Periodic solutions of Abel differential equations

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Abstract

For a class of polynomial non-autonomous differential equations of degree \( n \), we use phase plane analysis to show that each equation in this class has \( n \) periodic solutions. The result implies that certain rigid two-dimensional systems have at most one limit cycle which appears through multiple Hopf bifurcation.

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1. Introduction

We consider differential equations of the form

\[
\dot{z} = \frac{dz}{dt} = z^n + P_1(t)z^{n-1} + \cdots + P_{n-1}(t)z + P_n(t),
\]

(1.1)

where \( z \) is a complex-valued function and \( P_i \) are real-valued continuous functions. We denote by \( z(t, c) \) the solution of (1.1) satisfying \( z(0, c) = c \). Take a fixed real number \( \omega \); we define the set \( Q \) to be the set of all complex numbers \( c \) such that \( z(t, c) \) is defined for all \( t \) in the interval \([0, \omega]\); the set \( Q \) is an open set. On \( Q \) we define the displacement function \( q \) by

\[
q(c) = z(\omega, c) - c.
\]

Zeros of \( q \) identify initial points of solutions of (1.1) which satisfy the boundary conditions \( z(0) = z(\omega) \). We describe such solutions as periodic even when the functions \( P_i \) are not themselves periodic. However, if \( P_i \) are \( \omega \)-periodic then these solutions are also \( \omega \)-periodic. The main
concern is to estimate the number of periodic solutions. This problem was suggested by C. Pugh as a version of Hilbert’s sixteenth problem; it is listed by S. Smale as Problem 7 in [14]. Equations (1.1) have been studied in detail by Lloyd in [11], using the methods of complex analysis and topological dynamics.

Note that \( q \) is holomorphic on \( Q \). The multiplicity of a periodic solution \( \varphi \) is that of \( \varphi(0) \) as a zero of \( q \). It is useful to work with a complex dependent variable. The reason is that the number of zeros of a holomorphic function in a bounded region of the complex plane cannot be changed by small perturbations of the function. Hence, periodic solutions cannot then be destroyed by small perturbations of the right-hand side of the equation; periodic solutions can be created or destroyed only at infinity. Suppose that \( \varphi \) is a periodic solution of multiplicity \( k \). By applying Rouche’s theorem to the function \( q \), for any sufficiently small perturbations of the equation, there are precisely \( k \) periodic solutions in a neighborhood of \( \varphi \) (counting multiplicity). Upper bounds on the number of periodic solutions of (1.1) can be used as upper bounds on the number of periodic solutions when \( z \) is limited to be real-valued. This is the reason that \( P_i \) are not allowed to be complex-valued.

When \( n = 3 \), Eq. (1.1) is known as the Abel differential equation. It was shown in [11,13] that Abel differential equation has exactly three periodic solutions provided account is taken of multiplicity. We describe Eq. (1.1) as of Abel form. For \( n \geq 4 \), Lins Neto [10] has given examples which demonstrate that there is no upper bound, in terms of \( n \) only, on the number of periodic solutions. However, there are upper bounds for certain classes of equations. It was shown in [11], that each of the following equations has \( n \) periodic solutions,

\[
\dot{z} = z^n + \alpha(t)z, \\
\dot{z} = z^n + \alpha(t)z^{n+1} + \beta(t)z,
\]

where \( n \) is odd in the second equation. Ilyashenko, in [9], gave the following upper bound for the number of periodic solutions

\[
8 \exp\left((3C + 2) \exp\left(\frac{3}{2}(2C + 3)^n\right)\right),
\]

where \( C > 1 \) is an upper bound for the absolute values of the coefficients \( P_i(t) \). Although this estimate is non-realistic, it is the only known explicit estimate. In [12], Panov considered the equation

\[
\dot{z} = z^n + \alpha(t)z^2 + \beta(t)z + \gamma(t)
\]

and proved that the equation has at most three real periodic solutions if \( n \) is odd. Quartic equations having at least ten periodic solutions were described in [1]; the coefficients were polynomial functions in \( t \) of degree 3.

