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Symmetries and Ramsey properties of trees

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Abstract

In this paper we show the extent to which a finite tree of fixed height is a Ramsey object in the class of trees of the same height can be measured by its symmetry group.

1. Introduction

In the sequel, all the combinatorial structures referred to will be finite. For a natural number r, we write $[r]$ for the set $\{1,\ldots,r\}$. In this paper, a tree will be a poset T with a minimum element (the root) such that, for each $x \in T$, the set $\{y \in T\colon y < x\}$ is a linear order. The cardinality of this set is called the *height* of x and denoted by $ht(x)$. The height of a tree is the maximum height of any element of T. For $h \geqslant 0$, we denote the class of trees of height h by $\mathscr{T}_h$. If $T_1, T_2 \in \mathscr{T}_h$, an embedding μ of T_1 into T_2 is a poset embedding (i.e., μ is injective and $x < y$ iff $\mu(x) < \mu(y)$ for all x, $y \in T_1$) which is, in addition, height-preserving, i.e., $ht(\mu(x)) = ht(x)$ for all $x \in T_1$. For a tree T, the number of successors of an element x is called the arity of x and denoted by $ar(x)$. We denote the set of copies T_1 in T_2 by $[T_2, T_1]$. If T is of height h, a subtree of T is an image under an embedding of some tree S of height h.

For $T \in \mathscr{T}_h$, let $A(T)$ be the group of height-preserving automorphisms of T. Set

$$r(T) = \left(\prod_{i=1}^{k} n_i!\right) \Big/ |A(T)|,$$

where $n_1,\ldots,n_k$ are the arities of the elements of T. For reasons that will soon become clear, we call $r(T)$ the *Ramsey degree* of T.

We recall that if $\mathscr{C}$ is a class of finite structures for which we have a notion of a copy (image under an embedding) of an object A in an object B of $\mathscr{C}$, then A is said

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to be a *Ramsey object* in $\mathscr{C}$ if for each B in $\mathscr{C}$ and $r<\omega$, there is some object C such that for each partition $\chi:[C,A]\to[r]$, where $[C,A]$ is the set of copies of A in C, there is a copy B' of B in C such that all the elements of $[B',A]$ are in one block of the partition. The aim of this paper is to show, as was done in [3,4] for posets and bipartite graphs, respectively, the extent to which an object T is a Ramsey object in $\mathscr{T}_h$ can be measured. Our main theorem explains how the number $r(T)$ does it for a tree T.

Theorem 1. *For integers s, $h\geqslant 0$ and trees T, T_1, both of height h, there is a tree T_2 of the same height such that, for every $\chi:[T_2,T]\to[s]$, there is a copy, T_1', of T_1 in T_2 such that χ assumes at most $r(T)$ values on $[T_1',T]$. Moreover, for a given tree T of height h, and for $r\geqslant r(T)$, there is some $T_1\in\mathscr{T}_h$, such that, for any tree $T_2\in\mathscr{T}_h$ with $[T_2,T_1]\neq\emptyset$, there is some partition $\chi:[T_2,T]\to[r]$ with the property that χ assumes, on any copy of T_1 in T_2, at least $r(T)$ values.*

It follows that T will be a Ramsey object in $\mathscr{T}_h$ iff $r(T)=1$. A tree is *complete* when the arities of any two elements of *the same height* are the same. We shall soon see that $r(T)=1$ iff T is complete. It follows that the complete trees are the only Ramsey objects in $\mathscr{T}_h$. For some further results on the Ramsey properties of trees, the reader is referred to [1,2,6,7]. An analogous result for distributive lattices appears in [8].

2. Proof of main theorem

For each $i\geqslant 0$, the set of elements of T of height i is called the ith level of T and is denoted by $L_i(T)$. For $T\in\mathscr{T}_h$, let T^c be the complete tree such that, for every $i\leqslant h$ the arities of all $x\in L_i(T^c)$ will be $\max\{\mathrm{ar}(x)\colon x\in L_i(T)\}$. We call T^c the completion of T. An *orientation* of the tree T is a sequence O of the form $(O_1,\ldots,O_h)$, where, for some embedding μ of T into T^c, each O_i is the sequence of arities of $\mu(T)$, the image of T under μ, as they occur at level i of T^c, when we scan the latter from the left to the right. We call the pair (T,O) an oriented tree. We say in this case that the orientation O is induced by μ. (Note that a given orientation can be induced by more than one embedding of T into T^c.) We can visualise (T,O) as the tree T drawn in the plane in such a manner that the arities of the ith level of T appear as in the sequence O_i when $L_i(T)$ is scanned from the left to the right.

