Log Majorization and
Complementary Golden-Thompson Type Inequalities

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ABSTRACT

We obtain a log majorization result for power means of positive semidefinite matrices. This implies matrix norm inequalities for unitarily invariant norms, which are considered as complementary to the Golden-Thompson one. Other log majorization results are also obtained. We give logarithmic trace inequalities and determinant inequalities as applications of our log majorizations.

INTRODUCTION

The famous Golden-Thompson trace inequality, proved independently by Golden [7] and Thompson [13], is $\text{Tr} \, e^{H+K} \leq \text{Tr} \, e^H e^K$ for Hermitian matrices $H, K$. This inequality was extended by Lenard [10] and Thompson [14] to the weak majorization $e^{H+K} \prec e^{H/2} e^K e^{H/2}$, or equivalently $\|e^{H+K}\| \leq \|e^{H/2} e^K e^{H/2}\|$ for any unitarily invariant norm $\| \cdot \|$. 

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For positive semidefinite matrices $A, B \geq 0$ we write
\[
A \prec B \quad (\text{log})
\]
if $\prod_{i=1}^{k} \lambda_i(A) \leq \prod_{i=1}^{k} \lambda_i(B)$ for $k = 1, \ldots, n - 1$ and $\prod_{i=1}^{n} \lambda_i(A) = \prod_{i=1}^{n} \lambda_i(B)$, i.e., $\det A = \det B$, where $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ and $\lambda_1(B) \geq \cdots \geq \lambda_n(B)$ are the eigenvalues of $A$ and $B$ respectively. This notion is called the log majorization because it is equivalent to the usual majorization $\log A < \log B$ when $A, B > 0$ (strictly positive). Since
\[
A \prec B \quad \text{implies} \quad A \preceq \omega B, \quad (\text{log})
\]
so that $\|A\| \leq \|B\|$ for any unitarily invariant norm, log majorization gives a powerful technique for matrix norm inequalities. See [1, 5, 11] for theory of majorization for matrices.

Araki [2] used the log majorization method to extend the trace inequality of Lieb and Thirring. In fact, he showed the log majorization
\[
(A^{1/2}BA^{1/2})^r \prec A^{r/2}B^r A^{r/2} \quad (\text{log})
\]
for $A, B \geq 0$ and $r \geq 1$. This together with the Lie-Trotter formula strengthens the Golden-Thompson inequality as follows: For Hermitian $H, K$,
\[
\|\{\exp(pH/2) \exp(pK) \exp(pH/2)\}^{1/p}\|
\]
decreases to $\|\exp(H + K)\|$ as $p \downarrow 0$ for any unitarily invariant norm.

On the other hand, the Golden-Thompson trace inequality was complemented in [8]. More precisely, the following trace inequality was proved:
\[
\text{Tr}\{\exp(pH) \#_\alpha \exp(pK)\}^{1/p} \leq \text{Tr} \exp\{(1 - \alpha)H + \alpha K\}
\]
for Hermitian $H, K$ and $0 \leq \alpha \leq 1$, where $\#_\alpha$ denotes the $\alpha$-power mean, i.e., for $A, B > 0$
\[
A \#_\alpha B = A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}.
\]
The main purpose of this paper is to establish the log majorization

\[ A^r \#_\alpha B^r < (A \#_\alpha B)^r \]

for \( A, B \geq 0 \) and \( r \geq 1 \). This enables us to show that

\[ \| \exp( pH ) \#_\alpha \exp( pK ) \|^{1/p} \]

increases to \( \| \exp((1 - \alpha)H + \alpha K) \| \) as \( p \downarrow 0 \) for any unitarily invariant norm. We thus complete the complementary counterpart of the Golden-Thompson inequality.

After giving some preliminaries on log majorization in Section 1, we prove in Section 2 the main result stated above, using the method of compound matrices (or antisymmetric tensor powers) as in [2]. In Section 3 we transform the Furuta inequality into an equivalent log majorization. In Section 4 we present some other log majorizations. For instance, Cohen's generalization [4] of Bernstein's trace inequality [3] is restated as a log majorization. In Sections 5 and 6 we apply the log majorization results to obtain logarithmic trace inequalities and determinant inequalities.

1. PRELIMINARIES ON LOG MAJORIZATION

Throughout this paper we consider \( n \times n \) complex matrices. For Hermitian matrices \( H, K \) the weak majorization (or submajorization) \( H <_w K \) means that

\[ \sum_{i=1}^{k} \lambda_i(H) \leq \sum_{i=1}^{k} \lambda_i(K), \quad k = 1, 2, \ldots, n, \]

where \( \lambda_1(H) \geq \cdots \geq \lambda_n(H) \) and \( \lambda_1(K) \geq \cdots \geq \lambda_n(K) \) are the eigenvalues of \( H \) and \( K \) respectively. Further, the majorization \( H < K \) means that \( H <_w K \) and the equality holds for \( k = n \) in the above i.e., \( \text{Tr } H = \text{Tr } K \).

