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EVEN CYCLES WITH PRESCRIBED CHORDS IN PLANAR CUBIC GRAPHS

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The following result is being proved. *Theorem:* Let e be an arbitrary line of the 2-connected, 3-regular, planar graph G such that e does not belong to any cut set of size 2. Then G contains an even cycle for which e is a chord.

Introduction

With the help of the Four Colour Theorem it is fairly easy to show that the prism $P(G)$ of a 2-connected, 3-regular, planar graph G is Hamiltonian, [4]. To obtain the same result without using the Four Colour Theorem, a considerable effort is required, [1]. In fact, the material presented here is the positive outcome of an unsuccessful attempt to achieve that goal. That is, we prove the following theorem.

Theorem. *Let G be a 2-connected, 3-regular, planar graph. If e is a line of G such that $G-e$ is bridgeless, then there exists an even cycle for which e is a chord.*

In the proof of this theorem, use will be made of a fairly well known result about the nonexistence of a certain graph. This result was first proved by J.W. Moon, but it can be obtained as a consequence of a more general result as well, [2].

As for the terminology of this paper, we refer to [3] with the exception that we call a graph what is called a multigraph in [3]. That is, in our notion, a graph might contain multiple lines.

For a graph G having two totally disjoint cycles, the cyclic line-connectivity (shortly $\lambda_c(G)$) is defined as the size of the smallest line-cut S such that $G-S$ has at least two cyclic components.

Lemma. *If G is a 3-regular graph with at least four points and $K_4 \neq G \neq K_{3,3}$, then $\lambda_c(G)$ is defined.*

Proof (by contradiction). Clearly, for a 3-regular graph G having a bridge, $\lambda_c(G)=1$, i.e., $\lambda_c(G)$ is defined. Hence, a counterexample G to the Lemma must be bridgeless and—because of Petersen's Theorem—Hamiltonian. Let H be a Hamiltonian cycle of G and let ac, bd be two chords of H, $\{a, b, c, d\} \subset V(G)$. Since G is a counterexample, a, b, c, d must lie in this order on H. Since this is true for any two chords, it is immediate that G has two disjoint cycles if $|V(G)|>6$, and $G=K_4$, $G=K_{3,3}$ respectively, depending on whether $|V(G)|=4$ or $=6$. This proves the Lemma.

For an arbitrary graph G and $V_1 \subset V(G)$, the subgraph of G induced by V_1 is denoted by $\langle V_1\rangle$, that is, $V(\langle V_1\rangle)=V_1$, and $xy\in E(\langle V_1\rangle)$ iff $x, y\in V_1$ and $xy \in E(G)$.

For a face F of a plane graph, bd F denotes the boundary of F. The circumference of F equals $|E(\text{bd } F)|$. Finally, a path (cycle, respectively) in G is a sequence $v_1, v_1v_2, v_2, \ldots, v_{r-1}, v_{r-1}v_r, v_r$ with $v_i \in V(G)$, $i=1,\ldots,r$, $v_iv_{i+1}\in E(G)$, $i=1,\ldots,r-1$ and $v_j \neq v_k$ for $j\neq k$, $i\leq j$, $k\leq r$ ($1\leq j$, $k\leq r-1$ and $v_1=v_r$, respectively).

Proof of the Theorem. We give an indirect proof. If G has two points, or $G=K_4$, or $G=K_{3,3}$, then $G-e$ is Hamiltonian for every $e\in E(G)$, and the theorem holds in this case. Therefore, G must have at least four points, and $K_4\neq G\neq K_{3,3}$ must hold. By the Lemma, $\lambda_c(G)$ is defined. Next we eliminate the possibility $\lambda_c(G)=2$ where G is a counterexample to the theorem with as few points as possible. Thus, if G has a separating pair of lines $e_1=x_1x_2$, $e_2=y_1y_2$, then $G-\{e_1, e_2\}$ has exactly two components G_1' and G_2' (each of which contains a cycle). The notation can be chosen such that

$$G_i = G_i' \cup [x_i, y_i], \quad i=1,2$$

satisfies the hypothesis of the theorem and has fewer points than G. Since $e_1\neq e\neq e_2$ we assume $e\in E(G_1')$. Moreover, we can choose e_1, e_2 in such a way that G_2 is 3-line-connected: Among all possible choices for e_1, e_2 with $e\in E(G_1')$, take a pair with minimal $|E(G_2)|$. An even cycle C_1 having e as a chord exists in G_1 and must contain x_1y_1; otherwise, C_1 would be as required in G. On the other hand, G_2 contains an even cycle C_2 with $x_2z\neq x_2y_2$ as a chord; i.e., C_2 contains x_2y_2. (*Note*: G_2 has no separating pair of lines, and no separating pair of lines in G_1 contains e.) Therefore, each of the paths

$$P_i = C_i - \{x_iy_i\}, \quad i=1,2$$

has odd length, and obviously we have

$$E(P_1)\cap E(P_2)=\emptyset.$$

That is,

$$C=P_1\cup P_2\cup\{e_1, e_2\}$$

is an even cycle which, clearly, contains e as a chord. Thus, $\lambda_c(G)=2$ is impossible.

