Extraconnectivity of graphs with large minimum degree and girth

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Abstract

The extraconnectivity $\kappa(n)$ of a simple connected graph $G$ is a kind of conditional connectivity which is the minimum cardinality of a set of vertices, if any, whose deletion disconnects $G$ in such a way that every remaining component has more than $n$ vertices. The usual connectivity and superconnectivity of $G$ correspond to $\kappa(0)$ and $\kappa(1)$, respectively. This paper gives sufficient conditions, relating the diameter $D$, the girth $g$, and the minimum degree $\delta$ of a graph, to assure maximum extraconnectivity. For instance, if $D \leq g - n + 2(\delta - 3)$, for $n \geq 2\delta + 4$ and $g \geq n + 5$, then the value of $\kappa(n)$ is $(n + 1)\delta - 2n$, which is optimal. The corresponding edge version of this result, to assure maximum edge-extraconnectivity $\lambda(n)$, is also discussed.

1. Introduction

One of the most important properties to be taken into account when designing an interconnection network is its fault-tolerance; that is, the ability of the system to work even if some nodes and/or links fail. See the survey of Bermond et al. [1]. For instance, it is interesting to know when the graph that models the network is maximally connected or edge-connected, which means that the network remains connected if the number of elements that fail is less than its minimum degree, that is the minimum number of links incident with a node. This paper is devoted to the study of graph models for optimally connected networks with respect to the following fault-tolerance property: when some nodes or links fail, the surviving components of the network have to connect a given minimum number of nodes. This problem corresponds to the study of a kind of conditional graph connectivity introduced by Harary in [8].

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The standard graph theoretic terms not defined in this paper can be found in the book of Chartrand and Lesniak [4]. A simple connected graph $G$ with diameter $D$ is said to be $\ell$-geodetic if $\ell$ is the maximum integer, $1 \leq \ell \leq D$, such that for any $x, y \in V(G)$ there exists at most one $x \leftrightarrow y$ path of length less than or equal to $\ell$. When $\ell = D$, the graph $G$ is called strongly geodetic, see [3, 9]. If $G$ has girth $g$, then clearly $G$ is $\ell$-geodetic for $\ell = \lceil (g - 1)/2 \rceil$. Reciprocally, if $G$ is $\ell$-geodetic, then its girth $g$ is either $2\ell + 1$ or $2\ell + 2$.

Soneoka et al. [10, 11] and Fàbrega and Fiol [5] have given sufficient conditions, in terms of the girth — or, in the case of digraphs, a new parameter of a similar significance — and the diameter, for a (di)graph to be maximally connected. For an $\ell$-geodetic graph these sufficient conditions can be stated in the following way. If $G$ has minimum degree $\delta$, diameter $D$, connectivity $\kappa$, and edge-connectivity $\lambda$, then

$$\begin{align*}
\kappa &= \delta \quad \text{if } D \leq 2\ell - 1, \\
\lambda &= \delta \quad \text{if } D \leq 2\ell.
\end{align*}$$

(1)

Let $G$ be a maximally connected graph with minimum degree $\delta$, that is $\kappa = \delta$. If $G \neq K_{\delta+1}$ and $v$ is a vertex of degree $\delta$, then the set of vertices adjacent to $v, \Gamma(v)$, is a minimum order trivial disconnecting set. If every disconnecting set of vertices of cardinality $\delta$ is trivial, then $G$ is said to be super-$\kappa$, see [2]. Analogously, $G$ is super-$\lambda$ if all its minimum edge-disconnecting sets are trivial. In this context, let us define a nontrivial set of vertices or edges as a vertex or edge set that does not contain a trivial disconnecting one. Fiol et al. have proved in [7] that if $G$ is $\ell$-geodetic with minimum degree $\delta > 2$ and diameter $D \leq 2\ell - 2$, and $F \subset V(G)$, $|F| \leq 2\delta - 3$, is nontrivial, then $G - F$ is connected. Analogously, if $D \leq 2\ell - 1$ and $A \subset E(G)$, $|A| \leq 2\delta - 3$, is nontrivial, then $G - A$ is connected. Thus, $G$ is super-$\kappa$ if $D \leq 2\ell - 2$ and $G$ is super-$\lambda$ if $D \leq 2\ell - 1$.

Let us define $\kappa(1)$ as the minimum cardinality of a nontrivial set of vertices $F$, if any, such that $G - F$ is not connected. Define $\lambda(1)$ in a similar way. Then, $\kappa(1)$ and $\lambda(1)$ measure the superconnectivity and edge-superconnectivity of $G$ and, from the above results, we have that if $G$ is an $\ell$-geodetic graph with minimum degree $\delta > 2$ and diameter $D$, then

$$\begin{align*}
\kappa(1) &\geq 2\delta - 2 \quad \text{if } D \leq 2\ell - 2, \\
\lambda(1) &\geq 2\delta - 2 \quad \text{if } D \leq 2\ell - 1.
\end{align*}$$

(2)

If we have no further information about the structure of $G$, this result is best possible in the following sense. Suppose that $G$ contains an edge with end-vertices $u$ and $v$ of degree $\delta$ and such that $\Gamma(u) \cap \Gamma(v) = \emptyset$. Then, the set $F = \Gamma(u) \cup \Gamma(v) \setminus \{u, v\}$ could be an example of nontrivial disconnecting set with $2\delta - 2$ vertices. Thus, for such a graph $G$, $\kappa(1) \leq 2\delta - 2$ and, by the results given in (2), $D \leq 2\ell - 2$ is a sufficient condition for $\kappa(1) = 2\delta - 2$. The edge case can be discussed similarly.

Given a graph $G$ and a graph-theoretic property $\mathcal{P}$, Harary defined in [8] the conditional connectivity $\kappa(G; \mathcal{P})$ [edge-connectivity $\lambda(G; \mathcal{P})$] as the minimum cardinality of a set of vertices [edges], if any, whose deletion disconnects the graph and every
remaining component has property $\mathcal{P}$. In this paper the property $\mathcal{P}_n$ of having more than $n$ vertices is considered.

If $H$ is a subgraph of $G$ and $v \in V(H)$, let $N_H(v)$ denote the set $\Gamma(v) \setminus V(H)$ and let $N(H) = \bigcup_{v \in V(H)} N_H(v)$. Given a graph $G$ and a fixed integer $n \geq 0$, let us say that $F \subseteq V(G)$ is $n$-nontrivial if $F$ does not contain a set $N(H)$ for any subgraph $H \subseteq G$ is $k$ vertices, $1 \leq k \leq n$ (for $n = 0$, any $F \subseteq V$ is $0$-nontrivial). From this point of view, $\kappa(n) \equiv \kappa(G; \mathcal{P}_n)$ is the minimum cardinality of a $n$-nontrivial disconnecting set. As stated in the Introduction, $\kappa(0) [\lambda(0)]$ corresponds to the connectivity $\kappa$ [edge-connectivity $\lambda$], and $\kappa(1) [\lambda(1)]$ measures the superconnectivity [edge-superconnectivity] of $G$. In what follows it is supposed that, for the graphs considered, such a $\kappa(n)$ exists. Otherwise, it can be assumed, by convention, some kind of optimality for such value (as the case of the complete graph is dealt with respect to the standard connectivity $\kappa(0)$). The conditional edge-connectivity $\lambda(n)$ can be defined in a similar way. Moreover, note that if $F$ is $n$-nontrivial for a given $n$, then $F$ is also $n'$-nontrivial for any $n' \leq n$. Thus, $\kappa(n') \leq \kappa(n) [\lambda(n')] \leq \lambda(n)$.