We write \( A \geq 0 \) if \( A \) is a positive semidefinite matrix, and \( A > 0 \) if \( A \geq 0 \) is invertible (or strictly positive definite). For \( A, B \geq 0 \) let us write
A < B and refer to \( \log \) majorization if
\[
\prod_{i=1}^{k} \lambda_i(A) \leq \prod_{i=1}^{k} \lambda_i(B), \quad k = 1, 2, \ldots, n - 1,
\]
and
\[
\prod_{i=1}^{n} \lambda_i(A) = \prod_{i=1}^{n} \lambda_i(B), \quad \text{i.e.,} \quad \det A = \det B.
\]

Note that when \( A, B > 0 \) the \( \log \) majorization \( A < B \) is equivalent to \( \log A < \log B \).

In this section we list some preliminary results for later convenience.

**Lemma 1.1.** If
\[
A, B \geq 0 \quad \text{and} \quad A < B,
\]
then the following hold:

1. \( \text{Tr} f(A) \leq \text{Tr} f(B) \) for any continuous function \( f \) on an interval containing the eigenvalues of \( A, B \) such that \( f(e') \) is convex.
2. \( \| A \| \leq \| B \| \) for any unitarily invariant norm \( \| \cdot \| \).

In fact, it is well known [1, 11] that if \( H \) and \( K \) are Hermitian with \( H < K \), then \( f(H) < \_w f(K) \) for any convex function \( f \) on an interval containing the eigenvalues of \( H, K \). Also for \( A, B \geq 0, A < B \) holds if and only if \( \| A \| \leq \| B \| \) for any unitarily invariant norm \( \| \cdot \| \). Hence the above lemma follows when \( A, B > 0 \). When \( A, B \geq 0 \) we can choose sequences \( \{ A_n \} \) and \( \{ B_n \} \) such that \( A_n, B_n > 0, A_n < B_n, A_n \rightarrow A, \) and \( B_n \rightarrow B \).

For each matrix \( X \) and \( k = 1, 2, \ldots, n \) let \( C_k(X) \) denote the \( k \)th compound (or the \( k \)-fold antisymmetric tensor power) of \( X \). See e.g. [5, 11] for details. Then (1) and (2) below are basic facts, and (3) is easily seen from (2).

**Lemma 1.2.**

1. \( C_k(X^*) = C_k(X)^* \).
2. \( C_k(XY) = C_k(X)C_k(Y) \) for every pair of matrices \( X, Y \).
3. \( C_k(A^p) = C_k(A)^p \) for every \( A \geq 0 \) and \( p > 0 \).
Moreover, the following is a well-known consequence of the Binet-Cauchy theorem (see [11, pp. 503–504]), which says that to obtain $A < B$ for given $A, B > 0$ we may show that $\|C_k(A)\|_\infty \leq \|C_k(B)\|_\infty$ for $k = 1, \ldots, n$ as well as $\det A = \det B$. Here $\|\cdot\|_\infty$ denotes the operator (or spectral) norm.

**Lemma 1.3.** For every $A > 0$,

$$\prod_{i=1}^{k} \lambda_i(A) = \lambda_i(C_k(A)) \left[ = \|C_k(A)\|_\infty \right], \quad k = 1, \ldots, n.$$

The following is the well-known Löwner-Heinz inequality.

**Lemma 1.4.** If $A > B > 0$ then $A^p > B^p$ for every $0 < p < 1$.

We finally state the Lie-Trotter formula (see [12, p. 295]) and its continuous parameter version (see the remark after [8, Lemma 3.3]).

**Lemma 1.5.**

1. For every pair of matrices $S, T$,

$$\lim_{k \to \infty} \{\exp(S/k) \exp(T/k)\}^k.$$

2. For every Hermitian $H, K$,

$$\lim_{p \to 0} \left\{ \exp\left(\frac{pH}{2}\right) \exp\left(\frac{pK}{2}\right) \exp\left(\frac{pH}{2}\right) \right\}^{1/p}.$$

2. **INEQUALITIES FOR POWER MEANS**

Araki [2] proved the following:

**Theorem A.** For every $A, B > 0$,

$$\left( A^{1/2} BA^{1/2} \right)^{(log)}_r < A^{r/2} B^{r/2}, \quad r \geq 1,$$
or equivalently

\[
(A^{q/2}B^{q/2})^{1/q} \prec (A^{p/2}B^{p/2})^{1/p}, \quad 0 < q \leq p. \tag{2.1}
\]

Here note that

\[
\det (A^{1/2}BA^{1/2})^r = (\det A \det B)^r = \det(A^{r/2}B^{r/2}).
\]