In this paper, we consider the class of equations

\[
\dot{z} = z^n + \alpha(t)z^{n-1} + \beta(t)z^{n-2} \tag{1.2}
\]

as a generalized Abel differential equation. We show that if \( \beta(t) \leq 0 \), then this equation has at most two non-zero periodic solutions. We give conditions on \( \alpha \) and \( \beta \) that imply the equation has exactly two non-zero periodic solutions, one non-zero periodic solution, or no non-zero periodic solutions. Particular cases of this result, with \( n = 4 \) and \( n = 5 \), were given in [3,4]. In Section 2, we describe the phase portrait of (1.1) and recall some results from [11]. In Section 3, we state and prove our main result. In the final section, we use the result of Section 3 to show that a certain family of rigid two-dimensional systems has at most one limit cycle and this limit cycle appears through multiple Hopf bifurcation.
2. The phase portrait

We identify Eq. (1.1) with the \( n \)-tuples \((P_1, P_2, \ldots, P_n)\) and write \( \mathcal{L} \) for the set of all equations of this form. With the usual definitions of additions and scalar multiplications, \( \mathcal{L} \) is a linear space; it is a normed space if for \( P = (P_1, P_2, \ldots, P_n) \) we define

\[
\|P\| = \max \left\{ \max_{0 \leq t \leq \omega} |P_1(t)|, \max_{0 \leq t \leq \omega} |P_2(t)|, \ldots, \max_{0 \leq t \leq \omega} |P_n(t)| \right\}.
\]

The displacement function \( q \) is holomorphic on the open set \( Q \). Moreover, \( q \) depends continuously on \( P \) with the above norm on \( \mathcal{L} \) and the topology of uniform convergence on compact sets on the set of holomorphic functions.

The positive real axis and the negative real axis are invariant. Moreover, if \( \varphi \) is a non-real solution which is periodic, then so is \( \bar{\varphi} \), its complex conjugate.

In [11], it was shown that the phase portrait of (1.1) is as shown in Fig. 1. We refer to [11] for the details. There, the coefficients \( P_i(t) \) were \( \omega \)-periodic. It can be verified that the same methods are applicable to the study of the number of solutions that satisfy \( z(0) = z(\omega) \) whether the coefficients are periodic or not.

Note that the radius, \( \rho \), of the disc \( D \) is a sufficiently large number that depends only on \( \|P\| \) and \( \omega \). If \( z = re^{i\theta} \) then the sets \( G_k, k = 0, 1, \ldots, 2n - 3 \), which are the “arms” in the figure, are defined by

\[
G_k = \left\{ z \mid r > \rho, \frac{k\pi}{n-1} - a \frac{\pi}{r} < \theta < \frac{k\pi}{n-1} + a \frac{\pi}{r} \right\},
\]

where \( a = \max\{6, 6\|P\|\} \). Between the arms are the sets \( H_k, k = 0, 1, \ldots, 2n - 3 \), which are defined by

\[
H_k = \left\{ z \mid r > \rho, \frac{k\pi}{n-1} + a \frac{\pi}{r} \leq \theta \leq \frac{(k+1)\pi}{n-1} - a \frac{\pi}{r} \right\}.
\]

For even \( k \), trajectories can enter \( G_k \) only across \( r = \rho \), and for odd \( k \), trajectories can leave \( G_k \) only across \( r = \rho \). No solution can become infinite in \( H_k \) as time either increases or decreases. Every solution enters \( D \). Solutions become unbounded if and only if they remain in one of the arms \( G_k \), tending to infinity as \( t \) increases if \( k \) is even and as \( t \) decreases if \( k \) is odd.