It is clear that each complete tree has exactly one orientation. Moreover, if a tree has a unique orientation, then it must be complete. For if T were not complete, consider the first level where there are elements x and y with $\mathrm{ar}(x)\neq\mathrm{ar}(y)$. Take any embedding μ of T into T^c. Let σ be an automorphism of T^c that interchanges x and y. Clearly $\sigma\mu$ is another embedding of T into T^c. It is clear that the orientations induced by μ and $\sigma\mu$ are distinct.

If T' is a subtree of T, then T' inherits a unique orientation from any given orientation O of T, namely, if O is induced by the embedding $\mu:T\to T^c$, say, the inherited

orientation of T' is induced by the restriction of μ to T'. Here it is important to note that the inherited orientation of T' depends on O only. If (T, O^1) and (S, O^2) are oriented trees, an embedding $\mu:(T, O^1) \to (S, O^2)$ is a tree embedding such that $\mu(T)$ inherits the orientation O^1 from the orientation O^2 of S.

Lemma 1. *For a given tree T, the Ramsey degree, $r(T)$, is the number of distinct orientations of T. Hence $r(T) = 1$ iff T is complete.*

Proof. Let h be the height of T. The proof proceeds by induction on h, the case $h = 0$ being trivial. Let T be a tree of height h. We can represent, for a given orientation O of T, the oriented tree (T, O) as a sequence $(T_1, O^{(1)}), \ldots, (T_\ell, O^{(\ell)})$ of (oriented) trees of height $< h$. Here $T_1, \ldots, T_\ell$ are the components of the forest, such that, when viewed from the left to the right, (T, O) is exactly a root connected with the sequence $(T_1, O^{(1)}), \ldots, (T_\ell, O^{(\ell)})$ of trees. Suppose that there are exactly τ isomorphism types of trees among the T_i (now with their orientations ignored) with α_i of them being of the ith isomorphism type, for $i = 1, \ldots, \tau$. Let M be the multiset $\{1^{\alpha_1}, \ldots, \tau^{\alpha_\tau}\}$.

Each orientation of T is obtained by first finding orientations $O^1, \ldots, O^\ell$ for $T_1, \ldots, T_\ell$, respectively, and by then permuting the oriented trees $(T_1, O^1), \ldots, (T_\ell, O^\ell)$. Two such permutations yield the same orientation of T iff they induce the same multipermutation of M. Hence, by induction, writing $s(T)$ for the number of orientations of T, we have

$$\begin{aligned} s(T) &= \binom{\ell}{\alpha_1, \ldots, \alpha_\tau} s(T_1) \cdots s(T_\ell) \\ &= \binom{\ell}{\alpha_1, \ldots, \alpha_\tau} r(T_1) \cdots r(T_\ell). \end{aligned}$$

Moreover, it is easily seen that

$$|A(T)| = \alpha_1! \cdots \alpha_\tau! \; |A(T_1)| \cdots |A(T_\ell)|.$$

Consequently, $s(T) = r(T)$. Finally, we have already shown that $s(T) = 1$ iff T is complete.

If A is a set and $n \geqslant 0$ we write $\binom{A}{n}$ for the set of n-subsets of A. In particular, $\binom{A}{0}$ is the singleton $\{\emptyset\}$. For integers $n_1, \ldots, n_v \geqslant 0$ and $m, n, s \geqslant 1$, we write

$$n \overset{s}{\Rightarrow} [m, (n_1, \ldots, n_v)]$$

when the following holds:

For sets $Y_1, \ldots, Y_v$ with $|Y_i| \geqslant n$, and an s-colouring $\psi: \prod_{i=1}^{v} \binom{Y_i}{n_i} \to [s]$, there is, for each $i \in [v]$, some subset Z_i of Y_i, each set Z_i having exactly m elements, such that ψ assumes a constant value on the set $\prod_{i=1}^{v} \binom{Z_i}{n_i}$.