Now suppose $\lambda_c(G)=3$. Consider a separating line set consisting of exactly three lines $e_i=v_iw_i$, $i=1,2,3$, such that $G-\{e_1,e_2,e_3\}$ has (exactly) two components G_1^*, G_2^* each containing a cycle. Introduce the new points z_1, z_2 and form

$$G_i=G_i^*\cup\{z_i\}\cup\{z_iy_j \mid j=1,2,3\},\quad i=1,2$$

with $y_j=v_j$ for $i=1$ and $y_j=w_j$ for $i=2$ (w.l.o.g. $v_j\in G_1^*$).

Obviously G_i satisfies the hypothesis of the Theorem (since it is even three-connected) and has fewer points than G, $i=1,2$.

Case (a): *e does not belong to the above separating set.* Then, w.l.o.g., we can assume $e\in E(G_1^*)\subset E(G_1)$. An even cycle C_1 having e as a chord exists in G_1, and it passes through z_1; otherwise C_1 is already in G as required. Suppose w.l.o.g. $C_1\cap\{z_1v_j \mid j=1,2,3\}=\{z_1v_1, z_1v_2\}$. Applying now the Theorem to G_2 we find in G_2 an even cycle C_2 having z_1w_3 as a chord. Again, form paths (now of even length)

$$P_i=C_i-z_i,\quad i=1,2;$$

$$P_1\cap P_2=\emptyset \quad\text{and}\quad C=P_1\cup P_2\cup\{e_1,e_2\}$$

is an even cycle in G having e as a chord.

Case (b): *W.l.o.g.* $e=e_3$. Considering even cycles C_i in G_i with z_1v_3, z_2w_3 respectively, as chords and forming P_i, $i=1,2$, and C as in Case (a), we again obtain an even cycle in G having e $(=e_3)$ as a chord. Thus we can conclude that $\lambda_c(G)\geq 4$ must hold.

Now we consider a fixed embedding of G in the plane. For simplicity's sake we denote this embedding by the same letter G (this is no real loss of generality since the 3-connectedness of G implies its—basically—unique embedability). For the distinguished line e denote by F_1 and F_2 the faces having e as a boundary line. Suppose F_1 has even circumference and thus F_2 has odd circumference; otherwise e would be a chord of the even cycle $\text{bd}\,F_1\cup\text{bd}\,F_2-\{e\}$. Furthermore denote by $L_1,\ldots,L_{2k-1},\ldots,L_n$, $n\equiv 1\ (\text{mod}\ 2)$, the faces adjacent to F_1, F_2 respectively, in a cyclic order such that $L_1,\ldots,L_{2k-1}$ have a boundary line with F_1 in common. Because of $\lambda_c(G)\geq 4$ the above L_i's are n distinct faces; for the same reason, if $|i-j|>1$ and $L_i\cap L_j\neq\emptyset$, then L_i is adjacent to F_1 if and only if L_j is adjacent to F_2, and neither of them is adjacent to both F_1, F_2. Each of these L_i, $1\neq i\neq 2k-1$, must have even circumference; otherwise, $\text{bd}\,L_i\cup\text{bd}\,F_1\cup\text{bd}\,F_2$ contains an even cycle with e as a chord. Similarly,

$$|E(\text{bd}\,L_1)|\equiv 1\ (\text{mod}\ 2)\quad\text{if and only if}\quad |E(\text{bd}\,L_{2k-1})|\equiv 0\ (\text{mod}\ 2).$$

Thus we have arrived at the following situation: G contains a set of faces, namely $F_1, F_2, L_1,\ldots,L_n$, such that F_2 and (w.l.o.g.) L_1 have odd circumference while $F_1, L_2,\ldots L_n$ have even circumference; and F_2 and L_1 are adjacent (see Fig. 1).

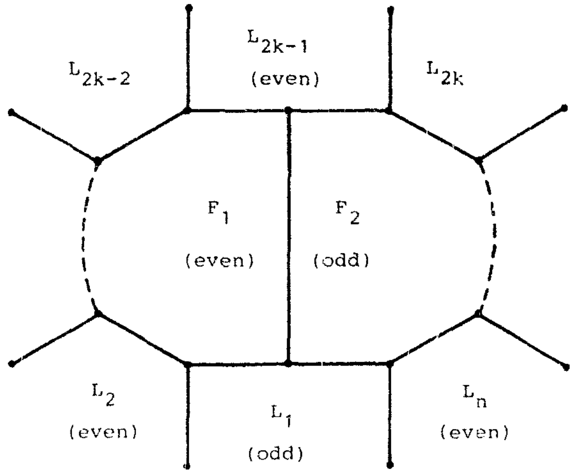

Fig. 1.

