ON A PROBLEM OF R. HALIN CONCERNING INFINITE GRAPHS II

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For every countable, connected graph \( A \) containing no one-way infinite path the following is shown: Let \( G \) be an arbitrary graph which contains for every positive integer \( n \) a system of \( n \) disjoint graphs each isomorphic to a subdivision of \( A \). Then \( G \) also contains infinitely many disjoint subgraphs each isomorphic to a subdivision of \( A \). In addition, corrections of errors are given that occur unfortunately in the forerunner of the present paper.

1. Introduction

In [7] Halin posed the following problem:

(1) Let \( A \) be an arbitrary graph and assume that a graph \( G \) contains for every positive integer \( n \) a system of \( n \) disjoint graphs each isomorphic to \( A \). Does then \( G \) necessarily contain infinitely many disjoint copies of \( A \)?

In [3] this question has been answered affirmatively if \( A \) is a graph that arises from \( n \) disjoint one-way infinite paths by identifying their initial vertices (where \( n \) is an arbitrary cardinal \( \neq 0 \)). Furthermore it was shown in [3] that the answer to (1) is affirmative if \( A \) is a countable tree with finite diameter. In [1], [8] and [9] it was independently shown by counterexamples that the answer to (1) is negative in the general case of an arbitrary graph \( A \).

The present paper deals with the following analogue to (1):

(2) Let \( A \) be an arbitrary graph and assume that a graph \( G \) contains for every positive integer \( n \) a system of \( n \) disjoint graphs each isomorphic to a subdivision of \( A \). Does then \( G \) necessarily contain infinitely many disjoint subgraphs each isomorphic to a subdivision of \( A \)?

In [7] this question was answered affirmatively if \( A \) is a graph such that a subdivision of \( A \) is isomorphic to a subgraph of \( S \), where \( S \) is the graph in Fig. 1 of [3]. This class of graphs includes every tree in which each vertex has degree not greater than 3. This result was sharpened in [2] by showing that the above result holds for all locally finite trees. Furthermore in [1] an example was given showing that the answer to (2) is negative in the general case of an arbitrary graph \( A \). In addition it is easy to see that the answer to (1) and (2) is affirmative if \( A \) is a finite graph (see [7]).
The purpose of the present paper is to prove the following theorem concerning problem (2):

For every countable, connected graph $A$ containing no one-way infinite path the answer to (2) is positive.

Furthermore we give three examples showing that this statement becomes false if we drop one of the conditions on $A$. In addition, we correct some errors that occur unfortunately in the forerunner of the present paper [3].

2. Definitions and notations

In this paper we consider only undirected graphs containing no loops or multiple edges. By $V(G)$ and $E(G)$ we denote the set of vertices and edges of the graph $G$, respectively. A path is a graph having exactly $n+1$ different vertices $v_0, \ldots, v_n$ and $n$ edges $e_i = (v_i, v_{i+1})$ ($i = 0, \ldots, n-1$). Let $A, B$ be graphs; if $P = (v_0, \ldots, v_n)$ is a path such that $P \cap A = v_0$ and $P \cap B = v_n$, then $P$ is called a $(A, B)$-path. We also write $P = P(a, b)$ for a path beginning in $a$ and ending in $b$. All vertices of $P(a, b)$ different from $a$ and $b$ are called inner vertices of $P(a, b)$. A one-way infinite path (briefly: 1-path) is a graph that consists of a sequence $v_0, v_1, \ldots$ of different vertices and the edges $e_i = (v_i, v_{i+1})$ ($i = 0, 1, \ldots$).

