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The M5-brane anomaly

A. Boyarsky [a,1], B. Kulik [b]

[a] *Spinoza Institute, Leuvenlaan 4, 3584 CE, Utrecht, The Netherlands*

[b] *C.N. Yang Institute for Theoretical Physics, SUNY at Stony Brook, Stony Brook, NY 11794-3840, USA*

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Abstract

The problem of the M5-brane anomaly cancellation is addressed. We reformulate FHMM construction [D. Freed et al., Adv. Theor. Math. Phys. 2 (1998) 601] making explicit the relation with the M5-brane SUGRA solution. We suggest another solution to the magnetic coupling equation which does not need anomalous $SO(5)$ variation of the 3-form potential and coincides with the SUGRA solution outside smoothed out core of the magnetic source. Chern–Simons term evaluated on this solution generates the same anomaly inflow as achieved by FHMM.

The potential anomaly of the normal bundle in the presence of the M5-brane has three contributions. The first one comes from a chiral theory of zero-modes on the brane world volume, second comes from the Chern–Simons coupling $\int G_4 \wedge I_7$, where G_4 is the 4-form field strength and I_7 is a gravitational Chern–Simons. The problem of finding the third contribution that cancels the previous two is addressed in [1] in the following way.

Instead of considering 11-d SUGRA in the background of the 11-d M5-brane solution the other theory is studied. It's 11-d SUGRA with a 4-form field-strength satisfying modified Bianchi identity corresponding to the (singular) magnetic coupling to a 6-dimensional submanifold (which represents the 5-brane in this picture).

To have a well-defined delta-function in the magnetic coupling equation the source is smoothed out and the 4-form G satisfies

$$dG = d\rho(r) \wedge e_4/2, \tag{1}$$

where ρ is a bump function. It equals to -1 on the brane ($r = 0$) and to 0 far away from the brane, e_4 is an angular form on the $SO(5)$ normal bundle. It is closed ($de_4 = 0$) and gauge invariant under $SO(5)$ transformations of the normal bundle. Locally $e_4 = de_3$. The tubular neighborhood of the brane is removed and the resulting effective action in the bulk is

$$\mathcal{L} = \lim_{\epsilon\to 0} \int\limits_{M_{11}-D_\epsilon(W_6)} \mathcal{L}_{\mathrm{SUGRA}}. \tag{2}$$

In [1] it is argued that a general solution to Eq. (1) is

$$G_4 = dC_3 + A\rho e_4/2 - B\, d\rho \wedge e_3/2. \tag{3}$$

The requirement for G to be regular at the origin gives $A = 0$ and $B = 1$ (since e_4 is not well-defined at

E-mail addresses: boyarsky@phys.uu.nl (A. Boyarsky), bkulik@insti.physics.sunysb.edu (B. Kulik).

[1] On leave of absence from Bogoliubov ITP, Kiev, Ukraine.

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the origin, but $d\rho = 0$ at $r = 0$). Thus

$$G_4 = dC_3 - d\rho \wedge e_3/2. \tag{4}$$

Since G should be gauge invariant under $SO(5)$ transformations but e_3 is not ($\delta e_3 = de_2^{(1)}$) C has a gauge variation

$$\delta C = -d\rho \wedge e_2^{(1)}/2. \tag{5}$$

This leads to an anomalous variation of the modified Chern–Simons term in the action Eq. (2) and produces the necessary anomaly inflow from the bulk. Chern–Simons should be modified since the relation between G_4 and dC_3 has changed.

To maintain $dS_{\mathrm{CS}} =$ (a closed form) Chern–Simons can be built out of

$$G_4 - \rho e_4/2 = d(C_3 - \rho e_3/2). \tag{6}$$

This is the choice of [1]. Thus the new Chern–Simons term is

$$S_{\mathrm{CS}} = \int\limits_{M_{11}-D_\epsilon(W_6)} (C_3 - \rho e_3/2) \wedge d(C_3 - \rho e_3/2) \wedge d(C_3 - \rho e_3/2). \tag{7}$$

In this approach the 5-brane is considered as a magnetic source for the 3-form and not as a solution to $D = 11$ SUGRA. "... the very important question of the relation of this approach to that based on a direct study of solutions to supergravity" [1] is left for the future.