The above (2.1) together with Lemmas 1.1 and 1.5 implies that if \( H \) and \( K \) are Hermitian, then

\[
\|\{\exp(\frac{pH}{2}) \exp(pK) \exp(\frac{pH}{2})\}\|^{1/p}
\]

decreases to \( \|\exp(H + K)\| \) as \( p \downarrow 0 \) for any unitarily invariant norm \( \| \cdot \| \). In particular, taking the trace norm, we have the strengthened Golden-Thompson trace inequality:

\[
\text{Tr} \exp(H + K) \leq \text{Tr}\{\exp(\frac{pH}{2}) \exp(pK) \exp(\frac{pH}{2})\}^{1/p}, \quad p > 0.
\]

In this section we establish a log majorization for power means of matrices. When \( 0 < \alpha < 1 \), the \( \alpha \)-power mean of \( A, B > 0 \) is defined and denoted by

\[
A \#_{\alpha} B = A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}.
\]

Further, \( A \#_{\alpha} B \) for \( A, B > 0 \) is defined by

\[
A \#_{\alpha} B = \lim_{\varepsilon \downarrow 0} (A + \varepsilon I) \#_{\alpha} (B + \varepsilon I).
\]

This \( \alpha \)-power mean is the operator mean corresponding to the operator monotone function \( t^\alpha \). See [6] for general theory of operator means. In particular, when \( \alpha = \frac{1}{2} \), \( A \#_{1/2} B = A \# B \) is the so-called geometric mean of \( A, B \geq 0 \). Note that \( A \#_{\alpha} B = B \#_{1-\alpha} A \) and if \( AB = BA \) then \( A \#_{\alpha} B = A^{1-\alpha}B^{\alpha} \), and that \((A, B) \mapsto A \#_{\alpha} B \) is jointly monotone.

**Theorem 2.1.** For every \( A, B \geq 0 \) and \( 0 \leq \alpha < 1 \),

\[
A^r \#_{\alpha} B^r \prec (A \#_{\alpha} B)^r, \quad r > 1,
\]
or equivalently

\[ (A \#_\alpha B)^r \not< A^r \#_\alpha B^r, \quad 0 < r \leq 1, \quad (2.3) \]

\[ (A^p \#_\alpha B^p)^{1/p} \not< (A^q \#_\alpha B^q)^{1/q}, \quad 0 < q \leq p. \quad (2.4) \]

**Proof.** The equivalence of (2.2)–(2.4) is immediate. To prove (2.2) we may assume that \( A, B > 0 \), and let \( r \geq 1 \). It is easy to see by Lemma 1.2 that for \( k = 1, \ldots, n \)

\[ C_k(A^r \#_\alpha B^r) = C_k(A)^r \#_\alpha C_k(B)^r. \]

\[ C_k((A \#_\alpha B)^r) = (C_k(A) \#_\alpha C_k(B))^r. \]

Also

\[ \det(A^r \#_\alpha B^r) = (\det A)^{r(1-\alpha)}(\det B)^{r\alpha} = \det(A \#_\alpha B)^r. \]

Hence it suffices by Lemma 1.3 to show that

\[ \lambda_{1}(A^r \#_\alpha B^r) \leq \lambda_{1}(A \#_\alpha B)^r. \quad (2.5) \]

To do so we may prove that \( A \#_\alpha B \leq I \) implies \( A^r \#_\alpha B^r \leq I \), because both sides of (2.5) have the same order of homogeneity for \( A, B \), so that we can multiply \( A, B \) by a positive constant.

First let us assume \( 1 \leq r \leq 2 \) and write \( r = 2 - \varepsilon \) with \( 0 \leq \varepsilon \leq 1 \). Let \( C = A^{-1/2}BA^{-1/2} \). Then \( B = A^{1/2}CA^{1/2} \) and \( A \#_\alpha B = A^{1/2}C^\alpha A^{1/2} \). If \( A \#_\alpha B \leq I \) then \( C^\alpha \leq A^{-1} \), so that

\[ A \leq C^{-\alpha}. \quad (2.6) \]