The existence of n, for given n_i, m and s is a well-known result (the so-called product Ramsey theorem, see for example [5]).

In the sequel, we shall think of complete trees as being represented in the plane. If T_1, T_2 are complete trees of the same height, an embedding $\mu: T_1 \to T_2$ is said to be

strong if, for each level L_i of T_1, if $x, y \in L_i$ and x is to the left of y then $\mu(x)$ will be to the left $\mu(y)$ on the ith level of T_2. We have introduced this idea in order to ensure that the induced orientations of subtrees of T_1 are preserved under μ.

For trees $T, T_1, T_2 \in \mathscr{T}_h$ where T_1 and T_2 are complete, for a natural number s and an orientation O of T we write

$$T_2 \overset{s}{\Rightarrow} [T_1,\ (T, O)]$$

when the following holds:

For every s-colouring χ of the copies of the oriented (T, O) in T_2, there is a strong embedding $\mu : T_1 \to T_2$ such that χ assumes a constant value on all the copies of (T, O) in T_2 which are contained in the image of T_1 under μ.

In order to prove Theorem 1 it suffices to prove

Lemma 2. *For $h \geqslant 0$, $s \geqslant 1$ and $T, T_1 \in \mathscr{T}_h$ with T_1 complete, there exists, for every orientation O of T, a complete $T_2 \in \mathscr{T}_h$ such that $T_2 \overset{s}{\Rightarrow} [T_1,\ (T, O)]$.*

We now deduce Theorem 1 from Lemma 2. Let $O^1, \ldots, O^r$ with $r = r(T)$ be the distinct orientations of T. Set $T^0 = T_1^c$ and apply Lemma 2 to find complete trees $T^1, \ldots, T^r$ such that

$$T^{i+1} \overset{s}{\Rightarrow} [T^i,\ (T, O^i)]$$

for $i = 0, \ldots, r-1$. Then a simple downward induction shows that $T_2 = T^r$ satisfies the conclusion of Theorem 1.

If (T, O) is an oriented tree and if S is complete, both of the same height, we write $[S, (T, O)]$ for the set of copies of T in S that inherit the orientation O from the (unique) orientation of S.

We now prove Lemma 2 by an induction on h, the case $h = 0$ being trivial. Assume Lemma 2 holds for some $h \geqslant 0$. Let $T, T_1 \in \mathscr{T}_{h+1}$ with T_1 complete. Let $T_1^{(1)}$ and $T^{(1)}$ be the first h levels of T_1 and T, respectively. Write $O^{(1)}$ for the orientation of $T^{(1)}$ which is the restriction of O to the first h levels. If $O = (O_1, \ldots, O_{h+1})$, say, we denote the sequence O_h by $(n_1, \ldots, n_v)$.

It follows from the induction hypothesis that there is a complete $T_2^{(1)} \in \mathscr{T}_h$ such that

$$T_2^{(1)} \overset{s}{\Rightarrow} [T_1^{(1)},\ (T^{(1)}, O^{(1)})]. \tag{1}$$

Enumerate the elements of $[T_2^{(1)}, (T^{(1)}, O^{(1)})]$ as $T^1, \ldots, T^\mu$.

Set $m_0 = k$, the arity of any element of $L_h(T_1)$. Define $m_1, \ldots, m_\mu$ iteratively by

$$m_{i+1} \overset{s}{\Rightarrow} [(m_i,\ (n_1, \ldots, n_v)],$$

$i = 0, \ldots, \mu - 1$. Let T_2 be a complete tree of height $h + 1$ whose first h levels are exactly those of $T_2^{(1)}$ and such that each element of the hth level of T_2 has arity m_μ. We shall show that T_2 has the properties as in the statement of Lemma 2.

Label the elements of $L_h(T_2)$ as they appear from left to right by the numbers $1,\dots,f$. In the sequel we shall identify $L_h(T_2)$ with $[f]$. The sets of successors of $1,\dots,f$ in T_2 are denoted by $X_1^0,\dots,X_f^0$, respectively.