By $|S|$ we denote the cardinality of a set $S$. Let $G$ be a graph. $G$ is called countable if $|V(G)| = |V(G)|$ is countable. $G$ is called connected if for every $a, b \in V(G)$ there is a path $P(a, b)$ in $G$. A maximal connected subgraph of $G$ is called a component of $G$. By $\mathcal{G}(G)$ we denote the set of components of $G$. By $G \subseteq V$ (for $V \subseteq V(G)$) we denote the graph that arises from $G$ by dropping every vertex $v \in V$ and every edge incident to $v$. (If $H$ is a subgraph of $G$, then we also write $G \supseteq H$ instead of $G \supseteq V(H)$.) If $G'$ is isomorphic to $G$, then we call $G'$ a copy of $G$. If $G'$ is a subgraph of $G$, then we write $G' \subseteq G$. For a cardinal $n$ we write $K_n$ for the complete graph with $n$ vertices. Let $E \subseteq E(G)$. Subdivide every $e \in E$ by inserting a finite number of new vertices of degree two on $e$. Then the arising graph $G'$ is called a subdivision of $G$ and the vertices of $G$ are called main vertices of $G'$. We write $H \subseteq G$ if there is a subgraph of $G$ isomorphic to a subdivision of $H$. By $\bigcup_{i \in I} G_i$ we denote the disjoint union of a family $(G_i)_{i \in I}$ of graphs. For a graph $A$ and a cardinal $n$ we write $nA$ for the graph $\bigcup_{i \in I} A_i$ if $A_i$ is isomorphic to $A$ for every $i \in I$ and $|I| = n$. We shall call a graph $A$ regular, if $nA \subseteq G$ for every $n \in \mathbb{N}$ always implies $\mathcal{X}_n A \subseteq G$. (By $\mathbb{N}$ we denote the set of positive integers.)

3. Proof of the regularity of every countable, connected graph containing no 1-path

Definition. For $n = 1, 2, \ldots$ let $\mathcal{A}_n = \{ A^{(n,m)} : m = 1, \ldots, n \}$ be a set of $n$ different graphs. Let $\mathcal{A} = \bigcup_{n \in \mathbb{N}} \mathcal{A}_n$. We call $\mathcal{A'} \subseteq \mathcal{A}$ a complete subsystem of $\mathcal{A}$ if for every
$k \in \mathbb{N}$ there is a $n_k \in \mathbb{N}$ such that $|\mathcal{W} \cap \mathcal{W}_{n_k}| \geq k$. Otherwise $\mathcal{W}$ is called incomplete subsystem of $\mathcal{W}$.

First we shall prove some lemmas.

**Lemma 1.** For every $n \in \mathbb{N}$, let $\mathcal{W}_n = \{A^{(n,m)} : m = 1, \ldots, n\}$ be a system of $n$ disjoint graphs such that there are not infinitely many disjoint graphs in $\mathcal{W} = \bigcup_{n=1}^{\infty} \mathcal{W}_n$. Then there exists a complete subsystem $\mathcal{W}'$ of $\mathcal{W}$ such that $\mathcal{W}(A') = \{A'' \in \mathcal{W} : A'' \cap A' = \emptyset\}$ is an incomplete subsystem of $\mathcal{W}$ for every $A' \in \mathcal{W}$.

**Proof.** Let us assume that Lemma 1 does not hold. Then for every complete subsystem $\mathcal{W}'$ of $\mathcal{W}$ there is an $A' \in \mathcal{W}'$ such that $\mathcal{W}'' = \{A'' \in \mathcal{W} : A'' \cap A'' = \emptyset\}$ is a complete subsystem of $\mathcal{W}$. By applying this successively we get a sequence of disjoint members of $\mathcal{W}$. This contradicts our assumption on $\mathcal{W}$, and thus Lemma 1 is proved.

**Lemma 2.** For every $n \in \mathbb{N}$, let $\mathcal{W}_n = \{A^{(n,m)} : m = 1, \ldots, n\}$ be a system of $n$ disjoint subgraphs of a graph $G$ having the following properties: $A^{(n,m)}$ is infinite, connected and has no 1-path. Furthermore let us assume that there are not infinitely many disjoint graphs in $\mathcal{W} = \bigcup_{n=1}^{\infty} \mathcal{W}_n$. Then there is a complete subsystem $\mathcal{W}'$ of $\mathcal{W}$ having the following property: For every $A^{(n,m)} \in \mathcal{W}$ there is a finite, connected subgraph $T^{(n,m)}$ of $A^{(n,m)}$ and a sequence of different $C^{(n,m)}_i \in \mathcal{E}(A^{(n,m)} - T^{(n,m)})$ ($i = 1, 2, \ldots$) such that:

1. $\mathcal{W}^{(n,m)} = \{A^{(q,p)} \in \mathcal{W} : A^{(q,p)} \cap \bigcup_{i=1}^{\infty} C^{(n,m)}_i = \emptyset\}$ is an incomplete subsystem of $\mathcal{W}$.
2. $\mathcal{W}^{(n,m)} = \{A^{(q,p)} \in \mathcal{W} : A^{(q,p)} \cap C^{(n,m)}_i \neq \emptyset\}$ is an incomplete subsystem of $\mathcal{W}$ ($i = 1, 2, \ldots$).