Hence by Lemma 1.4

\[ A^{1-\varepsilon} \leq C^{-\alpha(1-\varepsilon)}. \quad (2.7) \]
We now get
\[
A^r \#_{\alpha} B^r = A^{1-\varepsilon/2} \left( A^{-1+\varepsilon/2} B - B^{-\varepsilon} \cdot B A^{-1+\varepsilon/2} \right)^{\alpha/2} A^{1-\varepsilon/2}
\]
\[
= A^{1-\varepsilon/2} \left\{ A^{-1} (1-\varepsilon)/2 C A^{1/2} \left( A^{-1/2} C^{-1} A^{-1/2} \right)^{\varepsilon} \times A^{1/2} C^{-1} A^{-1/2} \right\}^{\alpha/2} A^{1-\varepsilon/2}
\]
\[
= A^{1/2} \left( A^{1-\varepsilon} \#_{\alpha} \left[ C \left( A \#_{\varepsilon} C^{-1} \right) C \right] \right) A^{1/2}
\]
\[
< A^{1/2} \left( C^{-\alpha (1-\varepsilon)} \#_{\alpha} \left[ C \left( C^{-\alpha} \#_{\varepsilon} C^{-1} \right) C \right] \right) A^{1/2},
\]
using (2.6), (2.7), and the joint monotonicity of power means. Since a direct computation yields
\[
C^{-\alpha (1-\varepsilon)} \#_{\alpha} \left[ C \left( C^{-\alpha} \#_{\varepsilon} C^{-1} \right) C \right] = C^\alpha,
\]
we get
\[
A^r \#_{\alpha} B^r \leq A^{1/2} C^{\alpha} A^{1/2} = A \#_{\alpha} B \leq I.
\]

When \( r > 2 \), writing \( r = 2^k (2 - \varepsilon) \) with \( k \in \mathbb{N} \) and \( 0 \leq \varepsilon \leq 1 \), we can proceed by induction. \( \blacksquare \)

The next lemma was given in [8, Lemma 3.3].

**Lemma 2.2.** If \( H \) and \( K \) are Hermitian and \( 0 \leq \alpha \leq 1 \), then

\[
\exp \{ (1 - \alpha) H + \alpha K \} = \lim_{p \to 0} \{ \exp(pH) \#_{\alpha} \exp(pK) \}^{1/p}.
\]

By Theorem 2.1, Lemma 2.2, and (2.1) we have:

**Corollary 2.3.** If \( H \) and \( K \) are Hermitian and \( 0 \leq \alpha \leq 1 \), then for every \( p, q > 0 \)

\[
\{ \exp(pH) \#_{\alpha} \exp(pK) \}^{1/p}
\]
\[
< \exp \{ (1 - \alpha) H + \alpha K \}
\]
\[
< \log \{ \exp \left( \frac{1 - \alpha}{2} qH \right) \exp(\alpha qK) \exp \left( \frac{1 - \alpha}{2} qH \right) \}^{1/q}.
\]
COROLLARY 2.4. If $H$ and $K$ are Hermitian and $0 \leq \alpha \leq 1$, then
\[
\|\{\exp(pH) \#_\alpha \exp(pK)\}^{1/p}\|
\]
increases to $\|\exp((1 - \alpha)H + \alpha K)\|$ as $p \downarrow 0$ for any unitarily invariant norm $\|\cdot\|$. In particular,
\[
\text{Tr}\{\exp(pH) \#_\alpha \exp(pK)\}^{1/p}
\]
increases to $\text{Tr}\exp((1 - \alpha)H + \alpha K)$ as $p \downarrow 0$, which gives the complemented Golden-Thompson trace inequality proved in [8].

3. LOG MAJORIZATION EQUIVALENT TO THE FURUTA INEQUALITY

In the following let us adopt the usual convention $A^0 = I$ for $A \geq 0$. Furuta [6] proved a remarkable generalization of Lemma 1.4, which can be stated in the following form.

**Theorem B.** If $A \geq B \geq 0$, then
\[
A^{\alpha(r+s)} \geq \left( A^{r/2}B^sA^{r/2} \right)^\alpha
\]
whenever $0 < \alpha < 1$, $r, s \geq 0$, and $(1 - \alpha)r \geq \alpha s - 1$.

The next theorem is a reformulation of the Furuta inequality in terms of log majorization.

**Theorem 3.1.** If $A \geq 0$ and $B \geq 0$, then
\[
A^{(1 - \alpha)/2}B^\alpha A^{(1 - \alpha)/2} \succ \left( A^{(1 - \alpha)q/2\alpha}B^pA^{(1 - \alpha)q/2\alpha} \right)^{1/p}
\]
whenever $0 < \alpha \leq 1$, $p \geq 0$, and $q \leq \min\{\alpha, \alpha p\}$. 
Proof. When $0 < \alpha < 1$, $r, s > 0$, and $(1 - \alpha)r \geq \alpha s - 1$, Theorem B says that $A^{-1/2}BA^{-1/2} \leq I$ implies

$$A^{-(r+s)\alpha} (A^{[r-a(r+s)]/2}B^\alpha A^{[r-a(r+s)]/2})$$

$$= A^{-\alpha(r+s)/2} (A^{r/2}B^\alpha A^{r/2})^\alpha A^{-\alpha(r+s)/2} \leq I.$$ 

Arranging the order of homogeneity and using Lemma 1.3 as in the proof of Theorem 2.1, we obtain

$$(A^{-1/2}BA^{-1/2})^\alpha > A^{-\alpha(r+s)}\#_\alpha (A^{[r-a(r+s)]/2}B^\alpha A^{[r-a(r+s)]/2}).$$

Now the result follows when we put $p = \alpha s$ and $q = \alpha[\alpha(r + s) - r]$ and replace $A, B$ by $A^{-\alpha} B^\alpha$, respectively.