Suppose that a tree $T$ with $n + 1$ vertices, $n \geq 0$, each of degree $6$ in $G$, is a subgraph of $G$. If $F = N(T)$, then $T$ is a component of $G - F$. Moreover, if $G - F$ is not connected and each other component has at least $n + 1$ vertices, then it is clear that $\kappa(n) \leq |F| = |N(T)| \leq (n + 1)\delta - 2n$. Note that the value $\tau(n) = (n + 1)\delta - 2n$ gives the maximum number of vertices of the neighborhood of a tree $T$ with $n + 1$ vertices, each of degree $\delta$ in $G$, and so it is the optimal value of the $n$-extraconnectivity. In particular, $\tau(0)$ is the minimum degree $\delta$ of the graph. In [6] the following sufficient conditions for $\kappa(n) [\lambda(n)]$ to be optimal, in this sense, were stated. Let $G$ be an $\ell$-geodetic graph with minimum degree $\delta \geq 3$ and diameter $D$, and let $n \geq 2$. Then

$$\kappa(n) \geq (n + 1)\delta - 2n \quad \text{if} \quad D \leq \begin{cases} 2\ell - n - 1, & n \text{ even}, \\ 2\ell - n - 2, & n \text{ odd}, \end{cases}$$

$$\lambda(n) \geq (n + 1)\delta - 2n \quad \text{if} \quad D \leq \begin{cases} 2\ell - n, & n \text{ even}, \\ 2\ell - n - 1, & n \text{ odd}. \end{cases}$$

(3)

In the following section we improve the above sufficient conditions for $\kappa(n) [\lambda(n)]$ to be optimal. These conditions will now relate the parameter $\ell$, the minimum degree $\delta$, and the diameter $D$.

2. Optimally $n$-extraconnected graphs

Let us define a graph $G$ as optimally $n$-extraconnected if the minimum order of every $n$-nontrivial disconnecting set of vertices is at least $(n + 1)\delta - 2n$. As mentioned above, the purpose of this section is to obtain sufficient conditions on the diameter of $G$ to assure that the graph is optimally $n$-extraconnected. To this end, in what follows $G$ is a graph with girth $g \geq n + 5$, minimum degree $\delta \geq 3$, $n$ stands for a non-negative
integer, \(\tau(n) = (n + 1)\delta - 2n\), and \(F \subset V(G)\), \(|F| < \tau(n)|\), is a \(n\)-non-trivial disconnecting set. So, \(G - F\) is non-connected and all its components have more than \(n\) vertices.

The two following lemmas give some information about the structure of any component of \(G - F\).

**Lemma 2.1.** In any component of \(G - F\) there is a path of length at least \(n + 3\). Moreover, any vertex \(v\) of \(G - F\) lies on a path of length at least \(\lceil(n + 3)/2\rceil\).

**Proof.** Let \(C\) denote the component to which \(v\) belongs. If \(C\) contains a cycle, then its length is at least \(g \geq n + 5\). So, the result clearly holds in this case. Suppose that the component \(C\) is a tree. Condition \(g \geq n + 5\) also implies that \(N_C(u) \cap N_C(u') = \emptyset\) for any pair of vertices \(u, u' \in V(C)\) such that their distance in \(C\) satisfies \(d(u, u') \leq n + 2\); if not we would have a cycle with length at most \(n + 4\). Hence, as \(C\) has more than \(n\) vertices, it must have diameter greater than \(n + 2\); otherwise \(|N(C)| = |F| \geq \tau(n)\). Then, component \(C\) contains at least one \(u \leftrightarrow u'\) shortest path of length greater than \(n + 2\). Consequently, for any vertex \(v\) there exists in \(G - F\) either a \(v \leftrightarrow u\) or \(v \leftrightarrow u'\) path of length at least \(\lceil(n + 3)/2\rceil\).

Note that, by the above lemma, \(F\) is a \(n'\)-non-trivial disconnecting set for \(n' = n, n + 1, n + 2, n + 3\) and we have \(\kappa(n) \leq \kappa(n + 1) \leq \kappa(n + 2) \leq \kappa(n + 3) \leq |F|\). In particular, if \(F\) is a minimum order \(n\)-nontrivial disconnecting set, then \(\kappa(n) = |F|\) and, therefore, \(\kappa(n) = \kappa(n + 1) = \kappa(n + 2) = \kappa(n + 3)\).

Given a component \(C\) of \(G - F\) let \(\mu(C) = \max_{v \in V(C)} d(v, F)\). We have the following result:

**Lemma 2.2.** For any component \(C\) of \(G - F\), \(\mu(C) \geq 2\).

**Proof.** The proof is by contradiction. Thus, assume that \(C\) is a component of \(G - F\) such that \(\max_{v \in V(C)} d(v, F) = 1\). Let \(P = u_0u_1 \cdots u_n\) be a path in \(C\) of length \(n\). For each vertex \(v \in N_P(u_i), 0 \leq i \leq n\), let \(f_i \in F\) be a vertex at minimum distance from \(v\) and let \(F_i \subset F\) be the set of such vertices \(f_i\). Note that either \(v = f_i\) or \(d(v, f_i) = 1\). Since \(g \geq n + 5\), we have \(|F_i| \geq \delta - 2, 1 \leq i \leq n - 1, \ |F_0| \geq \delta - 1, \ |F_n| \geq \delta - 1\) and \(F_i \cap F_j = \emptyset\) for \(0 \leq i, j \leq n\). Hence, \(|F| \geq \sum_{i=0}^{n} |F_i| \geq \tau(n)\), a contradiction.

A consequence of Lemma 2.2 is that the diameter \(D\) of \(G\) satisfies \(D \geq 4\). Therefore, we have the following result.

**Proposition 2.1.** Let \(G\) be a graph with diameter \(D\), girth \(g \geq n + 5\), and minimum degree \(\delta \geq 3\). Then

\[\kappa(n) \geq \tau(n)\] if \(D \leq 3\).

Notice that since \(g \geq n + 5\) then \(3 \leq 2\ell - n - 1\) if \(g = 2\ell + 1\) is odd, and \(3 \leq 2\ell - n\) if \(g = 2\ell + 2\) is even. Hence, for graphs with \(D \leq 3\) and \(g \geq n + 5\), this result with

n = 0 is equivalent to (1). If n = 1 we obtain \( \kappa(1) \geq 2\delta - 2 \), that is, \( G \) is optimally superconnected. In this case the result stated in (2) is improved. If \( n = 2 \) the bounds given in (3) are also improved.

Our main result is the following theorem, which deals with the cases \( n \geq 3 \).

**Theorem 2.1.** Let \( G \) be a graph with girth \( g \geq n + 5 \), with minimum degree \( \delta \geq 3 \), and diameter \( D \). Then

\[
\kappa(n) \geq \tau(n) \quad \text{if} \quad D \leq \begin{cases} 
2\ell - 5 & (3 \leq n \leq \delta + 2), \\
2\ell - 7 & (\delta + 3 \leq n \leq 2\delta + 1), \\
2\ell - 9 & (2\delta + 2 \leq n \leq 2\delta + 3), \\
2\ell - n + 2\delta - 5 & (n \geq 2\delta + 4), \\
2\ell - n + 2\delta - 4 & (n \geq 2\delta + 5, \ n \text{ odd}).
\end{cases}
\]

The above upper bounds on the diameter could also be written using the girth \( g \) instead of the parameter \( \ell \). For instance, since \( g \geq 2\ell + 1 \), we have that, if \( D \leq g - n - 2(\delta - 3) \) and \( n \geq 2\delta + 4 \), then \( \kappa(n) \geq \tau(n) \). Note that the minimum degree \( \delta \) of \( G \) appears explicitly in the upper bound on \( D \). Hence, for values of \( n \) large enough with respect to \( \delta \), the previous known sufficient conditions given in (3) for \( G \) to be optimally extraconnected are improved.