**Proof.** (I) Find a complete subsystem $\mathcal{W}'$ of $\mathcal{W}$ according to Lemma 1. Then for every $k \in \mathbb{N}$ there is a $n_k \in \mathbb{N}$ and a $\mathcal{W}'_k \subseteq \mathcal{W}' \cap \mathcal{W}_{n_k}$ with $|\mathcal{W}'_k| = k$. Obviously it suffices to prove Lemma 2 for $\mathcal{W}'_k$ ($k \in \mathbb{N}$). Thus we can assume without loss of generality:

3. $\{A^{(q,p)} \in \mathcal{W} : A^{(q,p)} \cap A^{(n,m)} = \emptyset\}$ is an incomplete subsystem of $\mathcal{W}$ for every $A^{(n,m)} \in \mathcal{W}$.

(II) We are going to show the following:

4. For every $A' \in \mathcal{W}$ and every complete subsystem $\mathcal{W}'$ of $\mathcal{W}$, there is a finite, connected subgraph $T'$ of $A'$, a sequence of different $C'_i \in \mathcal{E}(A' - T')$ ($i = 1, 2, \ldots$) and a complete subsystem $\mathcal{W}''$ of $\mathcal{W}$, $\mathcal{W}'' \subseteq \mathcal{W}'$, such that:

4.1. $A \cap \bigcup_{i=1}^{\infty} C'_i \neq \emptyset$ for every $A \in \mathcal{W}''$.

4.2. $\mathcal{W}'' = \{A \in \mathcal{W}'' : A \cap C'_i \neq \emptyset\}$ is an incomplete subsystem of $\mathcal{W}$ ($i = 1, 2, \ldots$).

**Proof of (4).** Let us assume that (4) does not hold for a certain pair $A'$, $\mathcal{W}'$. We
lead this assumption to a contradiction by constructing a 1-path in $A'$:

1. Let $D_1 = A'$; then by (3) $\mathcal{W}(1) = \{A \in \mathcal{W} : A \cap D_1 \neq \emptyset\}$ is a complete subsystem of $\mathcal{W}$.

2. Let us assume that for a certain $n \in \mathbb{N}$ the subgraphs $D_i$ ($i = 1, \ldots, n$) and $T_i$ ($i = 1, \ldots, n - 1$) of $A'$ have already been defined such that:
   
   (a) $T_i \subseteq D_i$ ($i = 1, \ldots, n - 1$)
   
   (b) $D_i \in \mathcal{C}(D_{i-1} \setminus T_{i-1})$ ($i = 2, \ldots, n$)
   
   (c) $T_i$ is finite and connected.
   
   (d) There is an edge $e_i = (a_i, b_i) \in E(A')$: $a_i \in T_i$, $b_i \in T_{i+1}$ ($i = 1, \ldots, n-2$).

   (e) There is a complete subsystem $\mathcal{W}(n)$ of $\mathcal{W}$, $\mathcal{W}(n) \subseteq \mathcal{W}'$,
   
   such that $A \cap D_n \neq \emptyset$ for every $A \in \mathcal{W}(n)$.

   Because of (e), $D_n$ is infinite. Furthermore, as a subgraph of $A'$, $D_n$ has no 1-path. Hence by the characterisation of the graphs containing no 1-path given by Halin in [5, Satz 1], there is a finite subgraph $T_n$ of $D_n$ such that $\mathcal{C}(D_n \setminus T_n)$ is infinite. Further $D_i$ is connected ($i = 1, \ldots, n$). (For $i = 1$ this holds by assumption on the members of $\mathcal{W}$ and for $i > 1$ by (b).) Hence we can assume that $T_n$ is connected and that there is an edge $e_n = (a_{n-1}, b_{n-1}) \in E(A')$ with $a_{n-1} \in T_{n-1}$ and $b_{n-1} \in T_n$.

   It remains to prove the following assertion:

   (\text{*}) There is a $D_{n+1} \in \mathcal{C}(D_n \setminus T_n)$ such that

   $\mathcal{W}(n+1) = \{A \in \mathcal{W}(n) : A \cap D_{n+1} \neq \emptyset\}$

   is a complete subsystem of $\mathcal{W}$.