When $q = \alpha$ the theorem becomes the following:

**Corollary 3.2.** For every $A, B \geq 0$ and $0 < \alpha \leq 1$,

$$A^{(1-a)/2}B^\alpha A^{(1-a)/2} > \{A^p \#_\alpha (A^{[1-a]/2}B^p A^{[1-a]/2})\}^{1/p}, \quad p \geq 1.$$ 

We obtain another log majorization from the Furuta inequality.

**Theorem 3.3.** If $A \geq 0$ and $B \geq 0$, then

$$A^{1/2} (A^p \#_\alpha B^p)^{q/p} A^{1/2} > A^{(1+q)/2} (A^{-p/2}B^p A^{-p/2})^{\alpha q/p} A^{(1+q)/2}$$

for every $0 \leq \alpha \leq 1$ and $0 < q < p$.

**Proof.** It suffices to show that $A^{-1} \geq (A^p \#_\alpha B^p)^{q/p}$ implies

$$A^{-(1+q)} \geq (A^{-p/2}B^p A^{-p/2})^{\alpha q/p}.$$ 

If $A^{-1} \geq (A^p \#_\alpha B^p)^{q/p}$, then by (3.1) with $r = p$, $s = p/q$, and $\alpha = q/p$ we have

$$A^{-(1+q)} \geq \{A^{-p/2} (A^p \#_\alpha B^p) A^{-p/2}\}^{q/p} = (A^{-p/2}B^p A^{-p/2})^{\alpha q/p},$$

as desired.
COROLLARY 3.4. If $A \succeq 0$ and $B \succeq 0$, then for every $0 < r \leq 1$

\[
(A^{1/2}BA^{1/2})^r > A^{r/2}B^rA^{r/2} > A^r(A^{-1/2}BA^{-1/2})^r A^r.
\]

Proof. The first log majorization is nothing but (2.1). For the second, take $\alpha = 1$ and $q = 1$ in Theorem 3.3, and then replace $A, B$ by $A', B'$, respectively, with $r = 1/p$. We can consider the converse direction from log majorizations to matrix inequalities. In fact, it is immediate to show the Furuta inequality from Theorem 3.1. Also we can transform Theorem 2.1 into the following matrix inequality.

THEOREM 3.5. If $A \succeq B \succeq 0$ with $A > 0$, then

\[
A^r \geq \left[ A^{r/2}(A^{-1/2}BP^{-1}A^{-1/2})^r A^{r/2} \right]^{1/p}, \quad r, p > 1.
\]

Proof. Theorem 2.1 shows that if $A > 0$, $B \succeq 0$, and $0 < \alpha \leq 1$, then $A^{-1} > (A^{-1/2}BA^{-1/2})^\alpha$ implies $A^{-r} > (A^{-r/2}B^rA^{-r/2})^\alpha$ for every $r > 1$. Putting $p = 1/\alpha$ and replacing $A^{-1}, (A^{-1/2}BA^{-1/2})^\alpha$ by $A, B$, respectively, we get the result.

When $r = 2$ the theorem becomes the following:

COROLLARY 3.6. If $A \succeq B \succeq 0$ with $A > 0$, then

\[
A^2 \geq |A^{-1/2}BP^{-1}A^{-1/2}|^{2/p}, \quad p > 1.
\]

4. OTHER LOG MAJORIZATION RESULTS

In this section we obtain some other log majorization results. The following log majorization for matrix exponentials is just a restatement of the spectral inequality by Cohen [4].

THEOREM C. For an arbitrary matrix $T$,

\[
|\exp(T)| \leq \exp(\text{Re } T)
\]

where $\text{Re } T = (T + T^*)/2$.
The following was also proved in [4]: For any matrix $X$,

$$|X^k| < |X|^k$$

(log)

\(i.e., \ X^* X^k < (X X)^k\), \(k \in \mathbb{N}\). \hspace{1cm} (4.1)

In fact, the argument using Lemma 1.3 shows (4.1) because \(\|X^k\|_\infty \leq \|X\|_\infty^k = \|X\|_\infty^k\). The above theorem immediately follows, as in [4], from (4.1) applied to \(X = \exp(T/k)\) and Lemma 1.5.

As a consequence we have

$$\|\exp(T)\| \leq \|\exp(\text{Re} T)\|$$

for any unitarily invariant norm \(\| \cdot \|\). This extends the trace inequality in [3].

The next theorem is a rather general type of log majorization.