The following concepts and notation are used to prove Theorem 2.1. Let \( T \) be a tree contained in a given component of \( G - F \). For every vertex \( v \) of \( T \) we will consider a path \( T^*(v) = t_0v_1 \cdots v_{s_v-1}v_s = v \), \( s_v \geq 1 \), \( v_1 \notin V(T) \), such that \( d(t_i, F) > d(v_i, F) \), \( 1 \leq i \leq s_v \), and \( d(h, F) \leq d(v_i, F) \) for every \( h \notin V(T^*(v)) \) adjacent to \( v_i \) (if such a path does not exist, let \( s_v = 0 \) and consider the trivial path \( T^*(v) = v \)). Given a path \( P \) in the graph \( G \), \( ||P|| \) will denote its length, and thus \( ||T^*(v)|| = s_v \). Moreover, define \( N^z_T(v) = \Gamma(v) \setminus \{v_{s_v-1}\} \) (if \( s_v = 0 \), then \( N^z_T(v) = N_T(v) \)) and let \( N^z_T(v) = \bigcup_{v \in V(T)} N^z_T(v) \). For any \( h \in N^z_T(v) \), let \( j_h \) denote a vertex in \( F \) such that \( d(h, j_h) = d(h, F) \). For any \( v \in V(T) \), let \( T \oplus T^*(v) \) denote a subgraph obtained by attaching to \( T \) the path \( T^*(v) \). Moreover, let \( T^* \) be the subgraph obtained by joining \( T^*(v) \) to each \( v \in V(T) \). That is, if \( V(T) = \{v_0, v_1, \ldots, v_r\} \), then \( T^* = T \oplus T^*(v_0) \oplus \cdots \oplus T^*(v_r) \). Given \( v, v' \in V(T) \), \( p_T(v, v') \) will represent the \( v \leftarrow v' \) path in \( T \). If the diameter of \( T^* \), which is at most \( \max_{u,v \in V(T)} ||T^*(u) \oplus p_T(u,v) \oplus T^*(v)|| \), is less than \( g \) then \( T^* \) is a tree. Moreover, if \( D_T < g - 2 \), then \( N^z_T(u) \cap N^z_T(v) = \emptyset \) for any \( u, v \in V(T) \). Notice that, in a certain sense, \( T^* \) is as far as possible from \( F \). Besides, \( D_T \) will stand for the diameter of the tree \( T \).

The proof of our results will use the following lemma, already used to prove Lemma 2.2.

**Lemma 2.3.** If any component \( C \) of \( G - F \) contain a tree \( T \) of order \( n + 1 \) such that \( N^z_T(u) \cap N^z_T(v) = \emptyset \) for any \( u, v \in V(T) \), then \( |N^z_T(v)| \geq \tau(n) \). Moreover, if the diameter of \( T^* \) is less than \( g - 2\mu(C) - 2 \), then \( |F| = \kappa(n) \geq \tau(n) \).
Proof. We have that \( N^*(T) = \bigcup_{v \in V(T)} N^*_T(v) \). Hence, by the hypothesis \( |N^*(T)| = \sum_{v \in V(T)} |N^*_T(v)| \geq \sum_{v \in V(T), s \geq 1} (\delta - 1) + \sum_{v \in V(T), s = 0} |N_T(v)| \geq (x(\delta - 1)) + (\tau(n-x)) = \tau(n) \), where \( x = \{v \in V(T) : s \geq 1 \} \). Moreover, since \( D_T^* < g - 2 \) we have that \( N^*_T(u) \cap N^*_T(v) = \emptyset \) for any \( u, v \in V(T) \). Therefore, as \( |F| < \tau(n) \) there exist \( h \in N^*_T(u), h' \in N^*_T(v) \), \( h \neq h' \), for some \( u, v \in V(T) \) such that \( f_h = f_{h'} = f \). In this way we find a closed walk \( W = f \rightarrow h T^*(u) \oplus p_T(u, v) \oplus T^*(v) h' \rightarrow f = f \rightarrow h u_s v_1 \rightarrow v_s h' \rightarrow f \), where \( f \rightarrow h \) and \( h' \rightarrow f \) are shortest paths. Since \( ||W|| \leq \mu(C) + 1 + D_T^* + \mu(C) + 1 \), the condition on the diameter of \( T^* \) implies that the length of the closed walk \( W \) is less than the girth \( g \) of the graph, arriving to a contradiction. The conclusion is that \( |F| \geq \tau(n) \) and hence \( G \) is optimally \( n \)-extraconnected. \( \square \)

An important point of the above reasoning is that, from \( W \), we get a cycle and not an acyclic walk. This is because \( h \in N^*_T(u) \), \( h' \in N^*_T(v) \) and thus, the vertex adjacent with \( h \) in the shortest path \( f \rightarrow h \) is not \( u_s \), and analogously the vertex adjacent with \( h' \) in the shortest path \( h' \rightarrow f \) is not \( v_s \).

The next proposition, to be compared with Lemma 2.2, will be used in the proof of Theorem 2.1.

Lemma 2.4. If \( n \geq \delta + 1 \) then any component \( C \) of \( G - F \) satisfies

\[
\mu(C) \geq 3 \quad \text{if} \quad \begin{cases} 
\delta \geq 5, \\
n - \delta + 9 < g \quad \text{for} \quad 3 \leq \delta \leq 4.
\end{cases}
\]

Proof. Assume that there exists a component \( C \) such that \( \mu(C) = 2 \). Let \( z \in V(C) \) be a vertex such that \( d(z, F) = 2 \) and denote by \( S_z \) the tree formed by \( z \) and \( \delta \) of its adjacent vertices. Clearly, \( S_z \) is contained in \( C \). Moreover, since \( n \geq \delta + 1 \), we can consider in \( C \) a tree \( T' \) with \( n \) vertices that contains \( S_z \), and such that \( D_{T'} \leq (n - \delta + 1) + 2 \). See Fig. 1. (In this figure and the following ones the vertices are drawn in different levels according to their distances to \( F \).) Then \( T'^* \) has diameter at most \( \max_{u, v \in V(T')} ||T'^*(u) \oplus p_T(u, v) \oplus T'^*(v)|| \leq (n - \delta + 1) + 2 \) — notice that \( ||T'^*(v)|| \leq 1 \) for any \( v \in V(T') \) because \( \mu(C) = 2 \). Now, we are going to add one vertex to \( T' \) in order to get another tree \( T \) with \( n + 1 \) vertices, in such a way that the diameter of \( T^* \) remains upper bounded by \( n - \delta + 3 \). To this end, if there exists a vertex \( s \in V(T') \) such that \( ||T'^*(s)|| = 1 \), then let \( T = T' \oplus T'^*(s) \). In this case we have that \( D_{T^*} \leq D_{T'^*} \).

Otherwise, if \( ||T'^*(s)|| = 0 \) for any \( s \in V(T') \), let \( T = T' \oplus s e \), where \( e \in N(T') \cap V(C) \), whose existence is assured because \( C \) has at least \( n + 1 \) vertices. Now we have that \( D_{T^*} \leq \max \{D_{T'^*}, \max_{u, v \in V(T')} ||p_T(u, e) \oplus T^*(e)|| \} \leq (D_{T'^*} + 1) + 1 \leq n - \delta + 3 \). Thus, if \( \delta \geq 5 \) then \( D_{T^*} < g - 2 \mu(C) - 2 \) because \( g \geq n + 5 \), and \( \mu(C) = 2 \). If \( 3 \leq \delta \leq 4 \) we have that \( D_{T^*} < g - 6 = g - 2 \mu(C) - 2 \) since \( n - \delta + 9 < g \). Moreover, for any \( u, v \in V(T) \), it is \( N^*_T(u) \cap N^*_T(v) = \emptyset \) because \( D_{T^*} < g - 2 \). Then, by Lemma 2.3 we have that \( |F| \geq \tau(n) \), a contradiction. \( \square \)