Now we conclude that G must still contain another face F_0 ($\neq F_2, L_1$) with odd circumference: Otherwise, the dual $D(G)$ is a triangulation of the plane with exactly two points of odd degree which are adjacent; however, such triangulation does not exist, [2].

By the above parity considerations, F_0 is not adjacent to F_2. In order to finish the proof we want to find a path P' in $D(G)$ with the following properties:

(1) P' contains F_2 and exactly one other point F_0 of odd degree in $D(G)$ which is an endpoint of P'.

(2) F_1 belongs to P'.

(3) $\langle V(P')\rangle = P'$.

(4) Subject to (1), (2), (3), the length of P' is minimal.

Clearly, if (1), (2), (3) can be fulfilled, then (4) can be fulfilled as well.

Once such P' has been found, an even cycle as required can be constructed; namely: Because of (3), those boundary lines of the faces corresponding to the points of P' which belong to exactly one of these faces, form a cycle C. By (1), C is even; and by (2), $e \notin E(C)$ but it is a chord of C as are all those boundary lines corresponding to the lines of P'. (*Note*: Because of (3), $F_1, F_2 \in V(P')$ implies $L_1, L_{2k-1} \notin V(P')$ which in turn implies together with (4) that $\operatorname{bd} F_i - \{e\} \subset C$ for $i = 1$ or $i = 2$, depending on whether P' has F_1 or F_2 as an endpoint.)

Because of $\lambda_c(G) \geq 4$, $D(G)$ is 4-connected. (*Note*: $\lambda_c(G) \geq 4$ implies $\kappa(G) = 3$; hence $\kappa(D(G)) \geq 3$. However, $\kappa(D(G)) = 3$ implies the existence of a separating triangle in the triangulation of the plane $D(G)$, which in turn implies $\lambda_c(G) = 3$, an obvious contradiction.) Therefore, for every point of odd degree $F_0^* \neq F_2, L_1$ in $D(G)$, there is a path $P^* = P(F_2, F_0^*)$ in $D(G)$ not containing L_1 and L_{2k-1}. For each of the possible choices of P^* consider $\langle V(P^*) \cup \{F_1\}\rangle$. Each of these induced subgraphs contains a shortest path $\bar{P}$ from F_2 to a point F_0 which has odd degree in $D(G)$. $\bar{P}$ satisfies property (1). Consider $G' = \langle V(\bar{P}) \cup \{F_1\}\rangle$.

If G' is a path, then we can take $P' = G'$ as a path satisfying properties (1), (2), (3). By the above, we could find C as required. So we have to assume G' not to be a path. However, because $\bar{P}$ is a shortest path, $\langle \bar{P} \rangle$ is a path, i.e., the lines of G' not contained in the path

$$\bar{\bar{P}} = F_1, F_1F_2, \bar{P}$$

join F_1 with points of $\bar{P}$. If $\bar{P}$ is written in the form

$$\bar{P} = F_2, F_2T_1, T_1, \ldots, T_r, T_rF_0, F_0,$$

then F_1T_1, $F_1F_0 \notin E(G')$; otherwise $T_1 = L_{2k-1}$, $F_0 = L_1$ respectively, must hold, contradicting the construction of $\bar{P}$. (*Note*: $r \geq 1$.) Therefore, if we take the maximal j such that $F_1T_j \in E(G')$, then

$$P' = \langle\{F_2, F_1, T_j, \ldots, T_r, F_0\}\rangle$$

is a path. P' satisfies property (1) since $\bar{P}$ does (possibly $j = r$); and obviously, properties (2), (3) are also fulfilled by P'. Therefore, we can find—as described above—an even cycle C with e being a chord of C. This finishes the proof of the theorem.

Final remarks

The following corollary is—obviously—equivalent to the Theorem; its statement was suggested by one of the referees.

Corollary. *Let G be a 2-connected, 3-regular, planar graph, and let $e_1, e_2 \in E(G)$. If G has an embedding in the plane such that e_1, e_2 belong to the same face boundary, then G has an even cycle containing both e_1 and e_2.*

As the other referee pointed out, the Theorem cannot be generalized to hold for arbitrary 2-connected planar graphs: In the graph of Fig. 2, the cycle having e as a chord is odd.

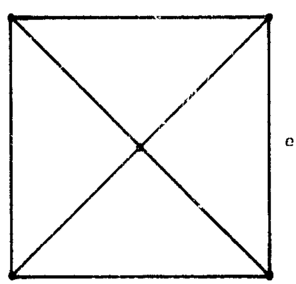

Fig. 2.

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