Let us compare Eq. (4) with the background M5-brane solution [2] [2]

$$ds^2 = \Delta^{-1/3}\eta_{MN}\, dx^M\, dx^N + \Delta^{2/3}\delta_{mn}\, dy^m\, dy^n, \tag{8}$$

$$G = \frac{1}{4!}\,\delta^{mn}\partial_m \Delta\bar{\varepsilon}_{npqrs}\, dy^p \wedge dy^q \wedge dy^r \wedge dy^s, \tag{9}$$

$$\Delta = 1 + \left(\frac{R}{r}\right)^3, \quad r = \sqrt{\delta_{mn} y^m y^n}. \tag{10}$$

Here Δ is a harmonic function such that $\Box\Delta = \delta(r)$ that is $(r^4\Delta')' = \delta(r)$ (prime $'$ denotes a derivative with respect to r). The 4-form G in the solution can be rewritten in the form

$$G = f(r)e_4/2 \tag{11}$$

(it is shown in Appendix A), where $f(r) = 1$ for $r > 0$ and it jumps at $r = 0$ since $f(r)' = \delta(r)$.

Therefore, the background solution of SUGRA for 5-brane satisfies the equation

$$dG = \delta(r)e_4/2. \tag{12}$$

Let us regularize the delta function in the spirit of [1], i.e., find a corresponding solution of the magnetic coupling Eq. (1).

Away from the brane G has the form $e_4/2$ for a magnetic brane with a charge 1. We see that dC in Eq. (3) should be equal to e_4 for the background solution. Thus C exists only locally and is equal to e_3 (up to a closed form). Therefore, the general solution of magnetic coupling equation $dG = d\rho \wedge e_4/2$ can better be written in the form

$$G = d\widetilde{C} + A\rho \wedge e_4/2 - B\, d\rho \wedge e_3/2 + (\text{closed 4-form}), \tag{13}$$

where $\widetilde{C}$ is already globally defined. To satisfy the asymptotic behavior of the background solution we have to take this closed 4-form to be e_4 (and $d\widetilde{C} = 0$ for background). Then if we set $A = 1$ and $B = 0$ we have

$$G = d\widetilde{C} + (\rho + 1)e_4/2. \tag{14}$$

It is still regular at $r = 0$ since $\rho(0) = -1$. In this case C does not have any anomalous variation under $SO(5)$ transformations since e_4 is gauge invariant.

Therefore, we just showed that there is a solution of the magnetic coupling equation which coincides with the classical solution to SUGRA outside a tubular neighborhood of the M5-brane (where $\rho = 0$) and is smooth near the brane. Let's see what anomaly inflow such a choice of the solution leads to. Evaluated on the class of solutions

$$G = d\widetilde{C} + e_4/2 \tag{15}$$

the gauge variation of the Chern–Simons has necessary surface term. Indeed, the potential for Eq. (15) can be defined locally as $C = \widetilde{C} + e_3/2$. Thus the variation of the Chern–Simons is

$$\delta S_{\mathrm{CS}} = \delta \int\limits_{M_{11}-D_\epsilon(W_6)} \big(\widetilde{C} + e_3/2\big) \wedge d\big(\widetilde{C} + e_3/2\big) \wedge d\big(\widetilde{C} + e_3/2\big) \tag{16}$$

[2] By $\bar{\varepsilon}_{npqrs}$ we denote a flat 5-d antisymmetric symbol. The 11-d index μ is split into (M, m), where M is in the direction of the brane world volume and m is in the direction transverse to the brane.

$$= \int_{M_{11}-D_\epsilon(W_6)} de_2^{(1)}/2 \wedge \bigl(d\widetilde{C} + e_4/2\bigr) \wedge \bigl(d\widetilde{C} + e_4/2\bigr). \qquad (17)$$

Integrating by parts and taking the limit and using the fact that $\widetilde{C}$ is smooth near the brane we obtain

$$\delta S_{\mathrm{CS}} = \int_{S_\epsilon(W_6)} e_2^{(1)}/2 \wedge e_4/2 \wedge e_4/2. \qquad (18)$$

This is in accord with [1]. The anomaly inflow is generated without any smoothing out of the magnetic source.