The following consequence improves, for \( n \geq \delta + 1 \) and \( \delta \geq 5 \), the result given in Proposition 2.1, because \( 5 \leq 2\ell - n + 1 \) if \( g \) is odd, or \( 5 \leq 2\ell - n + 2 \) if \( g \) is even.
Proposition 2.2. Let $G$ be a graph with diameter $D$, girth $g \geq n + 5$, and minimum degree $\delta \geq 5$. Let $n \geq \delta + 1$. Then,

$$\kappa(n) \geq \tau(n) \quad \text{if} \quad D \leq 5.$$

Corollary 2.1. In the hypothesis of Theorem 2.1 if $3 \leq \delta \leq 4$, any component $C$ of $G - F$ satisfies

$$\mu(C) \geq 3 \quad \text{if} \quad D \leq \begin{cases} 2\ell - 7 & (\delta + 3 \leq n \leq 2\delta + 1), \\ 2\ell - 9 & (2\delta + 2 \leq n \leq 2\delta + 3), \\ 2\ell - n + 2\delta - 5 & (n \geq 2\delta + 4), \\ 2\ell - n + 2\delta - 4 & (n \geq 2\delta + 5, \ n \text{ odd}) \end{cases}$$

Proof. Since $\ell \leq D$ and either $g = 2\ell + 1$ or $g = 2\ell + 2$, we always have that $g > n - \delta + 9$. Hence, by Lemma 2.4, $\mu(C) \geq 3$. \qed

The proof of Theorem 2.1 is organized in the following way. First, we will provide the proof for the first values of $n$, namely, $3 \leq n \leq 2\delta + 1$. In these cases the tree considered in the component of $G - F$ is directly obtained from the simple tree $S_z$, formed by a vertex $z$ at maximum distance from $F$ and $\delta$ of its adjacent vertices. For $n \geq 2\delta + 2$ a tree $T$ with a structure not so simple will be needed. After describing the structure of $T$, the diameter of $T^*$ will be studied. Then, we will assure that in any component exists a tree with more than $n$ vertices, in such a way that the diameter of $T^*$ is properly bounded. Finally, we will finish the proof of Theorem 2.1 for $n \geq 2\delta + 2$.

Proof of Theorem 2.1. ($3 \leq n \leq 2\delta + 1$)

From (3), the result holds for $n = 3, 4, 6$. We will extend this result to include the other values of $n$ not greater than $2\delta + 1$.

Assume first that $3 \leq n \leq \delta + 2$ and suppose $D \leq 2\ell - 5$, which implies $\ell \geq 5$ because $D \geq \ell$. The proof is by contradiction. Let $F$ be a $n$-nontrivial disconnecting set such that $|F| = \kappa(n) - \tau(n) - 1 \leq \delta^2 + \delta - 5$. Let $C_z$ and $C_y$ be two different components of $G - F$ and let $z \in C_z, \ y \in C_y$ be two vertices at maximum distance from $F$. It is clear that $D \geq d(z, y) \geq d(z, F) + d(y, F) = \mu(C_z) + \mu(C_y)$. Then, if $\mu = \mu(C_z) \leq \mu(C_y)$, we must have $\mu \leq \ell - 3$. Since $g \geq n + 5, \ \mu \geq 2$. Let $S_z$ be a tree formed by $z$ and $\delta$ of its adjacent vertices contained in $C_z$. The following different cases are considered:
(i) There exist at least two vertices \( z_1, z_2 \in \Gamma(z) \) such that \( \|S^*_z(z_1)\| = \|S^*_z(z_2)\| = 1 \). In this case, the tree \( T = S_z \oplus S^*_z(z_1) \oplus S^*_z(z_2) \) is contained in \( C_z \). The order of \( T \) is \( \delta + 3 \) and, the diameter of \( T^* \) satisfies \( \max_{u,v \in V(T)} \|T^*(u) \oplus p_T(u,v) \oplus T^*(v)\| \leq 4 \) since the path \( T^*(v) \) has length at most one for any \( v \in V(T) \), see Fig. 2. As \( 4 < g = 2 \mu - 2 \) because \( g \geq 2 \ell + 1 \) and \( \mu \leq \ell - 3 \), we have \( N^*_T(u) \cap N_T^*(v) = \emptyset \) for all \( u, v \in V(T) \), and from Lemma 2.3 we have \( |N^*(T)| \geq \tau(\delta + 2) = \delta^2 + \delta - 4 > |F| \). Moreover, Lemma 2.3 also gives \( |F| = \kappa(n) \geq \tau(n) \).

(ii) There exists only one vertex \( z_1 \in V(S_z) \) such that \( \|S^*_z(z_1)\| = \|z_1\| = 1 \). Since \( d(t,F) = \mu \), we have that \( \Gamma(t) \subset C_z \). Let us consider \( S_z \). If there exists \( w \in \Gamma(t) \), \( w \neq z_1 \), such that \( \|S^*_z(w)\| = 1 \), then \( S_z \) satisfies the conditions assumed in case (i) and the theorem holds. So, suppose that \( \|S^*_z(w)\| = 0 \) for any \( w \in \Gamma(t) \), \( w \neq z_1 \), and consider \( T = S_z \oplus z_1 tw \), see also Fig. 2. The tree \( T \) has order \( \delta + 3 \), and \( \Delta T^* \leq 4 \), because \( \|T^*(v)\| = 0 \) for any \( v \in V(T) \), \( v \neq z_1 \), and \( \|T^*(z_1)\| \leq 1 \). Hence, as in case (i) we have \( N^*_T(u) \cap N_T^*(v) = \emptyset \) for all \( u, v \in V(T) \), and \( |F| = \kappa(n) \geq \tau(n) \).

(iii) For any \( u \in V(S_z) \), it is \( \|S^*_z(u)\| = 0 \). If \( n < \delta + 1 \), then \( T = S_z \) is a tree of order at least \( n + 1 \) such that \( T^* = T \). Hence \( \Delta T^* \leq 2 \) and \( |N^*_T(T)| = |N(T)| \geq \tau(n) > |F| \). If \( \delta + 1 \leq n < \delta + 2 \), then \( |N(S_z) \cap V(C_z)| \geq 1 \), because the component \( C_z \) has more than \( n \) vertices. Now we have the following subcases:

- There exist \( e, e' \in N(S_z) \cap V(C_z) \), adjacent, respectively, to \( z_e, z_{e'} \in V(S_z) \), such that \( \|T^*(e)\| = \|T^*(e')\| = 0 \) where \( T = S_z \oplus z_e e \oplus z_{e'} e' \), see Fig. 3. Now \( T = T^* \) which implies that \( \Delta T^* \leq 4 \).

- There exists \( e \in N(S_z) \cap V(C_z) \), adjacent to \( z_e \in V(S_z) \), such that \( \|Q^*_e(e)\| = \|ee_1\| = 1 \) where \( Q_e = S_z \oplus z_e e \). Now, consider \( T = Q_e \oplus ee_1 \), that satisfies \( \Delta T^* = \Delta t \leq 4 \), see also Fig. 3.

- If \( \{e\} = N(S_z) \cap V(C_z) \), then \( \mu = 2 \). Denote by \( z_e \) the vertex of \( S_z \) to which \( e \) is adjacent and let \( e' \) be a vertex in \( V(C_z) \setminus V(S_z) \) adjacent to \( e \) (such a vertex exists because, by Lemma 2.1, in \( C_z \) there is a path with length at least \( n + 3 \)). Now, the tree \( T = S_z \oplus z_e e e' \) has order \( \delta + 3 \) and \( \Delta T^* \leq \|p_T(u,e') \oplus T^*(e')\| \leq 5 < g = 2 \), because \( g \geq 2 \ell + 1 \geq 11 \), see Fig. 3. Therefore, \( N^*_T(u) \cap N_T^*(v) = \emptyset \), for all \( u, v \in V(T) \), and by Lemma 2.3 \( |N^*(T)| > |F| \). Since \( N(S_z) \setminus \{e\} \subset F \), we must have a cycle such as \( f p_T(u,v) \oplus T^*(v) \rightarrow f \), where \( f = f_h \) for some \( h \in N^*(T) \) whose length is at most \( 1 + \Delta T^* + 1 + 2 \leq 9 \), which is a contradiction because \( g \geq 11 \), see Fig. 3.