**THEOREM 4.1.** For every \(A, B \geq 0\),

$$A^{p_1}B^{q_1} \cdots A^{p_k}B^{q_k} < A^{p_1 + \cdots + p_k}B^{q_1 + \cdots + q_k}$$

(log)

whenever

$$0 \leq p_1 \leq q_1 \leq p_1 + p_2 \leq q_1 + q_2 \leq \cdots$$

$$< q_1 + \cdots + q_{k-1} < p_1 + \cdots + p_k = q_1 + \cdots + q_k.$$

**Proof.** We can suppose that \(A, B > 0\) and \(p_1 + \cdots + p_k = q_1 + \cdots + q_k = 1\). Then it suffices as before to show that \(BA^2B \leq I\) implies

$$B^{q_k}A^{p_k} \cdots B^{q_1}A^{p_1}B^{q_1} \cdots A^{p_k}B^{q_k} \leq I.$$

But if \(BA^2B \leq I\), then since \(A^2 \leq B^{-2}\), \(B^2 \leq A^{-2}\), and \(0 \leq q_1 - p_1 \leq 1\), we get by Lemma 1.4

$$B^{q_1}A^{p_1}B^{q_1} \leq B^{q_1}B^{-2p_1}B^{q_1} = B^{2(q_1-p_1)} \leq A^{2(p_1-q_1)},$$

so that

$$A^{p_1}B^{q_1}A^{2p_1}B^{q_1}A^{p_2} \leq A^{2(p_1+p_2-q_1)} \leq B^{2(q_1-p_1-p_2)},$$
Repeating this argument yields

\[ B^{q_1} A^{p_1} \cdots B^{q_k} A^{p_1} B^{q_k} \cdots A^{p_k} B^{q_k} \leq B^{2(q_1 + \cdots + q_k - p_1 - \cdots - p_k)} = I, \]

as desired. \qed

As special cases of Theorem 4.1 we have:

**Corollary 4.2.** Let \( A, B > 0 \).

1. For every \( p_1, \ldots, p_k \geq 0 \),
   \[ |A^{p_1} B^{p_1} \cdots A^{p_k} B^{p_k}| < |A^{p_1 + \cdots + p_k} B^{p_1 + \cdots + p_k}|. \]

2. For every \( p, q, r > 0 \),
   \[ |A^p B^q A^r| < |A^{p+q} B^q|. \]

3. For every \( k \in \mathbb{N} \),
   \[ (AB)^k A < A^{(k+1)/2} B^k A^{(k+1)/2}. \]

**Proof.** (1) is obvious.

(2): By Theorem 4.1 with \( p_1 = p, p_2 = r, q_1 = p + r, \) and \( q_2 = 0 \), we get

\[ |A^p B^q A^r| = |A^p (B^{q/(p+r)})^{p+r} A^r| < |A^{p+r} (B^{q/(p+r)})^{p+r}| = |A^{p+r} B^q|. \]

(3): When \( k = 2m \) we get

\[ (AB)^{2m} A = |A^{1/2} (BA)^m|^2 \]

\[ = |A^{1/2} \left( (B^{2m}/(2m+1))^{(2m+1)/2} A \right)^m|^2 \]

\[ < |A^{(2m+1)/2} B^m|^2 \]
using Theorem 4.1 with

\[ p_1 = \frac{1}{2}, \quad p_2 = \cdots = p_{m+1} = 1, \]

\[ q_1 = \cdots = q_m = \frac{2m + 1}{2m}, \quad q_{m+1} = 0. \]

Moreover \(|A^{(2m+1)/2}B^m|^2\) is unitarily similar to

\[ |B^m A^{(2m+1)/2}|^2 = A^{(2m+1)/2} B^{2m} A^{(2m+1)/2}, \]

showing the result. When \(k = 2m - 1\) we can do similarly.

5. LOGARITHMIC TRACE INEQUALITIES

The following logarithmic trace inequalities were given in [S]: If \(A, B \geq 0\), then for every \(p > 0\)

\[ \frac{1}{p} \text{Tr} A \log( B^{p/2} A^{p} B^{p/2} ) \leq \text{Tr} A (\log A + \log B) \]

\[ \leq \frac{1}{p} \text{Tr} A \log( A^{p/2} B^{p} A^{p/2} ). \]

In this section we supplement these inequalities with related results.

**Theorem 5.1.** For every \(A, B \geq 0\),

\[ \frac{1}{p} \text{Tr} A \log( A^{p/2} B^{p} A^{p/2} ) \leq \text{Tr} A (\log A + \log B) \]

\[ \leq \frac{1}{p} \text{Tr} A \log( A^{p/2} B^{p} A^{p/2} ). \]

\( (5.1) \)

decreases to \(\text{Tr} A (\log A + \log B)\) as \(p \downarrow 0\).