   Proof of (\text{*}). Assume that $\{A \in \mathcal{W}(n) : A \cap D \neq \emptyset\}$ is an incomplete subsystem of $\mathcal{W}$ for every $D \in \mathcal{C}(D_n \setminus T_n)$. Let $T' = \bigcup_{i=1}^{n} T_i$ and $\mathcal{W}' = \{A \in \mathcal{W}(n) : A \cap (D_n \setminus T_n) \neq \emptyset\}$. Then, because of $|T_n| < \infty$ and (e), $\mathcal{W}'$ is a complete subsystem of $\mathcal{W}$, $\mathcal{W}' \subseteq \mathcal{W}'$.

   Furthermore notice that $\mathcal{C}(D_n \setminus T_n) \subseteq \mathcal{C}(A' \setminus T')$. Hence $T', \mathcal{C}(D_n \setminus T_n), \mathcal{W}'$ form a triplet having all properties described in (4). This is a contradiction to our assumption that (4) does not hold for $A'$ and $\mathcal{W}'$. Thus (\text{*}) is proved.

3. By 1 and 2 a sequence $T_n$ ($n \in \mathbb{N}$) of disjoint connected subgraphs of $A'$ is defined such that every $T_n$ is connected to $T_{n+1}$ by an edge of $A'$. Hence $A'$ contains a 1-path. Thus (4) is proved.

(III) For $A' \in \mathcal{W}$ and a complete subsystem $\mathcal{W}'$ of $\mathcal{W}$, we shall write $\Phi(A', \mathcal{W}') = (T', (C_i))_{i \in \mathbb{N}}$, $\mathcal{W}'$ for a triplet according to (4). Lemma 2 can now be easily proved by successive application of (4):

1. Let $\Phi(A'^{(1,1)}, \mathcal{W}) = (T'^{(1,1)}, (C_i^{(1,1)})_{i \in \mathbb{N}}$, $\mathcal{W}^{(1,1)})$ and $\mathcal{W}_1 := \{A^{(1,1)}\}$.

2. Let $\mathcal{W}_n = \{A^{(q,v_{n-1},i)} : i = 1, \ldots, n\}$, $T^{(q,v_{n-1},i)}$, $(C_i^{(q,v_{n-1},i)})_{i \in \mathbb{N}}$ be already defined such that $|\mathcal{W}_n| = n$ and $\mathcal{W}^{(n,n)}$ be a complete subsystem of $\mathcal{W}$. Choose

   $\mathcal{W}_{n+1} = \{A^{(q,v_{n-1},i)} : i = 1, \ldots, n+1\} \subseteq \mathcal{W}^{(n,n)}$

   with $|\mathcal{W}_{n+1}| = n + 1$. Furthermore define $T^{(q,v_{n-1},i+1)}$, $(C_i^{(q,v_{n-1},i+1)})_{i \in \mathbb{N}}$ ($i = 1, \ldots, n+1$) and $\mathcal{W}^{(n+1,n+1)}$ by

   $\Phi(A'^{(q,v_{n-1},i+1)}, \mathcal{W}^{(1,1)}) = (T^{(q,v_{n-1},i+1)}, (C_i^{(q,v_{n-1},i+1)})_{i \in \mathbb{N}}$, $\mathcal{W}^{(n+1,1)}))$
and
\[ \Phi(A_{\mathbb{R}_{n+1}}(n+1,i), \mathcal{G}_{n+1,i-1}) = \]
\[ = (T_{n+1,i}^{\mathbb{R}_{n+1}}(n+1,i), (C_{j}^{T_{n+1,i}^{\mathbb{R}_{n+1}}}(n+1,i))_{j \in \mathbb{N}} \mathcal{G}_{n+1,i}) \quad (i = 2, \ldots, n+1). \]

By 1 and 2, \( \mathcal{A}_n \) is defined for every \( n \in \mathbb{N} \) such that \( \mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n \) is a complete subsystem of \( \mathcal{A} \) which has the required property. This completes the proof of Lemma 2.

