows that $T_2 = T^r$ satisfies the conclusion of Theorem 1.

If $(T, O)$ is an oriented tree and if $S$ is complete, both of the same height, we write $[S, (T, O)]$ for the set of copies of $T$ in $S$ that inherit the orientation $O$ from the (unique) orientation of $S$.

We now prove Lemma 2 by an induction on $h$, the case $h = 0$ being trivial. Assume Lemma 2 holds for some $h \geq 0$. Let $T, T_1 \in \mathcal{T}_{h+1}$ with $T_1$ complete. Let $T^{(1)}_1$ and $T^{(1)}$ be the first $h$ levels of $T_1$ and $T$, respectively. Write $O^{(1)}$ for the orientation of $T^{(1)}$ which is the restriction of $O$ to the first $h$ levels. If $O = (O_1, \ldots, O_{h+1})$, say, we denote the sequence $O_h$ by $(n_1, \ldots, n_r)$.

It follows from the induction hypothesis that there is a complete $T^{(1)}_2 \in \mathcal{T}_h$ such that

$$T^{(1)}_2 \Rightarrow [T^{(1)}_1, (T^{(1)}, O^{(1)})].$$

(1)

Enumerate the elements of $[T^{(1)}_2, (T^{(1)}, O^{(1)})]$ as $T^{1}, \ldots, T^{\mu}$.

Set $m_0 = k$, the arity of any element of $L_h(T_1)$. Define $m_1, \ldots, m_\mu$ iteratively by

$$m_{i+1} \Rightarrow [(m_i, (n_1, \ldots, n_r))],$$

$i = 0, \ldots, \mu - 1$. Let $T_2$ be a complete tree of height $h + 1$ whose first $h$ levels are exactly those of $T^{(1)}_2$ and such that each element of the $h$th level of $T_2$ has arity $m_\mu$. We shall show that $T_2$ has the properties as in the statement of Lemma 2.
Label the elements of $L_h(T_2)$ as they appear from left to right by the numbers $1, \ldots, f$. In the sequel we shall identify $L_h(T_2)$ with $[f]$. The sets of successors of $1, \ldots, f$ in $T_2$ are denoted by $X^0_1 , \ldots , X^0_f$, respectively.

For given $\chi : [T_2, (T, O)] \to [s]$, we shall construct an induced colouring $\theta$ of $[T_2^{(1)}, (T^{(1)}, O^{(1)})]$ in stages, one stage for each copy $T'$ of $(T^{(1)}, O^{(1)})$ in $T_2^{(1)}$. For $i = 0, \ldots, \mu$, let $P_i$ be the following statement:

There are subsets $X^i_1, \ldots , X^i_f$ of $X^0_1, \ldots , X^0_f$ each of size $\geq m_{i-1}$, such that, for all extensions $T''$ of $T^1, \ldots , T^i$ to copies of $(T, O)$ in $T_2$ such that $L_{h+1}(T'') \subset \bigcup_{j=1}^i X^i_j$, the colour of $T''$ with respect to $\chi$ depends only on the copy, $S$, of $T^{(1)}$ of which it is an extension.

The statement $P_0$ is vacuously true. Next assume $P_i$ holds for some $i$ with $0 \leq i < \mu$. Let $p(1) < \cdots < p(v)$ be the subsequence of $[f]$ which are the points at the $h$th level of $T^{i-1}$. Set $Y_j = X^i_{p(j)}$ and define

$$
\psi : \left( Y_1 \atop n_1 \right) \times \cdots \times \left( Y_v \atop n_v \right) \to [s]
$$

as follows: For $(\tau_1, \ldots, \tau_r)$ in $\left( \begin{array}{c} 1 \\ n_1 \end{array} \right) \times \cdots \times \left( \begin{array}{c} 1 \\ n_v \end{array} \right)$, let $T_0$ be the copy of $(T, O)$ which is the extension of $T^{i-1}$ such that $\tau_j$ is the set of successors of $p(j)$, for $j = 1, \ldots, v$. Set $\psi(\tau_1, \ldots, \tau_r) = \chi(T_0)$.

Let $m_{i-1} \geq [m_{i-1} + 1, (n_1, \ldots, n_v)]$.

there are $Z_j \subset Y_j$ with $|Z_j| = m_{i-1}$, $j = 1, \ldots, v$, such that $\psi$ is monochromatic on

$$
\left( Z_1 \atop n_1 \right) \times \cdots \times \left( Z_v \atop n_v \right).
$$

Set

$$
X^i_{j+1} = \begin{cases} Z_k & \text{if } j = p(k) \in L_h(T^{i-1}), \\ X^i_j & \text{otherwise.} \end{cases}
$$

Then the sequence $X^i_{j+1}$, $j = 1, \ldots, f$ witnesses the truth of $P_{i+1}$.

Since $P_\mu$ holds, and $m_\mu = k$, the number of successors of any $x \in T_1$ at level $h$, there are sets $X^i_1, \ldots, X^i_f \subset L_{h+1}(T_2)$ with $|X^i_j| \geq k$ for all $j$, such that all the extensions $T''$ of an element $S$ of $[T_2^{(1)}, (T^{(1)}, O^{(1)})]$ such that $L_{h+1}(T'') \subset \bigcup_{j=1}^f X^i_j$ will have a colour that depends on $S$ only. We have thus found an induced $s$-colouring $\theta$ of $[T_2^{(1)}, (T^{(1)}, O^{(1)})]$.

By the relation (1), there is a copy, $R^{(1)}$, under a strong embedding of $T_1^{(1)}$ in $T_2^{(1)}$ such that $\theta$ assumes a constant value on $[R^{(1)}, (T^{(1)}, O^{(1)})]$. Write $q(1) < \cdots < q(t)$ for the elements of $L_h(R^{(1)})$. Let $R$ be the extension of $R^{(1)}$ to a complete tree of height $h + 1$ such that the successors of $q(i)$ are given by the set $X_{q(i)}^{(i)}$, $i = 1, \ldots, t$. Then $R$ contains a copy of $T_1$ and, moreover, $\chi$ is constant on $[R, (T, O)]$. This completes the proof of Lemma 2 and hence of the first part of Theorem 1.

We now prove the second part of Theorem 1: Choose $T \in \mathcal{F}_h$ and set $T_1 = T'$. Let $T_2$ be any tree of height $h$ with $[T_2, T_1] \neq \emptyset$. For $T' \in [T_2, T]$, set $\chi(T') = O$ when
$O$ is the orientation that $T'$ inherits from $T'^{\nabla}$. Then $\chi$ assumes on any set of the form $[T',T]$, with $T'$ a copy of $T_1$ in $T_2$, exactly $r(T)$ values.

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References