Let \( q(P, c) = z_P(\omega, c) - c \), where \( z_P(t, c) \) is the solution of \( P \in \mathcal{L} \) satisfying \( z_P(0, c) = c \). Suppose that \( (P_j) \) and \( (c_j) \) are sequences in \( \mathcal{L} \) and \( \mathbb{C} \), respectively, such that \( q(P_j, c_j) = 0 \). If \( P_j \to P \) and \( c_j \to c \) as \( j \to \infty \), then either \( q(P, c) = 0 \), in this case \( z_P(t, c) \) is a periodic

![Fig. 1. Phase portrait.](image)
solution, or \( z_P(t,c) \) is not defined for the whole interval \( 0 \leq t \leq \omega \). In the later case, we say that \( z_P(t, c) \) is a singular periodic solution. We also say that \( P \) has a singular periodic solution if \( c_j \to \infty \); in this case there are \( \tau \) and \( c \) such that the solution \( z_P \) with \( z_P(\tau) = c \) becomes unbounded in finite time as \( t \) increases and as \( t \) decreases. We summarize the results of [11].

**Proposition 2.1.**

(i) Let \( \mathcal{A} \) be the subset of \( \mathcal{L} \) consisting of all equations which have no singular periodic solutions. The set \( \mathcal{A} \) is open in \( \mathcal{L} \). All equations in the same components of \( \mathcal{A} \) have the same number of periodic solutions.

(ii) The equation \( \dot{z} = z^n \) has exactly \( n \) periodic solutions.

(iii) For each \( k \) and for each \( t_0 \) and \( r \geq \rho \), there is a unique \( \theta_0 \) such that the solution \( z(t,c) \) with \( z(t_0,c) = c \) remains in \( G_k \) for \( t \geq t_0 \) (\( k \) is even) or \( t \leq t_0 \) (\( k \) is odd) if and only if \( c = r e^{i\theta_0} \).

3. Main result

Assume that \( P_n(t) \equiv 0 \). We call the solution \( z = 0 \) a center if \( z(t,c) \) is periodic for all \( c \) in a neighborhood of 0. If the term \( z^n \) in (1.1) is replaced by \( P_0(t)z^n \), then there are equations with a center when \( P_0 \) has zeros. For cubic equations, this is related to the classical center problem of polynomial two-dimensional systems; we refer to [5] for details. However, when \( P_0 \) has no zeros then \( z = 0 \) is never a center. Particular cases of this result were given in [3] and [4] for \( n = 4 \) and \( n = 5 \), respectively. We give a brief proof, for the sake of completeness.

**Theorem 3.1.** The solution \( z = 0 \) is isolated as a periodic solution of (1.1) with \( P_n(t) \equiv 0 \).

**Proof.** Suppose, if possible, that there is an open set \( U \subset \mathbb{C} \) containing the origin such that all solutions starting in \( U \) are periodic. Then \( q \equiv 0 \) in the component of its domain of definition containing the origin. But the real zeros of \( q \) are contained in the disc \( D \). Thus

\[
\inf \{ c \in \mathbb{R} \mid c > 0, \ z(t,c) \text{ is not defined for } 0 \leq t \leq \omega \} < \infty.
\]

It follows that there is a real singular periodic solution; but a positive real periodic solution which tends to infinity can do so only as \( t \) increases. This is a contradiction, and the result follows. \( \Box \)

Now, we give the result about the number of periodic solutions.

**Theorem 3.2.** Suppose that \( \beta(t) \leq 0 \). Equation (1.2) has exactly \( n \) periodic solutions.

**Proof.** With \( z = re^{i\theta} \), we have

\[
\dot{\theta} = r^{n-1} \sin(n-1)\theta + r^{n-2} \alpha(t) \sin(n-2)\theta + r^{n-3} \beta(t) \sin(n-3)\theta.
\]

If \( |c| > \rho \) and is real then the real solution \( z(t,c) \) remains outside the disk \( D \) either when \( t \) increases or when \( t \) decreases, and will become infinite. Solutions that enter \( G_0 \) or \( G_{n-1} \) will leave \( G_0 \) or \( G_{n-1} \), except the solution that enters at \( \rho e^{i\theta_0} \) described in part (iii) of Proposition 2.1; this solution is real because any solution which is once real is always real. Therefore, the unique solution that becomes infinite is a real solution if \( k = 0 \) or \( k = n - 1 \). On the other hand, no real solution is unbounded as \( t \) increases and decreases. Hence, no singular periodic solution enters \( G_0 \) or \( G_{n-1} \) because singular periodic solutions are unbounded both as \( t \) increases and decreases.
Thus, a singular periodic solution enters $D$ from a $G_k$ with odd $k \neq n - 1$ and leaves $D$ to a $G_j$ with even $j \neq 0, n - 1$.