**Lemma 3.** Let \( G_j \ (j \in \mathbb{N}) \) be a sequence of disjoint, finite, connected graphs. For every \( i, j \in \mathbb{N} \) \((i \neq j)\) let \( v^{(j,i)} \) be a vertex of \( G_i \). Let \( H \) be the graph defined by \( V(H) = \bigcup_{j=1}^{\infty} V(G_j) \) and \( E(H) = \bigcup_{j=1}^{\infty} E(G_j) \cup \{ (v^{(j,i)}, v^{(i,j)}); i, j \in \mathbb{N}, i \neq j \} \). Then \( H \supseteq K_{\mathbb{N}_0} \).

**Proof.** In 1 and 2 we shall define for every \( n \in \mathbb{N} \) a subdivision \( K_n^* \) of \( K_n \) such that \( K_n^* \subseteq K_{n+1}^* \subseteq H \). Then obviously \( \bigcup_{n=1}^{\infty} K_n^* \) is a subdivision of \( K_{\mathbb{N}_0} \) in \( H \).

1. Let \( K_1^* = a_1 \) for \( a_1 \in V(G_1) \) such that \( a_1 = v^{(1,1)} \) for infinitely many \( j \in \mathbb{N} \).

2. Let \( K_n^* \) be already defined such that for the main vertices \( a_1, \ldots, a_n \) of \( K_n^* \) the following holds: There is an infinite \( J \subseteq \mathbb{N} \) such that \( a_m = v^{(s_m,j)} \) for every \( j \in J \) and certain \( s_m \in \mathbb{N} \) \((m = 1, \ldots, n)\). Choose a \( G_j \) with \( K_n^* \cap G_j = \emptyset \). Further, pick \( a_{n+1} \in V(G_j) \) and an infinite \( J' \subseteq J \) such that \( a_{n+1} = v^{(l,j)} \) for every \( j \in J' \). Now find \( n \) different \( G_{j(k)} \) with \( j(k) \in J' \) and \( G_{j(k)} \cap K_n^* = \emptyset \) \((k = 1, \ldots, n)\). Because \( G_{j(k)} \) is connected we have: There is a path \( W(a_{k}, a_{n+1}) \) in \( W_k \) such that the inner vertices of \( W_k \) belong to \( G_{j(k)} \). Hence \( K_{n+1}^* = K_n^* \cup \bigcup_{k=1}^{n} W_k \) is a subdivision of \( K_{n+1} \). This completes the proof of Lemma 3.

**Theorem.** Every countable, connected graph \( A \) containing no 1-path is regular.

**Proof.** (I) Since every finite graph is regular, suppose \( A \) is infinite. Let \( G \) be a graph that contains for every \( n \in \mathbb{N} \) a system \( \mathcal{A}_n = \{ A^{(n,m)}: m = 1, \ldots, n \} \) of \( n \) disjoint subdivisions of \( A \). Let \( \mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n \). We have to show that \( G \supseteq \mathbb{N}_0 A \). Thus by Lemma 2 we can assume without loss of generality that for every \( A^{(n,m)} \in \mathcal{A} \), there is a finite connected subgraph \( T^{(n,m)} \) of \( A^{(n,m)} \) and a sequence of different \( C^{(n,m)}_i \subseteq \mathcal{G}(A^{(n,m)} - T^{(n,m)}) \) \((i \in \mathbb{N})\) such that:

1. \( \mathcal{A}^{(n,m)} = \{ A^{(q,p)}(i) \in \mathcal{A} : A^{(q,p)}(i) \cap \bigcup_{j=1}^{n} C^{(n,m)}_j = \emptyset \} \) is an incomplete subsystem of \( \mathcal{A} \).

2. \( \mathcal{A}^{(n,m)} = \{ A^{(q,p)}(i) \in \mathcal{A} : A^{(q,p)}(i) \cap C^{(n,m)}_i \neq \emptyset \} \) is an incomplete subsystem of \( \mathcal{A}(i = 1, 2, \ldots) \).

(II) In 1 and 2 we shall define a subgraph \( G' \) of \( G \) which is isomorphic to a subdivision of a graph \( H \) having the structure described in Lemma 3. Then by Lemma 3 and since \( A \) is countable: \( \mathbb{N}_0 A \subseteq K_{\mathbb{N}_0} \subseteq H \subseteq G \). Thus the theorem is proved.