**Proof.** We may suppose \(B > 0\) as in the proof of [8, Theorem 3.5]. Then we can suppose \(A > 0\) as well, because \(\text{Tr} A \log( A^{p/2} B^{p} A^{p/2} )\) is continuous in \(A \geq 0\). So let us write \(A = e^{H}\) and \(B = e^{-K}\) with Hermitian \(H, K\). For
0 ≤ α ≤ 1 Corollary 2.4 implies that
\[ \text{Tr}\{\exp(\alpha H) \# \exp(\alpha K)\}^{1/\alpha} \]
increases as \( \alpha \downarrow 0 \). Since this trace is equal to \( \text{Tr} \, e^H \) independently of \( p > 0 \) when \( \alpha = 0 \), it follows that
\[
\frac{d}{d\alpha} \left. \text{Tr}\{\exp(\alpha H) \# \exp(\alpha K)\}^{1/\alpha} \right|_{\alpha=0} = \frac{1}{p} \text{Tr} \, e^H \log\{\exp(-pH/2) \exp(\alpha K) \exp(-pH/2)\}
\]
increases as \( p \downarrow 0 \) (the above differentiation was computed in [8]). Thus (5.1) decreases as \( p \downarrow 0 \). Apply Lemma 1.5 for the limit.

**Theorem 5.2.** For every \( A, B \geq 0 \) and \( p > 0 \),
\[
\frac{1}{p} \text{Tr} A \log(A^p \# B^p)^2 \leq \text{Tr} A (\log A + \log B).
\] (5.2)

**Proof.** Let \( P \) and \( Q \) be the support projections of \( A \) and \( B \), respectively. Then the support of \( A^p \# B^p \) is \( P \wedge Q \) by [9, Theorem 3.7], so that the left-hand side of (5.2) is \(-\infty\) unless \( P \leq Q \). Thus we may suppose \( B > 0 \), considering the restrictions of \( A, B \) on the range of \( Q \). Further, \( \text{Tr} A = 1 \) can be assumed. Then for any Hermitian \( H \) we have, as in the proof of [8, Theorem 1.3],
\[
\text{Tr} A (\log A + \log B) \geq \text{Tr} AH - \log \text{Tr} \exp(H - \log B)
\]
\[
\geq \text{Tr} AH - \log \text{Tr}\{B^{-p/2} \exp(\alpha H) B^{-p/2}\}^{1/\alpha}.
\]
For \( H = (1/p) \log(A^p \# B^p)^2 \) this yields
\[
\text{Tr} A (\log A + \log B) \geq \frac{1}{p} \text{Tr} A \log(A^p \# B^p)^2
\]
\[
- \log \text{Tr}\{B^{-p/2} (A^p \# B^p)^2 \}^{1/\alpha}.
\]
But

\[
\text{Tr}\left\{ B^{-p/2}(A^p \# B^p)^2 B^{-p/2}\right\}^{1/p}
\]

\[
= \text{Tr}\left\{ (B^{-p/2}A^p B^{-p/2})^{1/2} B^p (B^{-p/2}A^p B^{-p/2})^{1/2}\right\}^{1/p}
\]

\[
= \text{Tr}\{B^{p/2} (B^{-p/2}A^p B^{-p/2})^{1/2} B^{p/2}\}^{1/p}
\]

\[
= \text{Tr}\{B^{p/2} (B^{-p/2}A^p B^{-p/2}) B^{p/2}\}^{1/p}
\]

\[
= \text{Tr} A = 1.
\]

This completes the proof. \(\square\)

**Theorem 5.3.** If \(A > 0\) and \(B > 0\), then for every \(0 < \alpha < 1\) and \(P > 0\)

\[
\frac{1}{p} \text{Tr} A \log(A^p \# B^p) + \frac{\alpha}{p} \text{Tr} A \log(A^{p/2} B^{-p/2} A^{p/2}) \geq \text{Tr} A \log A.
\]

**Proof.** By continuity we can suppose \(A > 0\) as well. Then Theorem 3.3 gives for \(0 < q < p\)

\[
\text{Tr} A(A^p \#_q B^p)^q \geq \text{Tr} A^{1+q}(A^{-p/2} B^{p/2} A^{-p/2})^{\alpha q/p}.
\]

Since both sides in the above are equal to \(\text{Tr} A\) when \(q = 0\), we get

\[
\frac{d}{dq} \left|_{q=0} \text{Tr} A\left\{ (A^p \#_q B^p)^{1/p}\right\}^{q} \right| \geq \frac{d}{dq} \left|_{q=0} \text{Tr} A^{1+q}\left\{ (A^{-p/2} B^{p/2} A^{-p/2})^{\alpha q/p}\right\}^{q}\right|
\]

Simple computations of both differentiations yield the desired inequality. \(\square\)

Taking \(\alpha = \frac{1}{2}\) in Theorem 5.3, we have:

**Corollary 5.4.** If \(A > 0\) and \(B > 0\), then for every \(p > 0\)

\[
\frac{1}{p} \text{Tr} A \log(A^p \# B^p)^2 + \frac{1}{p} \text{Tr} A \log(A^{p/2} B^{-p/2} A^{p/2}) \geq 2 \text{Tr} A \log A.
\]
6. DETERMINANT INEQUALITIES

The next lemma is a variation of Lemma 1.1(1) when we apply it to \( \log f \).

