Discrete Applied Mathematics 44 (1993) 305-310
North-Holland

Covering the complete graph with plane cycles

Alan Hartman and Yoav Medan*

IBM Israel Science and Technology and Scientific Center, Haifa, Israel

Received 15 April 1990
Revised 5 January 1991

Abstract

Hartman, A. and Y. Medan, Covering the complete graph with plane cycles, Discrete Applied Mathematics 44 (1993) 305-310.

Let n vertices be distributed on the circumference of a circle in the plane. We find, for every n, the minimum number of cycles with no crossing edges such that every pair of vertices is adjacent on at least one cycle. The problem arises from the design of a train shuttle service between n cities with continuous guaranteed service at all times, and minimum number of rail lanes.

Introduction

A train shuttle service between n metropolitan cities has to be established. Unlike scheduled service, which can afford reuse of the same rail tracks for mutually exclusive scheduled trains, a shuttle service requires a guaranteed free city-to-city rail track connection at all times. Each shuttle terminal (vertex) has at least $n-1$ service gates to provide at least one direct connection link to each of the $n-1$ possible destinations.

If the cities are located on the circumference of a circle, then the problem is to assign each of the inter-city links to a rail track segment such that the total number of rail track circles (rail width) is minimized.

We denote the complete graph with n vertices by K_n. All undefined terminology and notation is taken from [1], however we use the notation $(v_1, v_2, \ldots, v_j)$ to denote the cycle which, in the notation of [1] would be denoted by $(v_1, v_2, \ldots, v_j, v_1)$. We consider the n vertices embedded in the plane on the circumference of a circle. A

Correspondence to: Dr. A. Hartman, IBM Science and Technology, MATAM, Haifa 31905, Israel. E-mail: Hartman@haifasc3.vnet.ibm.com

* E-mail: Medan@israearn.bitnet.

0166-218X/93/$06.00

cycle is *plane* with respect to an embedding if the straight-line segments joining its vertices intersect only at the vertices. Let m_n denote the minimum number of plane cycles which cover the edges of K_n.

There is an extensive literature on covering the edges of K_n with cycles. In the last century (see [3]), it was shown that K_n can be covered by $\lfloor n/2 \rfloor$ Hamilton cycles, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Thus $m_n \geq \lfloor n/2 \rfloor$, but the Hamilton cycle decomposition can only have one of its cycles plane. At the other extreme, every 3-cycle is a plane, and Fort and Hedlund [2] showed that there exists a 3-cycle cover of K_n with $\lceil n/3 \lceil (n-1)/2 \rceil \rceil$ 3-cycles, where $\lceil x \rceil$ is the least integer greater than or equal to x. So this provides an upper bound for m_n.

In this paper we prove the following.

Theorem. *For all* $n \geq 3$, *if* n *is odd then* $m_n = (n^2-1)/8$, *and if* n *is even then* $m_n = \lceil (n^2+1)/8 \rceil$.

The lower bound

Let the vertices be numbered $0, 1, \ldots, n-1$ around the circle and let $c = (v_1, v_2, \ldots, v_{j-1}, v_j)$ with $v_1 < v_2 < \cdots < v_j$ be a plane cycle. We will denote the sum $(v_2 - v_1) + (v_3 - v_2) + \cdots + (v_j - v_{j-1}) + (n + v_1 - v_j)$ by $\Sigma(c)$. Clearly $\Sigma(c) = n$ for all plane cycles.

Now sum $\Sigma(c)$ over all plane cycles in a cover of the complete graph with m plane cycles. The total is clearly mn. Now rearrange terms—since each edge occurs at least once we have $mn \geq n(1+2+\cdots+(n-1)/2)$ when n is odd, since there are n edges which contribute either j or $n-j$ for each $j = 1, 2, \ldots, (n-1)/2$. Thus for odd $n = 2k+1$ we have

$$m_{2k+1} \geq \frac{k^2+k}{2}. \tag{1}$$

When n is even—note that, as well as each edge appearing at least once, there must be at least $n/2$ edges which appear at least twice—since any multigraph whose edges are formed from a set of cycles has every vertex of even degree, and every vertex of K_n has odd degree. Now repeating the above argument we have $mn \geq n(1+2+3+\cdots+(n-2)/2) + (n/2)(n/2) + n/2$, since there are n edges which contribute j or $n-j$ for all $j = 1, 2, \ldots, (n-2)/2$, there are $n/2$ edges which contribute $n/2$ to the sum (the diameters), and an additional $n/2$ edges which contribute at least 1 to the sum (the repeated edges). This gives the bound