1. \( G_1 = T^{(1,1)} \).
2. Assume that we have already defined disjoint graphs $G_j = T^{(q_j, p_j)} (j = 1, \ldots, n)$. Assume furthermore that there is an $I \subseteq \{(i, j) : 1 \leq i < j \leq n\}$ such that for every $(i, j) \in I$ we have defined a $(G_i, G_j)$-path $W(i, j)$ for which the following holds:

\((*)\) No inner vertex of $W(i, j)$ belongs to $\bigcup_{k=1}^{n} G_k$ or to a path $W(k, m)$ different from $W(i, j)$.

Since $F = \bigcup_{i=1}^{n} G_i \cup \bigcup_{(i, j) \in I} W(i, j)$ is finite, $C = \bigcup_{j=1}^{n} \{ C^{(q_j, p_j)}_i : F \cap C^{(q_j, p_j)}_i \neq \emptyset \}$ is finite. Hence by (2)

$$\mathcal{D}_1 = \{ A^{(q, p)} \in \mathcal{A} : A^{(q, p)} \cap C \neq \emptyset \text{ for a } C \in \mathcal{C} \}$$

is an incomplete subsystem of $\mathcal{A}$. Furthermore,

$$\mathcal{D}_2 = \{ A^{(q, p)} \in \mathcal{A} : A^{(q, p)} \cap F \neq \emptyset \}$$

is an incomplete subsystem of $\mathcal{A}$. By (1),

$$\mathcal{D}_3 = \bigcup_{j=1}^{n} \mathcal{A}^{(q_j, p_j)}_j$$

is an incomplete subsystem of $\mathcal{A}$.

**Case 1.** $I = \{(i, j) : 1 \leq i < j \leq n\}$. Then choose $A^{(q_{n+1}, p_{n+1})} \in \mathcal{A} \setminus \mathcal{D}_2$ and let $G_{n+1} = T^{(q_{n+1}, p_{n+1})}$. Then $G_{n+1} \cap F = \emptyset$.

**Case 2.** $I \subset \{(i, j) : 1 \leq i < j \leq n\}$. Then choose $A' \in \mathcal{A} \setminus \bigcup_{k=1}^{n} \mathcal{D}_k$. Pick a certain pair $(i, j) \notin I(1 \leq i < j \leq n)$. From $A' \notin \mathcal{D}_3$ it follows: There is a $C^{(q, p)}_i$ and a $C^{(q, p)}_j$ with $A' \cap C^{(q, p)}_i \neq \emptyset$ and $A' \cap C^{(q, p)}_j \neq \emptyset$. Since $A'$ is connected, it follows that there is a $(G_i, G_j)$-path $W(i, j)$ such that all inner vertices of $W(i, j)$ are contained in $C^{(q, p)}_i \cup A' \cup C^{(q, p)}_j$. Since $A' \notin \mathcal{D}_2$, $A' \cap F = \emptyset$ holds. Since $A' \notin \mathcal{D}_1$, $C^{(q, p)}_i \cap F \neq C^{(q, p)}_j \cap F = \emptyset$ holds. Thus $W(i, j)$ satisfies ($*$).

3. By 1 and 2 a sequence $G_n(n \in \mathbb{N})$ and corresponding $(G_i, G_j)$-paths $W(i, j)$ are defined such that $G' = \bigcup_{n=1}^{n} G_n \cup \bigcup_{i<j} W(i, j)$ is a subdivision of a graph $H$ as in Lemma 3. This completes the proof of our theorem.

**4. Some examples**

Considering the theorem of the present paper one may ask, if the statement remains true if one drops one of the three conditions on $A$. The following three examples show that the answer is negative: The theorem becomes false if only one of the three conditions is dropped.

**Example 1.** The graph in Fig. 1 is an example for a countable, connected graph containing a $1$-path and being not regular. (For the proof see [1].)
Example 2. We give an example for a countable, disconnected, nonregular graph $A$ containing no 1-path.

Construction of $A$ (see Figs. 2 and 3). Let $C_n (n \in \mathbb{N})$ be a sequence of disjoint graphs defined as follows:

$$V(C_n) = \{a^n_0, \ldots, a^n_m, b^n_1, \ldots, b^n_n, c^n_1, \ldots, c^n_4, d^n_1, \ldots, d^n_4\}$$

and

$$E(C_n) = \{(a^n_i, a^n_{i+1}) : i = 0, \ldots, n-1\} \cup \{(a^n_i, b^n_i) : i = 1, \ldots, n\} \cup \{(a^n_i, c^n_i), (a^n_i, d^n_i) : i = 1, \ldots, 4\}.$$ 

Let $C_0$ be a graph that arises from $(C_n)_{n \in \mathbb{N}}$ as follows: Let $C_n = C_0 \cup \{c^n_i \cup \{d^n_i : i = 1, \ldots, 4\}\}$, identify the vertices $a^n_i (n \in \mathbb{N})$ and let $C_n = \bigcup_{i=0}^{n} C_n$. Let $A = \bigcup_{n=0}^{\infty} C_n$.