Now, if $k$ is odd and $1 \leq k \leq n - 3$, let $\theta_1 = \frac{k\pi}{n-2}$, $\theta_2 = \frac{(k-1)\pi}{n-2}$, and consider

$$
\dot{\theta}(\theta_1) = r^{n-1} \sin((n-1)\theta_1) + r^{n-2} \alpha(t) \sin((n-2)\theta_1) + r^{n-3} \beta(t) \sin((n-3)\theta_1),
$$

$$
\dot{\theta}(\theta_2) = r^{n-1} \sin((n-1)\theta_2) + r^{n-2} \alpha(t) \sin((n-2)\theta_2) + r^{n-3} \beta(t) \sin((n-3)\theta_2).
$$

But,

$$
\sin((n-1)\theta_1) = \sin\left(k\pi + \frac{k\pi}{n-2}\right) < 0,
$$

$$
\sin((n-3)\theta_1) = \sin\left(k\pi - \frac{k\pi}{n-2}\right) > 0,
$$

$$
\sin((n-1)\theta_2) = \sin\left((k-1)\pi + \frac{(k-1)\pi}{n-2}\right) > 0,
$$

$$
\sin((n-3)\theta_2) = \sin\left((k-1)\pi - \frac{(k-1)\pi}{n-2}\right) < 0,
$$

$$
\sin((n-2)\theta_1) = \sin((n-2)\theta_2) = 0.
$$

Under the above hypotheses, $\dot{\theta}(\theta_1) < 0$ and $\dot{\theta}(\theta_2) > 0$. Since $\rho$ is a sufficiently large number, we assume that $\rho > \frac{\sigma(n-2)(n-1)}{\pi}$. This condition on $\rho$ guarantees that the arc of intersection of $G_k$ with $D$ lies inside the sector

$$
\{r < \rho, \theta_2 < \theta < \theta_1\}.
$$

Precisely, the condition implies that

$$
\frac{k\pi}{n-2} > \frac{k\pi}{n-1} + \frac{a}{\rho}, \quad \frac{(k-1)\pi}{n-2} < \frac{k\pi}{n-1} - \frac{a}{\rho}.
$$

Hence, solutions do not leave the sector (see Fig. 2). Therefore, no singular periodic solution can enter $D$ from $G_k$ and leave $D$ to $G_{k-1}$ or $G_{k+1}$. Since the phase portrait is symmetric about the $x$-axis, if $k$ is odd and $n \leq k \leq 2n - 3$, no singular periodic solution can enter $D$ from $G_k$ and leaves $D$ to $G_{k-1}$ or $G_{k+1}$. It follows that the equation does not have a singular periodic solution.

Now, consider the class of equations

$$
\dot{z} = z^n + s\alpha(t)z^{n-1} + s\beta(t)z^{n-2},
$$

Fig. 2. Sector around $G_k$, odd $k$. 
with $0 \leq s \leq 1$. Since, $s\beta(t) \leq 0$, any equation in this family does not have singular periodic solutions. The equation $\dot{z} = z^n$ belongs to this family and has $n$ periodic solutions. By part (i) of Proposition 2.1, each of these equations has $n$ periodic solutions. □

**Remark 3.3.** Consider the differential equation

$$\dot{z} = z^n + \alpha(t)z^m + \beta(t)z^l$$

with $0 < l < m < n$. Following the steps in the proof of Theorem 3.2, let $\theta_1 = \frac{k\pi}{j}$ when $k$ is odd and $0 \leq k \leq n - 3$. It is required that

$$\sin((n - 1)\theta_1) > 0, \quad \sin((m - 1)\theta_1) = 0, \quad \sin((l - 1)\theta_1) < 0.$$ 

These conditions imply that $j = m - 1$, $m \geq n - 1 - \frac{1}{n - 2}$, and $l \geq n - 2 - \frac{1}{n - 3}$. Hence, we should assume that $m = n - 1$ and $l = n - 2$.