- The case that remains to be considered is when \( \|Q^*_e(e)\| = \|ee_1 e_2\| = 2 \) for any \( e \in N(S_z) \), with at most one exception \( e' \), in which case \( \|Q^*_e(e')\| = 0 \). If

---

(i) (ii)

Fig. 2. Cases (i) and (ii).
such a vertex $e'$ does not exist, consider the tree $T = Q_e \oplus Q^*_e(e)$, where $e$ is any vertex in $N(S_z)$. Now $D_T \leq 5$ and then $|N^*(T)| > |F|$. Hence, we have that $f_h = f_{h'} = f$ for some $h, h' \in N^*(T)$, $h \neq h'$. Then a cycle with length at most $\| f \leftrightarrow h'uz^xee_1e_2h \leftrightarrow f \| \leq (\mu - 2) + 7 + \mu = 2\mu + 5 \leq 2\ell - 1$ exists in $G$ because $h' \in N(S_z)$. So we get a contradiction, see Fig. 3. On the other hand, if $e'$ exists, then $d(z_{e'}, F) = \mu$. Otherwise if $d(z_{e'}, F) = \mu$, then $d(s, F) \geq \ell - 1$ for any $s \in \Gamma(z_{e'}) \setminus \{e', z\}$, and then it would be $\| Q^*_s(s) \| \leq 1$, contradicting that $\| Q^*_s(s) \| = 2$ because $s \in N(S_z)$. Consider now the tree $T = Q_e \oplus Q^*_e(e)$, $e \neq e'$, see Fig. 3. Then a cycle of length at most $\| f \leftrightarrow e'z_{e'}zz^xee_1e_2h \leftrightarrow f \| \leq (\mu - 1) + 7 + \mu = 2\mu + 6$ exists in $G$, again a contradiction.

Now, assume that $D \leq 2\ell - 7$ and $\delta + 3 \leq n \leq 2\delta + 1$. In this case $\ell \geq 7$ which implies that $g \geq 15$. Then the disconnecting set has order $|F| \leq \tau(n) - 1 \leq 2\delta^2 - 2\delta - 3$ and $\mu = \mu(C_z) \leq \ell - 4$. In this case, by Lemma 2.4 and Corollary 2.1 $\mu \geq 3$, which implies that $|N(S_z) \cap V(C_z)| \geq \delta(\delta - 1)$. Now we have the following subcases:

(iv) There exists one vertex $z_1 \in \Gamma(z)$ such that $\| S^*_z(z_1) \| = \| z_1t \| = 1$. Let us consider the tree $S_t$ and a vertex $u \in \Gamma(z_1)$, $u \neq z, t$; whose existence follows from $d(z_1, F) \geq \mu - 1$. In this case, $T = S_z \oplus S_t \oplus z_1u$ is contained in $C_z$, where $S_z \oplus S_t$ is the tree obtained by joining $S_z$ and $S_t$ as shown in Fig. 4. The order of $T$ is $2\delta + 2$, the diameter of $T^*$ is upper bounded by $\max_{u,v \in \Gamma(T)} \| T^*(u) \oplus p_T(u,v) \oplus T^*(v) \| \leq 6$, see Fig. 4. As $g \geq 15$, $D_T < g - 2$, and therefore, $N^*_T(u) \cap N^*_T(v) = \emptyset$ for any $u, v \in \Gamma(T)$. Then, $|N^*_T(T)| \geq \tau(2\delta + 1) = 2\delta^2 - 2\delta - 2 > |F|$. Moreover, since $6 < g - 2\mu - 2$, because $g \geq 2\ell + 1$ and $\mu \leq \ell - 4$, by Lemma 2.3 we obtain that $|F| = \kappa(n) \geq \tau(n)$. Once more we get a contradiction.

(v) For any $u \in V(S_z)$, it is $\| S^*_z(u) \| = 0$. Note that $|N(S_z) \cap V(C_z)| \geq \delta(\delta - 1)$ because $\mu \geq 3$. Let us consider the following subcases:

- There exists some $u \in V(S_z)$, $u \neq z$, such that $d(u, F) = \mu$. Consider the tree $T' = S_z \oplus S_u$ which has order $2\delta$. If there exists some $w \in V(S_u)$ such that $\| T'^*(w) \| = 1$, then...
Su satisfies the conditions assumed in (iv) and the theorem holds. Thus, we can suppose that \(|T^*(w)|=0\) for any \(w \in V(T')\). If \(n \leq 2\delta - 1\), then \(T = T'\) is a tree of order at least \(n + 1\) such that \(D_{T'} \leq D_T = 3\). If \(n \geq 2\delta\), then let \(e_1, e_2 \in N(T') \cap V(C_z)\), adjacents to a vertex \(w \in V(T')\), and consider the tree \(T = T' \oplus w_1 \oplus w_2\) contained in \(C_z\). Then \(T\) has order \(2\delta + 2\) and \(D_{T'} \leq 6\), see Fig. 5.

- Assume that \(d(u, F) = \mu - 1\) for any \(u \in V(S_z), u \neq z\). Then we have \(d(h, F) \leq \mu - 1\) for any \(h \in N_z(u)\) because \(|S_z^*(u)|=0\). Suppose first that there exists \(e \in N(S_z) \cap V(C_z)\), adjacent to \(z \in V(S_z)\), such that \(1 \leq |Q_e(e)| \leq 2\), where \(Q_e = S_z \oplus z \oplus e\). If \(e \in e_1\) is an edge of \(Q_e^*(e)\), then consider \(T = S_z \oplus z \oplus e \oplus S_{e_1}\), see Fig. 5. It has order \(2\delta + 2\), and as \(D_{T'} \leq 7\) we have that \(|N^*(T)| > |F|\), that is, there exist \(h, h' \in N^*(T)\), \(h \neq h'\) such that \(f_h = f_h' = f\). Moreover, \(|N^*(T)| = D_T \rightarrow 1\) if and only if \(u \in N(z)\), \(u \neq z\), and \(v \in N(e_1)\). Therefore, from \(f \leftrightarrow h \leftrightarrow v \leftrightarrow T^*(v)h' \leftrightarrow f\) we find a cycle whose length is at most \((\mu - 1) + 1 + D_{T'} + 1 + \mu \leq 2\mu + 8 \leq 2\ell\), because \(\mu \leq \ell - 4\), a contradiction since \(0 \geq \ell + 1\). Finally, let us suppose that \(|N^*(e)|=0\), for any \(e \in N(S_z) \cap V(C_z)\). As \(\mu \geq 3\), clearly \(|N(S_z) \cap V(C_z)| > \delta + 1\) and therefore the tree \(T\) is obtained by joining \(\delta + 1\) of these vertices to \(S_z\).