For given $\chi:[T_2,(T,O)]\to[s]$, we shall construct an induced colouring θ of $[T_2^{(1)},(T^{(1)},O^{(1)})]$ in stages, one stage for each copy T^i of $(T^{(1)},O^{(1)})$ in $T_2^{(1)}$. For $i=0,\dots,\mu$, let $\boldsymbol{P}_i$ be the following statement:

> There are subsets $X_1^i,\dots,X_f^i$ of $X_1^0,\dots,X_f^0$ each of size $\geqslant m_{\mu-i}$, such that, for all extensions T'' of $T^1,\dots,T^i$ to copies of (T,O) in T_2 such that $L_{h+1}(T'')\subset\bigcup_{j=1}^f X_j^i$, the colour of T'' with respect to χ depends only on the copy, S, of $T^{(1)}$ of which it is an extension.

The statement $\boldsymbol{P}_0$ is vacuously true. Next assume $\boldsymbol{P}_i$ holds for some i with $0\leqslant i<\mu$. Let $p(1)<\cdots<p(v)$ be the subsequence of $[f]$ $(=L_h(T_2))$ which are the points at the hth level of T^{i+1}. Set $Y_j=X_{p(j)}^i$ and define

$$\psi:\binom{Y_1}{n_1}\times\cdots\times\binom{Y_v}{n_v}\to[s]$$

as follows: For $(\tau_1,\dots,\tau_v)$ in $\binom{Y_1}{n_1}\times\cdots\times\binom{Y_v}{n_v}$, let T_0 be the copy of (T,O) which is the extension of T^{i-1} such that τ_j is the set of successors of $p(j)$, for $j=1,\dots,v$. Set $\psi(\tau_1,\dots,\tau_v)=\chi(T_0)$. Since

$$m_{\mu-i}\overset{s}{\Rightarrow}[m_{\mu-(i+1)},(n_1,\dots,n_v)],$$

there are $Z_j\subset Y_j$ with $|Z_j|=m_{\mu-(i-1)}$, $j=1,\dots,v$, such that ψ is monochromatic on

$$\binom{Z_1}{n_1}\times\cdots\times\binom{Z_v}{n_v}.$$

Set

$$X_j^{i-1}=\begin{cases}Z_k & \text{if } j=p(k)\in L_h(T^{i-1}),\\ X_j^i & \text{otherwise.}\end{cases}$$

Then the sequence X_j^{i+1}, $j=1,\dots,f$ witnesses the truth of $\boldsymbol{P}_{i+1}$.

Since $\boldsymbol{P}_\mu$ holds, and $m_0=k$, the number of successors of any $x\in T_1$ at level h, there are sets $X_1^\mu,\dots,X_f^\mu\subset L_{h+1}(T_2)$ with $|X_j^\mu|\geqslant k$ for all j, such that all the extensions T'' of an element S of $[T_2^{(1)},(T^{(1)},O^{(1)})]$ such that $L_{h+1}(T'')\subset\bigcup_{j=1}^f X_j^\mu$ will have a colour that depends on S only. We have thus found an induced s-colouring θ of $[T_2^{(1)},(T^{(1)},O^{(1)})]$.

By the relation (1), there is a copy, $R^{(1)}$, under a strong embedding of $T_1^{(1)}$ in $T_2^{(1)}$ such that θ assumes a constant value on $[R^{(1)},(T^{(1)},O^{(1)})]$. Write $q(1)<\cdots<q(t)$ for the elements of $L_h(R^{(1)})$. Let R be the extension of $R^{(1)}$ to a complete tree of height $h+1$ such that the successors of $q(i)$ are given by the set $X_{q(i)}^\mu$, $i=1,\dots,t$. Then R contains a copy of T_1 and, moreover, χ is constant on $[R,(T,O)]$. This completes the proof of Lemma 2 and hence of the first part of Theorem 1.

We now prove the second part of Theorem 1: Choose $T\in\mathscr{T}_h$ and set $T_1=T^c$. Let T_2 be any tree of height h with $[T_2,T_1]\neq\emptyset$. For $T'\in[T_2,T]$, set $\chi(T')=O$ when

O is the orientation that T' inherits from T_2^c. Then χ assumes on any set of the form $[T_1', T]$, with T_1' a copy of T_1 in T_2, exactly $r(T)$ values.

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