$$m_{2k} \geq \left\lceil \frac{k^2+1}{2} \right\rceil. \tag{2}$$

We define the *excess multigraph* of a plane cycle cover to be the graph with edges formed from the union of the cycles and then removing one copy of each edge of the complete graph. If the number of vertices, n, is odd then a cover meeting the bound in (1) has an excess multigraph with no edges, and each edge contributes $j<n/2$ to the sum. If n is even, then the excess multigraph has every vertex of odd degree, and thus must contain at least $n/2$ edges. If a plane cycle cover with $n\equiv 2 \pmod 4$ meets the bound in (2), this implies that the number of edges in the excess multigraph is precisely $n/2$ and each of these edges has the form $i, i+1$. Moreover each edge contributes $j\leq n/2$ to the sum. If $n\equiv 0 \pmod 4$ in a cover meeting the bound in (2) then the number of edges in the excess multigraph is not uniquely determined. The minimum value of the total sum is exceeded by n, but this can be made up of contributions from edges in the excess multigraph and also possibly from edges covered precisely once, but contributing more than $n/2$.

The upper bound

We will show that in all cases, the lower bound can be achieved by the following constructions of plane cycle covers.

A minimal plane cycle cover for odd n. Let $n=2k+1$ and label the points of the circle $0, 1, \ldots, 2k$ in order round the cycle. Construct the k 3-cycles $(0, i, k+i)$ for $1\leq i\leq k$. Now construct the following 4-cycles $(i, i+j, i+k, i+j+k)$ for $1\leq i\leq k-j$, and $1\leq j\leq k-1$.

It is not difficult to verify that this is indeed a plane cycle cover with $(k^2+k)/2$ cycles, and thus by (1), it is minimal.

Example 1. When $n=5$ we obtain the following minimum plane cycle cover: (0, 1, 3), (0, 2, 4), (1, 2, 3, 4).

Example 2. When $n=7$ we obtain the following minimum plane cycle cover: (0, 1, 4), (0, 2, 5), (0, 3, 6), (1, 2, 4, 5), (2, 3, 5, 6), (1, 3, 4, 6).

A minimal plane cycle cover for $n\equiv 2 \pmod 4$. Let $n=4k+2$ and label the points of the circle $0, 1, \ldots, 4k+1$. We reduce the labels modulo $4k+2$ throughout the construction.

Construct the 3-cycles $(2i, 2i+1, 2i+2k+1)$ for $i=0, 1, \ldots, 2k$, and the 4-cycles $(i, i+j, i+2k+1, i+j+2k+1)$ for $i=0, 1, \ldots, 2k$, and $j=2, 3, \ldots, k$. The 4-cycles cover all edges of the form $x, x+y$ with $0\leq x<4k+2$, and $2\leq y\leq 2k-1$. The 3-cycles cover all edges with $y=1$ and x even, with $y=2k$ and x odd, and all those with $y=2k+1$.

To complete the cover we construct another k 4-cycles and a single cycle of length $2k+3$. The long cycle is $(0, 1, 2, \ldots, 2k+2)$ and the remaining 4-cycles are $(2k(2j), 2k(2j+1), 2k(2j+2), 2k(2j+2)+1)$ for $j=0, 1, \ldots, k-1$ reducing the values modulo $4k+2$.

This is a plane cycle cover with $2k^2+2k+1$ cycles, and thus by (2) it is minimal. Note that k is relatively prime to $2k+1$, so that the set of values $2k(2j)$, and $2k(2j+1)$ for $j=0,1,\ldots,k-1$ reduced modulo $4k+2$ covers the set of all the even integers $0,2,\ldots,4k$, with the sole exception of $2k+2$. This guarantees that the last $k+1$ cycles cover all the edges of the forms $2i, 2i+2k$ and $i, i+1$, since the edge $2k+2, 0$ is covered by the long cycle. The excess multigraph contains the edges $x, x+1$ with x even, each with multiplicity 1.