From now on, assume \(n \geq 2\delta + 2\) and consider a vertex \(z\) in a given component \(C_z\) such that \(d(z, F) = \mu(C_z) = \mu\). By Lemma 2.1, we know that \(z\) belongs to a path \(P_z'\) in \(G - F\) of length at least \(\lceil (n + 3)/2 \rceil\). In order to complete the proof of Theorem 2.1, we can assume that \(\mu \geq 3\) by Lemma 2.4 and Corollary 2.1. Then we can consider a subpath \(P\) of \(P_z'\) that contains vertex \(z\) as an internal vertex and with length \(4 \leq p \leq \lfloor (n - 2\delta + 4)/2 \rfloor\). Moreover, we can assume that the distance in \(P\) from \(z\) to the endvertices of this path is at least two, and \(d(v, F) > 1\) for every internal vertex \(v\) of \(P\). We get a tree \(T'\) in the following way. Attach to vertex \(z\) all the paths of length two of
the form \(zz'z_j, \ z_j \notin V(P)\), \(1 \leq j \leq \delta - 1, \ 1 \leq i \leq \delta - 2\). Moreover at least \(\delta - 2\) edges \(vw, \ w \in \Gamma(v) \setminus V(P)\), can be attached to each internal vertex \(v \neq z\). In this way, we obtain a tree \(T'\) that has diameter \(D_{T'} = p\), and, since \(q \geq n + 5 > p\), the order of \(T'\) is \(n_{T'} = p(\delta - 1) + (\delta - 2)^2 + 1\). The structure of \(T'\) is as shown in Fig. 6. Then the structure of the tree \(T\) — contained in \(C_z\) and such that it is rooted at vertex \(z\) — which we consider is as follows according to whether \(p\) attains its extreme values or not:

**Type (a).** If \(p = 4\) then \(n_{T'} = \delta^2 + 1\) and therefore a tree \(T = T'\) on \(n + 1\) vertices is contained in \(C_z\) if \(2\delta + 2 \leq n \leq 2\delta + 3\), since \(2\delta + 4 \leq \delta^2 + 1\).

If \(p = \lfloor (n - 2\delta + 4)/2 \rfloor\) then \(n \geq 2\delta + 4\). Keeping in mind that \(p(\delta - 1) = 2p + (\delta - 3)p\), we have that \(n_{T'} = n - 1 + (\delta - 3)^2 + (\delta - 3)p\) if \(n\) is odd, and \(n_{T'} = n + (\delta - 3)^2 + (\delta - 3)p\) if \(n\) is even. Therefore, \(n_{T'} \geq n - 1\), except for \(\delta = 3\), in which case \(n_{T'} = 2\lfloor n/2 \rfloor\). In this case \(T = T'\).

**Type (b).** If \(4 < p < \lfloor (n - 2\delta + 4)/2 \rfloor\) then \(n \geq 2\delta + 6\). Notice that the endvertices of \(P, t\) and \(t'\), satisfy \(d(t, F) = d(t', F) = 1\) and \(p \geq 2(\mu - 1)\). Now, if \(\delta \geq 4\) and the order of \(T'\) is less than \(n + 1\), then let \(T\) be a tree of order at least \(n + 1\) that contains \(T'\). Otherwise let \(T = T'\). On the other hand, if \(\delta = 3\) and the order of \(T'\) is less than \(2\lfloor n/2 \rfloor\), then let \(T\) be a tree of order at least \(2\lfloor n/2 \rfloor\) that contains \(T'\). Otherwise, \(T = T'\). As any component of \(G - F\) has more than \(n\) vertices, the existence of such a tree \(T\) is assured in any case. Hence, if \(\delta \geq 4\), the diameter \(D_T\) of \(T\) is at most \(p + (n - (\delta - 1)p - (\delta - 2)^2) = n - (\delta - 2)p - (\delta - 2)^2\), and if \(\delta = 3\), then \(D_T \leq p + (2\lfloor n/2 \rfloor - 2p + 2) = 2\lfloor n/2 \rfloor - p - 2\).

The characteristics of \(T\) are summarized in Table 1.

Now, we consider \(T^*\). Note that, for any given vertex \(v\) of \(T\), the length of \(T^*(v) = v_1 \cdots v_s\), is at most \(\|p_T(v, z)\|\) because \(d(v_i, F) > d(v_{i-1}, F), \ 1 \leq i \leq s_r\) or \(d(z, F) = \mu\) and \(\mu\) is the maximum possible distance to \(F\) from a vertex in the component. The following result allows us to bound the diameter of \(T^*\).

**Lemma 2.5.** According to the type of tree \(T\) the diameter \(D_{T^*}\) of \(T^*\) satisfies

- **Type (a):** \(D_{T^*} \leq 2D_T \leq \begin{cases} 8, & 2\delta + 2 \leq n \leq 2\delta + 3, \\ 2 \left\lfloor \frac{n - 2\delta + 4}{2} \right\rfloor, & n \geq 2\delta + 4. \end{cases}\)

- **Type (b):** \(D_{T^*} \leq 2(\mu - 1) + D_T \leq \begin{cases} n - \delta^2 + 8, & \delta \geq 4, \\ 2 \left\lfloor \frac{n}{2} \right\rfloor - 2, & \delta = 3. \end{cases}\)

Moreover, for any pair \(u, v\) of different vertices of \(T\), \(N^*_T(u) \cap N^*_T(v) = \emptyset\).
Table 1
Characteristics of the tree $T$

<table>
<thead>
<tr>
<th>Type (a)</th>
<th>$p$</th>
<th>$n$</th>
<th>$n_T \geq$</th>
<th>$D_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4</td>
<td>$[2\delta + 2, 2\delta + 3]$</td>
<td>$n + 1$</td>
<td>$p$</td>
</tr>
<tr>
<td></td>
<td>$\left\lceil \frac{n-2\delta+4}{2} \right\rceil$</td>
<td>$\geq 2\delta + 4$</td>
<td>if $\delta \geq 4$, $n + 1$</td>
<td>$p$</td>
</tr>
</tbody>
</table>

| Type (b) | $\left\lceil \frac{n-2\delta+4}{2} \right\rceil$ | $\geq 2\delta + 6$ | if $\delta \geq 4$, $n + 1$ | $n-(\delta-2)p-(\delta-2)^2$ |
|          | $\delta = 3, 2\left\lceil \frac{\delta}{3} \right\rceil$ | | | $2\left\lceil \frac{\delta}{3} \right\rceil - p - 2$ |

Proof. We have to bound the length of the path

$$T^*(u) \oplus p_T(u,v) \oplus T^*(v) = u_s \cdots u_1 p_T(u,v)v_1 \cdots v_s, \quad (4)$$

for any pair $u, v$ of different vertices of $T$. According to $T$, consider the following cases:

Type (a): First, suppose that $p_T(u,z)$ and $p_T(z,v)$ have a common subpath of length $k > 0$, and assume $\|p_T(u,z)\| \geq \|p_T(z,v)\|$. As stated above, the length of the path $T^*(u)$ is at most $\|p_T(u,z)\|$ and, analogously, $\|T^*(v)\| \leq \|p_T(v,z)\|$. Thus, the length of the path given in (4) is upper bounded by $2\|p_T(u,z)\| + 2\|p_T(z,v)\| - 2k \leq 2(\|p_T(u,z)\| + 1) \leq 2D_T$ because, by the structure of $T$, $\|p_T(z,v)\| \leq k + 1$. On the other hand, if $p_T(u,z)$ and $p_T(z,v)$ are edge disjoint paths, then clearly $\|p_T(u,z)\| + \|p_T(z,v)\| = \|p_T(u,v)\| \leq D_T$, and we find that the length of the path (4) is now bounded by $2(\|p_T(u,z)\| + \|p_T(v,z)\|) \leq 2D_T$.

Type (b): In this case $\|T^*(u)\|$ and $\|T^*(v)\|$ are at most $\mu - 1$ and $p \geq 2(\mu - 1)$, $\mu \geq 3$. Besides, $\|p_T(u,v)\| \leq D_T$. Hence, the length of (4) is bounded by $2(\mu - 1) + D_T \leq p + D_T$. Thus, if $\delta \geq 4$, then $D_T \leq n - p(\delta - 3) - (\delta - 2)^2 \leq n - \delta^2 + 8$ because $p \geq 4$. If $\delta = 3$, we have $D_T \leq 2n/2 - 2$.