**Lemma 6.1.** If

\[
A, B \succeq 0 \quad \text{and} \quad A < B, \tag{log}
\]

then \( \det f(A) \leq \det f(B) \) for any continuous function \( f \geq 0 \) on an interval containing the eigenvalues of \( A, B \) such that \( \log f(e^t) \) is convex.

This lemma supplies a number of determinant inequalities via the log majorizations given in Sections 2–4. For instance, let us consider the function \( f(t) = \frac{1}{\log t} \) on \((1, \infty)\). Then we obtain sample results as follows.

**Theorem 6.2.** Let \( H \) and \( K \) be Hermitian.

1. If \( e^H \geq e^{-K} \) then

\[
\det \left( \frac{1}{p} \log \{ \exp( pH ) \exp( pK ) \} \right) \tag{6.1}
\]

increases to \( \det(H + K) \) as \( 1 \leq p \downarrow 0 \).

2. If \( e^H \leq e^{-K} \) and \( n \) is even [odd], then (6.1) increases [decreases] to \( \det(H + K) \) as \( 1 \geq p \downarrow 0 \).

**Proof.** (1): Since for \( k \in \mathbb{N} \)

\[
\{ \exp( pK/2 ) \exp( pH ) \exp( pK/2 ) \}^k = \exp( pK/2 ) \{ \exp( pH ) \exp( pK ) \}^k \exp( -pK/2 ),
\]

we have by power series expansion

\[
\log \{ \exp( pK/2 ) \exp( pH ) \exp( pK/2 ) \} = \exp( pK/2 ) \log \{ \exp( pH ) \exp( pK ) \} \exp( -pK/2 ),
\]

so that (6.1) is equal to \( \det \log A(p) \), where

\[
A(p) = \{ \exp( pK/2 ) \exp( pH ) \exp( pK/2 ) \}^{1/p}.
\]
Since (2.1) gives

\[ A(p) < e^{K/2}e^H e^{K/2} \]  

for \( 0 < p \leq 1 \), if \( e^H \geq e^{-K} \), i.e. \( e^{K/2}e^H e^{K/2} \geq I \), then \( A(p) \geq I \) for \( 0 < p \leq 1 \). We may suppose that the eigenvalues of \( A(p) \) are greater than one, because otherwise \( \det \log A(p) = 0 \). Then using (2.1) and Lemma 6.1 with \( f(t) = 1/\log t \) on \((1, \infty)\), we see that \( \det [\log A(p)]^{-1} \) decreases as \( 1 \geq p \downarrow 0 \), so that \( \det \log A(p) \) increases as \( 1 \geq p \downarrow 0 \). This together with Lemma 1.5 gives the conclusion.

(2): Replace \( H, K \) by \(-K, -H\), respectively, in (1). Then

\[
\det \left( -\frac{1}{p} \log \{ \exp(-pK) \exp(-pH) \} \right)
\]

increases to \( \det[-(H+K)] \) as \( 1 \geq p \downarrow 0 \). But (6.2) is equal to

\[
(-1)^n \det \left( -\frac{1}{p} \log \{ \exp(pH) \exp(pK) \} \right),
\]

completing the proof.

The next theorems can similarly be proved via the log majorizations in Theorem 2.1 and Theorem C. We omit the details.

**Theorem 6.3.** Let \( H \) and \( K \) be Hermitian and \( 0 \leq \alpha \leq 1 \).

(1) If \((1 - \alpha)H + \alpha K \geq 0\) then

\[
\det \left( -\frac{1}{p} \log \{ \exp(pH) \#_{\alpha} \exp(pK) \} \right)
\]

(6.3)

decreases to \( \det[(1 - \alpha)H + \alpha K] \) as \( p \downarrow 0 \).

(2) If \((1 - \alpha)H + \alpha K \leq 0\) and \( n \) is even [odd], then (6.3) decreases [increases] to \( \det[(1 - \alpha)H + \alpha K] \) as \( p \downarrow 0 \).

**Theorem 6.4.** Let \( T \) be an arbitrary matrix.

(1) If \( \text{Re} \ T \geq 0 \) then

\[
\det \log |e^T| \geq \det \text{Re} \ T.
\]
(2) If $\Re T \leq 0$ then

$$\det \log |e^T| \geq \det \Re T \quad \text{if } n \text{ is even},$$

$$\det |\log |e^T| | \leq \det \Re T \quad \text{if } n \text{ is odd}.$$