**Lemma 3.4.** Let $A = \int_0^\omega \alpha(t) \, dt$ and $B = \int_0^\omega \beta(t) \, dt$. The solution $z = 0$ of (1.2) has multiplicity

- (i) $n - 2$ if $B \neq 0$,
- (ii) $n - 1$ if $B = 0$ and $A \neq 0$,
- (iii) $n$ if $A = B = 0$.

**Proof.** We write

$$z(t, c) = \sum_{k=1}^\infty a_k(t)c^k$$

and substitute directly into Eq. (1.2). This gives a recursive set of linear differential equations for the $a_k(t)$ with initial conditions $a_1(0) = 1$ and $a_k(0) = 0$ if $k > 1$. The multiplicity is $K$ if $a_1(\omega) = 1$, $a_k(\omega) = 0$ for $2 \leq k \leq K - 1$ and $a_K(\omega) \neq 0$. Direct computations give

$$a_1(t) \equiv 1; \quad a_k(t) \equiv 0, \quad 2 \leq k \leq n - 3,$$

$$\dot{a}_{n-2} = \beta; \quad \dot{a}_{n-1} = \alpha; \quad \dot{a}_n = 1.$$ 

Solving the last three equations gives $a_{n-2}(\omega) = B$, $a_{n-1}(\omega) = A$, and $a_n(\omega) = \omega$. The result follows. □

**Corollary 3.5.** Consider Eq. (1.2) with $\beta(t) \leq 0$. The equation has

- (i) two non-zero periodic solutions if $B < 0$; at most one is positive and at most one is negative. If $B$ is small and negative, then there are two non-zero real periodic solutions;
- (ii) one non-zero periodic solution if $B = 0$ and $A \neq 0$; this solution is a real solution. It is positive if $A < 0$ and it is negative if $A > 0$;
- (iii) no non-zero periodic solutions if $A = B = 0$.

**Proof.** Consider the case $B < 0$. The multiplicity of the origin is $n - 2$. There are two non-zero periodic solutions. In a neighborhood $c = 0$, $q(c) = Bc^{n-2} + O(c^{n-1})$. If $n$ is odd then $q(c)$ has the sign of $c$ if $|c|$ is large. Therefore, $q$ has a positive solution and a negative solution if it is defined for large $|c|$. If $n$ is even then $q(c) > 0$ if $|c|$ is large. Again, $q$ has a positive solution and a negative solution if it is defined for large $|c|$.
In the case $B = 0$ and $A \neq 0$, the multiplicity of the origin is $n - 1$. There is only one non-zero solution which is a real solution because complex solutions occur in conjugates pairs. The argument used in the first case implies that this solution is positive if $A < 0$ and is negative if $A > 0$.

If we start with $A = B = 0$, then the multiplicity of $z = 0$ is $n$. Perturb the equation, so that $B \neq 0$ and $A \neq 0$. Since the total number of periodic solutions is unchanged by small perturbations, a real solution will bifurcate out of the origin. Now, we make a second perturbation so that $B$ is negative and such that $|B|$ is small compared to $|A|$. A second real non-zero solution will bifurcate out of the origin. If the equation is perturbed such that $A = 0$ but $B$ is negative and small, then the stability will be reversed and two non-zero real periodic solutions bifurcate out of the origin; one is positive and one is negative.