Since, in any case, the diameter of $T^*$ is at most $n$, these results imply that all the vertices in the path (4) must be different and that $N^*_T(u) \cap N^*_T(v) = \emptyset$. Otherwise we would have $g \leq n + 2$, contradicting $g \geq n + 5$. \qed

In order to get in any component of $G - F$ a tree $T$ satisfying $|N^*(T)| \geq \tau(n)$, we need the order of $T$ to be at least $n + 1$. As seen in Table 1, this is not necessarily the case for trees of type (a) or type (b) when $\delta = 3$ and $n \geq 2\delta + 4$. However, we have the results stated in the following lemmas.

Lemma 2.6. Consider in a component of $G - F$ a tree $T$ of type (a) or type (b), with order less than $n + 1$. It is possible to extend $T$ to a tree $Q$ with order $n + 1$ in
such a way that the diameter of \( Q^* \) satisfies

- **Type (a):** \( D_{Q^*} \leq 2D_T \)
- **Type (b):** \( D_{Q^*} \leq 2(\mu - 1) + D_T \)

Moreover, for any pair \( u, v \) of different vertices of \( Q \), \( N_Q^*(u) \cap N_Q^*(v) = \emptyset \).

**Proof.** Let \( T \) be either of type (a) or type (b). We have \( \delta = 3 \) and \( n \geq 2\delta + 4 \). We have to keep in mind that, for any vertex \( s \in V(T) \), a tree of type (a) verifies \( ||T^*(s)|| \leq ||p_T(s,z)|| \leq D_T - 2 \), and for a tree of type (b), \( ||T^*(s)|| \leq \mu - 1 \), where \( \mu \geq 3 \). Assume that the order of \( T \) is \( n - 1 \). We will see that it is possible to add two vertices in order to get a tree \( Q \) of order \( n + 1 \) such that, if \( T \) is a tree of type (a), then the diameter of \( Q^* \) verifies \( D_{Q^*} \leq 2D_T \), or, if \( T \) is a tree of type (b), then \( D_{Q^*} \leq D_T + 2(\mu - 1) \). Let us consider the following cases:

1. (i) Either there exist two vertices \( s, s' \in V(T) \) such that \( ||T^*(s)|| = ||T^*(s')|| = 1 \), in which case the searched tree is \( Q = T \oplus T^*(s) \oplus T^*(s') \), or there exists a vertex \( s \) such that \( ||T^*(s)|| \geq 2 \) and \( Q = T \oplus T^*(s) \). Obviously, \( Q \) satisfies \( D_{Q^*} \leq D_T \).

2. (i) There only exists one vertex \( s \in V(T) \) such that \( ||T^*(s)|| = ||s_s_1|| = 1 \). Then \( \Gamma(s_1) \subset C \), where \( C \) denotes the component that contains tree \( T \). Let \( h \in \Gamma(s_1) \), \( h \neq s \), and consider \( Q = T \oplus ss_1h \). We have \( D_{Q^*} \leq ||Q^*(h)|| + D_T + 2 \). If \( T \) is of type (a), then \( ||Q^*(h)|| \leq ||p_T(s,z)|| \), and \( D_{Q^*} \leq 2D_T \) because \( ||p_T(s,z)|| \leq D_T - 2 \). On the other hand, if \( T \) is of type (b), then \( D_{Q^*} \leq (\mu - 1) + 2 + D_T \leq 2(\mu - 1) + D_T \) because \( \mu \geq 3 \).

3. (iii) Suppose that \( ||T^*(s)|| = 0 \) for any \( s \in V(T) \). It must be \( |N(T) \cap V(C)| \geq 1 \). Consider the following subcases:

   - There exists \( e \in N(T) \cap V(C) \), adjacent to \( s \in V(T) \), such that \( ||T^*(e)|| = ||e_1 \cdots e_r|| \geq 1 \), where \( T' = T \oplus se \). In this case let \( Q = T \oplus se_1 \). If \( T \) is of type (a), then \( ||Q^*(e)|| \leq 1 + ||Q^*(e_1)|| \leq 1 + ||p_T(s,z)|| \leq D_T - 1 \) because, in the worst case, \( d(e_1, F) = d(s,F) \). Thus, \( D_{Q^*} \leq 2D_T \). If \( T \) is of type (b), then \( ||Q^*(e)|| \leq \mu - 1 \), and, hence, \( D_{Q^*} \leq D_T + 2(\mu - 1) \).

   - There exist \( e_1, e_2 \in N(T) \cap V(C) \), adjacent respectively to \( s_1, s_2 \in V(T) \), such that \( ||Q^*(e_1)|| = ||Q^*(e_2)|| = 0 \) where \( Q = T \oplus s_1e_1 \oplus s_2e_2 \). Then \( D_{Q^*} \leq D_T + 2 \).

   - If \( \{e\} = N(T) \cap V(C) \), then, by the construction of \( T \), it must be \( \mu = 3 \). As the component \( C \) has more than \( n \) vertices, there must exist a vertex \( e' \in C \) adjacent to \( e \). In this way, \( Q = T \oplus see' \) has order \( n + 1 \). Moreover, \( ||Q^*(e')|| \leq 2 \) because \( \mu = 3 \). Thus, \( D_{Q^*} \leq ||p_Q(s', e')|| + ||Q^*(e')|| \leq D_T + 4 \) where \( s' \in V(Q) \). Therefore, if \( T \) is of type (a), then \( D_{Q^*} \leq 2D_T \) because \( D_T \geq 4 \). If \( T \) is of type (b), then \( D_{Q^*} \leq 2(\mu - 1) + D_T \) because \( \mu = 3 \) (Fig. 7).

![Fig. 7. A tree of type (b) with \( \mu = 3 \).](image-url)
In any case $D_T \leq n$ and we conclude, in the same way as in Lemma 2.6, that $N_Q^*(u) \cap N_Q^*(v) = \emptyset$, for any pair $u,v$ of different vertices of $Q$. □

All the above results allow us to state the following corollary.

**Corollary 2.2.** In any component of $G - F$ there exist a tree $T$ of order at least $n + 1$ such that $T^*$ have diameter $D_{T^*}$ bounded as follows:

| Type (a) | \( D_{T^*} \leq \begin{cases} 
8, & 2\delta + 2 \leq n \leq 2\delta + 3, \\
2 \left\lceil \frac{n - 2\delta + 4}{2} \right\rceil, & n \geq 2\delta + 4;
\end{cases} |
| Type (b) | \( D_{T^*} \leq \begin{cases} 
n - \delta^2 + 8, & \delta \geq 4 \text{ and } n \geq 2\delta + 6, \\
2 \left\lceil \frac{n}{2} \right\rceil - 2, & \delta = 3 \text{ and } n \geq 2\delta + 6.
\end{cases} |

Moreover, for any pair $u,v$ of different vertices of $T$, $N_T^*(u) \cap N_T^*(v) = \emptyset$.

Now, we can finish the proof of Theorem 2.1.

**Proof of Theorem 2.1 (continuation).** \((n \geq 2\delta + 2)\)

Again, the proof is by contradiction. Let $F$ be a $n$-nontrivial disconnecting set such that $|F| = \kappa(n) \leq \tau(n) - 1$. Let $C_z$ and $C_y$ be two different components of $G - F$ and let $z \in C_z$, $y \in C_y$ be two vertices at maximum distance from $F$. It is clear that $D \geq d(z,y) \geq d(z,F) + d(y,F) = \mu(C_z) + \mu(C_y)$. Assume that $\mu = \mu(C_z) \leq \mu(C_y)$.

Consider a tree $T$ with order $n + 1$ and $D_{T^*}$ given by Corollary 2.2. Hence for any pair $u,v$ of different vertices of $T$, $N_T^*(u) \cap N_T^*(v) = \emptyset$ and therefore, by Lemma 2.3 $|N^*(T)| \geq \tau(n) > |F|$. Thus, it suffices to state in each case that $D_{T^*} < g - 2\mu - 2$, and then $|F| = \kappa(n) \geq \tau(n)$.