Example 3. When $n=6$ we obtain the following minimum plane cycle cover: (0, 1, 3), (2, 3, 5), (4, 5, 1), (0, 1, 2, 3, 4), (0, 2, 4, 5).

Example 4. When $n=10$ we obtain the following minimum plane cycle cover: (0, 1, 5), (2, 3, 7), (4, 5, 9), (6, 7, 1), (8, 9, 3), (0, 2, 5, 7), (1, 3, 6, 8), (2, 4, 7, 9), (3, 5, 8, 0), (4, 6, 9, 1), (0, 1, 2, 3, 4, 5, 6), (0, 4, 8, 9), (8, 2, 6, 7).

A minimal plane cycle cover for $n\equiv 0 \pmod 4$. For $n=4$ we have the cover (0, 1, 2, 3), (0, 2, 3), (1, 2, 3). For $n=8$ we have the cover (0, 2, 4, 6), (1, 3, 5, 7), (0, 4, 5), (1, 4, 5), (2, 6, 7), (3, 6, 7), (0, 3, 4, 7), (1, 2, 5, 6), (0, 1, 2, 3). For $n\geq 12$, let $n=4k$ with $k\geq 3$, and label the points of the circle $0, 1, \ldots, 4k-1$. We reduce the labels modulo $4k$ throughout the construction.

Construct the 4-cycles $(i, i+j, i+2k, i+2k+j)$ for $0\leq i<2k$, and $3\leq j<k$, and $(i, i+k, i+2k, i+3k)$ for $0\leq i<k$. These cycles cover all edges of the form $x, x+y$ with $0\leq x<4k$, and $3\leq y\leq 2k-3$.

Construct the 3-cycles $(2i+\varepsilon, 2i+2k, 2i+2k+1)$ for $i=0,1,\ldots,k-1$, and $\varepsilon=0,1$. These cycles cover all edges with $y=2k$, and with $y=2k-1$ and x odd. The edges with $y=1$ and $x\geq 2k$, x even are covered twice.

Now add the 4-cycles $(2i, 2i+2k-1, 2i+2k, 2i+4k-2)$ with $0\leq i<2k$, and the single $2k$-cycle $(1, 3, 5, \ldots, 4k-1)$. These cycles cover all edges with $y=2$, all the remaining edges with $y=2k-1$, and the edges with $y=1$ and x odd. The edges with $y=2k-2$ and x even are also covered.

To complete the construction we construct the following set of 6-cycles, $(2i-1, 2i, 2i+1, 2i+2k-1, 2i+2k+1)$ with $0\leq i<k$. These cycles cover all the edges with $y=2k-2$ and x odd, and all the edges with $y=1$. So the excess multigraph has edges $x, x+1$ with multiplicity 1 when x is odd, and multiplicity 2 when x is even and $2k\leq x\leq 4k-2$.

This is a plane cycle cover with $2k^2+1$ cycles, and thus by (2) it is minimal.

Example 5. When $n=12$ we obtain the following minimum plane cycle cover.

(0, 3, 6, 9), (1, 4, 7, 10), (2, 5, 8, 11), (0, 6, 7), (1, 6, 7), (2, 8, 9), (3, 8, 9),

(4, 10, 11), (5, 10, 11), (0, 5, 6, 10), (2, 7, 8, 0), (4, 9, 10, 2), (6, 11, 0, 4), (8, 1, 2, 6), (10, 3, 4, 8), (1, 3, 5, 7, 9, 11), (11, 0, 1, 5, 6, 7), (1, 2, 3, 7, 8, 9), (3, 4, 5, 9, 10, 11).

Example 6. When $n = 16$ we obtain the following minimum plane cycle cover.

(0, 3, 8, 11), (1, 4, 9, 12), (2, 5, 10, 13), (3, 6, 11, 14), (4, 7, 12, 15), (5, 8, 13, 0), (6, 9, 14, 1), (7, 10, 15, 2), (0, 4, 8, 12), (1, 5, 9, 13), (2, 6, 10, 14), (3, 7, 11, 15), (0, 8, 9), (1, 8, 9), (2, 10, 11), (3, 10, 11), (4, 12, 13), (5, 12, 13), (6, 14, 15), (7, 14, 15), (0, 7, 8, 14), (2, 9, 10, 0), (4, 11, 12, 2), (6, 13, 14, 4), (8, 15, 0, 6), (10, 1, 2, 8), (12, 3, 4, 10), (14, 5, 6, 12), (1, 3, 5, 7, 9, 11, 13, 15), (15, 0, 1, 7, 8, 9), (1, 2, 3, 9, 10, 11), (3, 4, 5, 11, 12, 13), (5, 6, 7, 13, 14, 15).