As for the case of the regularized solution of Eq. (14), the Chern–Simons should be modified since G is not a closed form. One can make the same choice as in [1], namely, take a closed form $G - \rho e_4/2$ instead (see Eq. (6)). In this case $G - \rho e_4/2 = e_4/2 + d\widetilde{C}$ and the modified Chern–Simons reads

$$S_{\mathrm{CS}} = \int_{M_{11}-D_\epsilon(W_6)} \bigl(\widetilde{C} + e_3/2\bigr) \wedge d\bigl(\widetilde{C} + e_3/2\bigr) \wedge d\bigl(\widetilde{C} + e_3/2\bigr). \qquad (19)$$

This is the Chern–Simons (see Eq. (16)) evaluated on the class of M5-brane solution Eq. (15).

Note that the function ρ introduced for the regularization of the magnetic source does not enter in the modified Chern–Simons.

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Appendix A. Derivation of $G = f(r)e_4/2$

The value of G in the solitonic solution of Eq. (9) can be written as

$$G = \frac{1}{4!}\bar{\varepsilon}_{\mu\nu\gamma\lambda\rho}\delta^{\rho\sigma}\partial_\sigma \Delta\, dy^\mu \wedge dy^\nu \wedge dy^\gamma \wedge dy^\lambda = \bar{*}n, \quad n = d\Delta. \qquad (A.1)$$

We denote by $\bar{*}$ a 5-d (transverse) dual with respect to the flat metric. We want to show that a geometrical meaning of $\bar{*}n$ from the point of view of the embedding $W^6 \subset M^{11}$ is $\bar{*}n = f(r)e_4$, where $f(r) = 128\pi^2 r^4 \Delta'$. In [1] e_4 was

$$e_4 = \frac{1}{4!}\frac{1}{64\pi^2}\bar{\varepsilon}_{klmnp}\hat{y}^k\, d\hat{y}^l \wedge d\hat{y}^m \wedge d\hat{y}^n \wedge d\hat{y}^p, \qquad (A.2)$$

where all indexes are contracted with the flat metric, $|y| = r$, $\hat{y}^k = y^k/r$ and $d\hat{y}^k = dy^l/r(\delta_l^k - \delta_{lm}y^m y^k/r^2)$. In the angular form e_4 there is in general some part that depends on the connection in the normal bundle. In the background solution it is obviously zero (it can also be easily checked by direct calculations). To find a relation between the normal form n and the angular form e_4 we rewrite e_4 in the basis of dy^m instead of $d\hat{y}^m$

$$e_4 = \frac{1}{4!}\frac{1}{64\pi^2}\bar{\varepsilon}_{klmnp}\frac{1}{r^5}y^k\, dy^{l'} \wedge dy^{m'} \wedge dy^{n'} \wedge dy^{p'} D^l_{l'} D^m_{m'} D^n_{n'} D^p_{p'}, \qquad (A.3)$$

where

$$D^n_m \equiv \delta^n_m - y_m y^n/r^2, \qquad (A.4)$$

$$d\hat{y}^m = \frac{1}{r}D^m_n\, dy^n. \qquad (A.5)$$

Since $y^{[m}y^{n]} = 0$ only the first term in each of D^n_m contributes in Eq. (A.3)

$$e_4 = \frac{1}{4!}\frac{1}{64\pi^2}\bar{\varepsilon}_{klmnp}\frac{1}{r^5}y^k\, dy^l \wedge dy^m \wedge dy^n \wedge dy^p. \qquad (A.6)$$

This expression should be compared to $\bar{*}n$

$$n = d\Delta,$$
$$\bar{*}n = \frac{1}{4!}\delta^{rk}\bar{\varepsilon}_{rlmnp}\partial_k\Delta\, dy^l \wedge dy^m \wedge dy^n \wedge dy^p. \qquad (A.7)$$

Thus for $f(r)e_4/2$ to be equal to $\bar{*}n$ the function $f(r)$ should be taken as

$$f(r)y^m/\bigl(128\pi^2 r^5\bigr) = \delta^{mn}\partial_n\Delta(r) = \Delta' y^m/r. \qquad (A.8)$$

It implies

$$f(r) = 128\pi^2 r^4 \Delta'(r). \qquad (A.9)$$

This is what we intended to prove. Since $r^4\Delta' = \mathrm{const}$ for $r > 0$ parameter R in Δ can be chosen such that $f(r) = 1$.

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