Construction of $G$. Now we construct a graph $G$ such that $nA \subseteq G$ for every $n \in \mathbb{N}$ but not $\aleph_0 A \subseteq G$. Let $C_{n,j} (j = 1, 2, \ldots)$ be a sequence of disjoint copies of $C_n$ such that $C_{n,j} \cap C_0 = \emptyset$. The vertices of $C_{n,j}$ will be called $a^{n,j}_i, b^{n,j}_i$ etc. Let $D_n$ be the graph that arises from $C_0$ and $(C_{n,j})_{j \in \mathbb{N}}$ by identifying $c^{n,j}_i \in C_{n,j}$ with $c_i \in C_0 (j = 1, 2, \ldots)$. Let

$$G = \bigcup_{n=1}^{\infty} (D_n \cup nC_n).$$

The proof that $A$ and $G$ have the asserted properties will be left to the reader.

Example 3. Let $A'$ be the graph in Fig. 2 of [3]. In the following we shall use the notations given in Fig. 2 of [3]. Let $A^\ast$ be the graph that arises from $A'$ by adding $\aleph_2$ new vertices to $A'$ and joining them to $a \in V(A')$ by edges. Let $A$ be the graph that arises from $A^\ast$ by replacing every edge of $A^\ast$ by "an edge of thickness $\aleph_1"$

Fig. 2.

Fig. 3.
i.e.: Let $e = (x, y) \in E(A^*)$; then drop $e$ and join $x$ and $y$ by $\aleph_1$ disjoint paths of length 2. Then $A$ is a connected, uncountable, nonregular graph without 1-path. For the proof one has to construct $G$ analogously to the construction of $G$ in Example 3.9 in [3]. Again we leave all further details to the reader.

5. Corrigendum

We would like to correct the errors that occur in the forerunner of the present paper ([3]).

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<td>$\mathcal{B}_{n-1}, \mathcal{B}_1^{(1)}$</td>
<td>$\mathcal{B}_{n-1}, \mathcal{B}_1^{(1)}$</td>
</tr>
<tr>
<td>21</td>
<td>$\mathcal{B}_{n+1}$</td>
<td>$\mathcal{B}_{n+1}$</td>
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<tr>
<td>p. 7, 29</td>
<td>$\mathcal{B}_1^{(k)}$</td>
<td>$\mathcal{B}_1^{(k)}$</td>
</tr>
<tr>
<td>30</td>
<td>$\mathcal{B}_1^{(k)}$</td>
<td>$\mathcal{B}_1^{(k)}$</td>
</tr>
<tr>
<td>p. 8, 5</td>
<td>$\mathcal{B}_j$</td>
<td>$\mathcal{B}_j$</td>
</tr>
<tr>
<td>17</td>
<td>$\mathcal{B}_j^{*}$</td>
<td>$\mathcal{B}_j^{*}$</td>
</tr>
<tr>
<td>26</td>
<td>$s$</td>
<td>$s_0$</td>
</tr>
<tr>
<td>p. 9, 16</td>
<td>$\mathcal{B}_m := \mathcal{B}_m \setminus \mathcal{S}_m$</td>
<td>$\mathcal{S}_m := \mathcal{B}_m \setminus \mathcal{S}_m$</td>
</tr>
<tr>
<td>17</td>
<td>$\bar{D}^{(n,m)} := \bigcup_{v \in \mathcal{S}_m} V$</td>
<td>$\bar{D}^{(n,m)} := \bigcup_{v \in \mathcal{S}_m} V$</td>
</tr>
<tr>
<td>23, 24, 30</td>
<td>$\bar{D}$</td>
<td>$\bar{D}^{(n,m)} := \bigcup_{v \in \mathcal{S}_m} V$</td>
</tr>
</tbody>
</table>

References

On a problem of R. Halin, II