Directions for further research

In the above discussion of the train shuttle problem, we assumed that the cities were located on a circular rail network. We can generalize the problem as follows.

Let G and H be graphs with the same vertex set. A *G-shuttle* consists of a cycle $C = (v_1, v_2, \ldots, v_j)$ of G and a set of distinguished vertices of the cycle $R(C) = (v_{i1}, v_{i2}, \ldots, v_{ik})$ called the *stops* of the shuttle. The cycle of stops $R(C)$ of a G-shuttle is called its *route*.

A *G-shuttle cover* of H consists of a collection of G-shuttles with the property that every edge of H is an edge in at least one of the routes. Intuitively, the edges of H represent the vertices which require a direct shuttle connection, and G is the graph of possible rail links. If a vertex on a shuttle is not a stop then trains pass through this vertex without stopping.

The *size* of a G-shuttle cover is the number of shuttles. The *rail width* of a set of G-shuttles is the maximum over all edges of G of the number of shuttles containing an edge. The *vertex width* of a set of G-shuttles is the maximum over all vertices of G of the number of shuttles containing a vertex.

We generalize the problem solved above in three ways. For given graphs G and H with the same vertex set:

(1) Determine the minimum size of a G-shuttle cover of H. We will denote this quantity by $\mathrm{MS}(G, H)$.

(2) Determine the minimum rail width of a G-shuttle cover of H. We will denote this quantity by $\mathrm{MRW}(G, H)$.

(3) Determine the minimum vertex width $\mathrm{MVW}(G, H)$, of a G-shuttle cover of H.

Our Theorem determines the value of $\mathrm{MS}(C_n, K_n)$, using C_n for each of the shuttles, and the plane cycles as the set of routes. Since C_n is the only cycle in C_n we have $\mathrm{MS}(C_n, K_n) = \mathrm{MRW}(C_n, K_n) = \mathrm{MVW}(C_n, K_n)$. The Hamilton cycle decomposition of the complete graph shows that $\mathrm{MRW}(K_n, K_n) = 1$ if n is odd, $\mathrm{MRW}(K_n, K_n) = 2$ if n is even, and that $\mathrm{MS}(K_n, K_n) = \mathrm{MVW}(K_n, K_n) = \lfloor n/2 \rfloor$.

It would be interesting to characterize graphs G such that $\mathrm{MRW}(G, K_n) = \mathrm{O}(n^2)$ or $\mathrm{O}(n)$ or $\mathrm{O}(1)$, and similarly for $\mathrm{MVW}(G, K_n)$ and $\mathrm{MS}(G, K_n)$.

Analogous questions can be raised about paths rather than cycles, defining *path-G-shuttle*, *path route*, and *path cover* in the analogous way. Let P_n denote the path with n vertices. It is a simple exercise to show that the minimum number of path-P_n-shuttles which cover K_n is $\lfloor n^2/4 \rfloor$, and this is also the minimum rail width and vertex width.

The minimum rail width and minimum vertex width of a path shuttle cover are related to problems in the design of VLSI circuits, particularly when the graph G is a regular grid (see e.g. [4]). However problems in VLSI design are usually complicated by additional constraints on the admissible paths in a cover.

Acknowledgement

We would like to thank both the anonymous referees who gave valuable suggestions for the improvement of the presentation.

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Elsevier, New York, 1976).

[2] M.K. Fort and G.A. Hedlund, Minimal coverings of pairs by triples, Pacific J. Math. 8 (1958) 709–719.

[3] E. Lucas, Recréations Mathématiques Vol. 2 (Gauthier-Villars, Paris, 1890).

[4] M.P. Vecchi and S. Kirkpatrick, Global wiring by simulated annealing, IEEE Trans. Comput. Aided Design 2 (1983) 215–222.