### 4. Rigid systems

Consider the system

$$
\begin{align*}
\dot{x} &= \lambda x - y + x(R_{n-1}(x, y) + R_{n-2}(x, y) + \cdots + R_1(x, y)), \\
\dot{y} &= x + \lambda y + y(R_{n-1}(x, y) + R_{n-2}(x, y) + \cdots + R_1(x, y)),
\end{align*}
$$

(4.1)

where $R_i$ is a homogeneous polynomial of degree $i$. The system in polar coordinates becomes

$$
\begin{align*}
\dot{r} &= r^n R_{n-1}(\cos \theta, \sin \theta) + r^{n-1} R_{n-2}(\cos \theta, \sin \theta) + \cdots + r^2 R_1(\cos \theta, \sin \theta) + \lambda r, \\
\dot{\theta} &= 1.
\end{align*}
$$

This system is called a rigid system because the derivative of the angular variable is constant. It was shown in [2] that if the origin is a center then \( \int_{0}^{2\pi} R_k(\cos \theta, \sin \theta) d\theta = 0 \) for all \( 1 \leq k \leq n - 1 \). In fact, these definite integrals are the first focal values of the system. It is clear that the origin is the only critical point and if it is a center then it is a uniformly isochronous center; see [6]. Limit cycles of (4.1) correspond to positive $2\pi$-periodic solutions of

$$
\frac{dr}{d\theta} = R_{n-1}r^n + R_{n-2}r^{n-1} + \cdots + R_1r^2 + \lambda r.
$$

In [7], the family of rigid systems

$$
\begin{align*}
\dot{x} &= \lambda x - y + xR_{n-1}(x, y), \\
\dot{y} &= x + \lambda y + yR_{n-1}(x, y)
\end{align*}
$$

was considered. It was shown that if \( \lambda \int_{0}^{2\pi} R_{n-1}(\cos \theta, \sin \theta) d\theta < 0 \), then there is at most one limit cycle. Cubic rigid systems were considered in [8]. It was shown that there are at most two limit cycles.

Now, we consider the rigid system

$$
\begin{align*}
\dot{x} &= -y + x(R_{n-1}(x, y) + R_{n-2}(x, y) + R_{n-3}(x, y)), \\
\dot{y} &= x + y(R_{n-1}(x, y) + R_{n-2}(x, y) + R_{n-3}(x, y)).
\end{align*}
$$

(4.2)

In polar coordinates, the system becomes

$$
\frac{dr}{d\theta} = R_{n-1}r^n + R_{n-2}r^{n-1} + R_{n-3}r^{n-2}.
$$
If a function $R_i$ does not change sign, then it is necessary to assume that $i$ is even. We assume that $n$ is odd. Hence, $R_{n-2}$ is a homogeneous polynomial in $\sin \theta$ and $\cos \theta$ of odd degree; therefore, $\int_0^{2\pi} R_{n-2}(\cos \theta, \sin \theta) \, d\theta = 0$. On the other hand, real periodic solutions occur in pairs. If $\varphi(\theta)$ is a real periodic solution then so is $-\varphi(\theta + \pi)$. Let $B = \int_0^{2\pi} R_{n-3}(\cos \theta, \sin \theta) \, d\theta$. The Liapunov quantities of the system are $B$ and $1$. If $B = 0$ then the origin is unstable. By perturbing the coefficients of $R_{n-3}$ such that $B < 0$ and is small enough, the origin will be unstable and a small-amplitude limit cycle appears through multiple Hopf bifurcation. The following result follows directly from these remarks and Theorem 3.2.

**Theorem 4.1.** Suppose that $n$ is odd, $R_{n-1} \equiv 1$, and $R_{n-3} \leq 0$. Let $B = \int_0^{2\pi} R_{n-3}(\cos \theta, \sin \theta) \, d\theta$.

(i) The origin is not a center for system (4.2).

(ii) If $B < 0$ then system (4.2) has at most one limit cycle. If $B$ is small enough then the system has a unique limit cycle; this limit cycle is unstable.

(iii) If $B = 0$ then system (4.2) does not have a limit cycle.

**Remark 4.2.** If the leading coefficient $R_{n-1}$ does not vanish anywhere then the transformation of the independent variable

$$
\theta \mapsto \exp \left( \int_0^\theta R_{n-1}(\cos u, \sin u) \, du \right)
$$

reduces the polar equation into a similar equation but with a leading coefficient equals one.

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**References**


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