(i) If $D \leq 2\ell - 9$, then $\mu \leq \ell - 5$, for $2\delta + 2 \leq n \leq 2\delta + 3$. By Corollary 2.2, we have that $D_{T^*} + 2\mu + 2 \leq 2\ell - 10 + 2 < 2\ell + 1 \leq g$.

(ii) If $D \leq 2\ell - n + 2\delta - 4$, $n \geq 2\delta + 5$, and $n$ is odd, then $\mu \leq \ell + \delta - 2 - (n + 1)/2$.

Therefore, by Corollary 2.2 we have:

- If $T$ is of type (a), then $D_{T^*} \leq n - 2\delta + 3$. So, $2\mu + 2 + D_{T^*}$ would be bounded by $(2\ell + 2\delta - 5) + 2 + (n - 2\delta + 3) = 2\ell < g$.

- If $T$ is of type (b) and $\delta \geq 4$, then $D_{T^*} \leq n - \delta^2 + 8$, and, again, $2\mu + 2 + D_{T^*} \leq (2\ell + 2\delta - 5 + n) + 2 + (n - \delta^2 + 8) \leq 2\ell$. Otherwise, if $\delta = 3$, then $D_{T^*} \leq n - 3$.

Thus, once more, $2\mu + 2 + D_{T^*}$ would be at most $(2\ell - n + 1) + 2 + (n - 3) = 2\ell$.

(iii) If $D \leq 2\ell - n + 2\delta - 5$ and $n \geq 2\delta + 4$, the reasoning is similar to the above case. □

3. Edge-extraconnectivity

As in the vertex case, let us define a graph $G$ as *optimally $n$-edge-extraconnected* if the minimum order of every $n$-nontrivial edge-disconnecting set is at least $\tau(n)=$
(n + 1)\delta - 2n$. Let $G$ be a graph with girth $g \geq n + 5$, and let $E \subset A(G)$, be a minimum $n$-nontrivial edge-disconnecting set such that $|E| = \lambda(n) \leq \tau(n) - 1$. It follows that $G - E = C_1 \cup C_2$, where $C_1, C_2$ are two different connected components, and the only edges between them are those of $E$. Now we consider $F = \{ f \in V(C_1) : f f' \in E \}$, and $F' = \{ f' \in V(C_2) : f f' \in E \}$. The notations and concepts are the same as in the preceding section. It is clear that Lemma 2.1 holds in $G - E$, and also holds for the first part of Lemma 2.3, that is, if $C_1$ contain a tree $T$ of order $n + 1$ such that $\gamma*(u) \cap \gamma*(v) = \emptyset$ for any $u, v \in V(T)$, then $|\gamma*(T)| \geq \tau(n)$. Let $\mu = \max_{i \in \gamma(C_1)} d(v, F)$ and $\mu' = \max_{i \in \gamma(C_2)} d(F', v)$, and suppose that $\mu \leq \mu'$. Then we have the following result which is analogous to Lemmas 2.2-2.4.

**Lemma 3.1.** We have $\mu \geq 1$. Moreover, if $C_1$ contain a tree $T$ of order $n + 1$ such that $DT < g - 2\mu - 2$, then $|E| = \gamma(n) \geq \tau(n)$. If $n \geq \delta + 1$ then,

$$\mu \geq 3 \quad \text{if} \quad \left\{ \begin{array}{l} \delta \geq 5, \\ n - \delta + 9 < g \quad \text{for} \quad 3 \leq \delta \leq 4. \end{array} \right.$$

**Proof.** First suppose that $\mu = 0$. Notice that in this case $V(C_1) = F$. Given $x \in V(C_1)$ let us denote by $\omega_E(x)$ the edges of $E$ with endvertex $x$, and by $\omega_E(H) = \bigcup_{x \in H} \omega_E(x)$, for any $H \subset V(C_1)$. Consider a path $P = x_0 x_1 x_2 \ldots x_n$ in $C_1$ of length $n$. We have that, for any $1 \leq i \leq n - 1$, $|\omega_E(x_i)| > |\omega_E(N_p(x_i) \cap V(C_1))| \geq \delta - 2$, and for $i = 0, n$, $|\omega_E(x_i)| > |\omega_E(N_p(x_i) \cap V(C_1))| \geq \delta - 1$. Since for any $v \in V(C_1)$, $|\omega_E(v)| \geq 1$ and $g \geq n + 5$ we conclude that $|E| > \sum_{i=0}^{n} |\omega_E(x_i)| + |\omega_E(N_p(x_i) \cap V(C_1))| > \sum_{i=0}^{n} |\omega_E(x_i)| + |(N_p(x_i) \cap V(C_1))| > \tau(n)$, which is a contradiction and then $\mu > 1$.

To prove the second part, since $DT < g - 2$ we have that $N_F^*(u) \cap N_F^*(v) = \emptyset$ for any $u, v \in V(T)$. Assume that $DT = |T^*(u) \cup p_T(u, v) \cup T^*(v)|$ for some $u, v \in V(T)$. If $u \in F$ (or $v \in F$) it could be that $\gamma^*(u) \cap F' \neq \emptyset$, but in this case $|T^*(u)| = 0$. Therefore, if $|F'| \leq |E| < \tau(n)$, there exists $h \in N^*_F(v)$ such that $f_h = f \in F'$ and we find a closed walk $f p_T(u, v) \cup T^*(v) h \leftarrow f$ whose length is at most $1 + DT + \mu + 2 < g$, since $DT < g - 2\mu - 2$ and $\mu \geq 1$, a contradiction. On the other hand, if $u, v \notin F$ then as $|F| \leq |E| < \tau(n)$ there exist $h \in N^*_F(u), h' \in N^*_F(v), h \neq h'$, such that $f_h = f_{h'} = f \in F$. Now we find a closed walk $f \leftarrow h T^*(u) \cup p_T(u, v) \cup T^*(v) h' \leftarrow f$ whose length is at most $\mu + 1 + DT + \mu + 1 < g$, a contradiction. Thus, $|E| \geq \tau(n)$.

Now, assume that $\mu \leq 2$ and let $z \in V(C_1)$ be a vertex such that $d(z, F) = \mu$. As $n \geq \delta + 1$ we can consider in $C_1$ a tree $T'$ of order $n$ that contains $z$. The diameter of $T'^*$ is at most $DT + 2 \leq n - \delta + 3$. Furthermore, it is possible to extend $T'$, by adding one vertex, to a tree of order $n + 1$ such that $DT - \leq n - \delta + 3$. Thus, if $\delta \geq 5$ then $DT < g - 2\mu - 2$ because $g \geq n + 5$, and $\mu \leq 2$. If $3 \leq \delta \leq 4$ we have $DT < g - 6$ since $n - \delta + 9 < g$. Hence, for any $u, v \in V(T)$, we have $N^*_F(u) \cap N^*_F(v) = \emptyset$. Then, $|E| > \tau(n)$, a contradiction. \[\square\]

The following edge version of Theorem 2.1 derives from the above lemma and all the results of the vertex-case. Notice that this theorem improves the previous
known sufficient conditions given in (3) for G to be optimally edge-extra-connected.

**Theorem 3.1.** Let G be a graph with girth \( g \geq n + 5 \), with minimum degree \( \delta \geq 3 \), and diameter \( D \). Then,

\[
\lambda(n) \geq \tau(n) \quad \text{if} \quad \begin{cases}
D \leq 2\ell - 4 & (3 \leq n \leq \delta + 2), \\
D \leq 2\ell - 6 & (\delta + 3 \leq n \leq 2\delta + 1), \\
D \leq 2\ell - 8 & (2\delta + 2 \leq n \leq 2\delta + 3), \\
D \leq 2\ell - n + 2\delta - 4 & (n \geq 2\delta + 4), \\
D \leq 2\ell - n + 2\delta - 3 & (n \geq 2\delta + 5, \ n \ odd).
\end{cases}
\]

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