Pergamon

Computers Math. Applic. Vol. 34, No. 10, pp. 49–59, 1997
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0898-1221/97 $17.00 + 0.00

PII: S0898-1221(97)00206-X

A Family of Simultaneous Zero-Finding Methods

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(Received February 1997; accepted April 1997)

Abstract—Applying Hansen-Patrick's formula for solving the single equation $f(z) = 0$ to a suitable function appearing in the classical Weierstrass' method, two one-parameter families of iteration functions for the simultaneous approximation of all simple and multiple zeros of a polynomial are derived. It is shown that all the methods of these families have fourth-order of convergence. Some computational aspects of the proposed methods and numerical examples are given.

Keywords—Zeros of polynomials, Iteration functions, Simultaneous methods, Convergence order.

1. INTRODUCTION

About twenty years ago, Hansen and Patrick proposed in [1] a family of iteration methods with cubic convergence for finding a single (simple or multiple) zero of a given function f. Based on their formulas, we construct two one-parameter classes of iteration functions for the simultaneous approximation of all zeros of a polynomial. First, we apply Hansen-Patrick's formula to Weierstrass' correction to derive a family of iteration function for finding all simple zeros of a polynomial. In the second part of the paper, we use a similar approach to furnish a one-parameter family of iteration functions for the simultaneous determination of multiple zeros of a polynomial. The obtained methods involve polynomial derivatives and have a different structure in reference to the methods for simple zeros. Both classes of methods provide

(1) simultaneous determination of all zeros of a given polynomial, and
(2) the acceleration of the order of convergence from *three* to *four*.

We have found no previous derivation of the proposed iteration methods except the particular case $\alpha = -1$ for simple zeros. Apart from very fast convergence, the new methods are simple for the implementation on digital computers. A number of numerical experiments showed that the proposed methods possess very good convergence properties in the case of crude initial approximations.

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Let f be a function of z and let α be a fixed parameter. Hansen and Patrick derived in [1] one parameter family of iteration functions for finding simple zeros of f in the form

$$\hat{z} = z - \frac{(\alpha+1)f(z)}{\alpha f'(z) \pm \sqrt{f'(z)^2 - (\alpha+1)f(z)f''(z)}}. \tag{1}$$

Here z is a current approximation and $\hat{z}$ is a new approximation to the sought zero. This family includes the Ostrowski ($\alpha = 0$), Euler ($\alpha = 1$), Laguerre ($\alpha = 1/(\nu - 1)$), and Halley methods ($\alpha = -1$) and, as a limiting case ($\alpha \to \infty$), Newton's method. All the methods of the family (1) have cubic convergence to a simple zero except Newton's method which is quadratically convergent.

The methods of the family (1) converge only linearly to a zero of multiplicity $m > 1$ except in the case $\alpha = 1/(m-1)$ when the order of convergence is 1.5 (see [1]). When the multiplicity m is known, Hansen and Patrick [1] proposed the following modification of (1):

$$\hat{z} = z - \frac{m(m\alpha+1)f(z)}{m\alpha f'(z) \pm \sqrt{m(m\alpha - \alpha + 1)f'(z)^2 - m(m\alpha+1)f(z)f''(z)}}. \tag{2}$$

The iteration method (2) converges cubically to a zero of multiplicity m for finite constant α. The family (2) includes the multiple zero counterpart of Laguerre's method (by letting $\alpha = 1/(n-m)$ in (2)). The case $\alpha = -1/m$, which requires a limiting operation in (2), gives the counterpart of Halley's method in the form

$$\hat{z} = z - \frac{f(z)}{(m+1)/(2m)f'(z) - (f(z)f''(z))/(2f'(z))} \tag{3}$$

(see [1]). In the limit case when $\alpha \to \infty$ the iteration formula (2) gives the well known Schröder's generalization [2]

$$\hat{z} = z - m\frac{f(z)}{f'(z)} \tag{4}$$

of Newton's method with quadratic convergence.

For brevity, in the sequel, we will omit indices in the products $\prod$ and the sums $\sum$ assuming that they run from 1 to n in the case of simple zeros and from 1 to ν ($\nu \le n$) for multiple zeros.

2. CLASS OF METHODS FOR SIMPLE ZERO

In this paper, we will consider a special case when the function f is an algebraic polynomial. First let P be a monic polynomial of degree n with simple zeros $\zeta_1, \dots, \zeta_n$ and let $z_1, \dots, z_n$ be n pairwise distinct approximations to these zeros. One of the most efficient iteration methods for the simultaneous determination of all simple zeros of a polynomial is Weierstrass' method of the second order $\hat{z}_i = z_i - W_i$ ($i \in I_n := \{1, \dots, n\}$) (see, e.g., [3–6]), where

$$W_i = \frac{P(z_i)}{\prod_{j\ne i}(z_i - z_j)} \tag{5}$$

is the so-called *Weierstrass' correction.* For simplicity, we will use in the sequel the following abbreviations:

$$G_{1,i} = \sum_{j\ne i} \frac{W_j}{z_i - z_j}, \qquad G_{2,i} = \sum_{j\ne i} \frac{W_j}{(z_i - z_j)^2}.$$

Using Weierstrass' corrections $W_1, \dots, W_n$ and approximations $z_1, \dots, z_n$, by the Lagrangian interpolation we can represent the polynomial P for all $z \in \mathbb{C}$ as

$$P(z) = \prod_{j=1}^{n}(z - z_j) + \sum_{k=1}^{n} W_k \prod_{\substack{j=1 \\ j\ne k}}^{n}(z - z_j). \tag{6}$$

Let us introduce the function $z \mapsto h_i(z)$ by $h_i(z) := P(z)/\prod_{j\neq i}(z-z_j)$. Using (6), we get

$$h_i(z) = W_i + (z - z_i)\left(1 + \sum_{j\neq i} \frac{W_j}{z - z_j}\right). \tag{7}$$

Note that any zero ζ_i of P is also a zero of the function $h_i(z)$. Starting from (7), we find

$$\begin{aligned} h_i(z_i) &= W_i, \qquad h_i'(z_i) = 1 + \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{W_j}{z_i - z_j} = 1 + G_{1,i}, \\ h_i''(z_i) &= -2\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{W_j}{(z_i - z_j)^2} = -2G_{2,i}. \end{aligned} \tag{8}$$

Applying Hansen-Patrick's formula (1) to the function $h_i(z)$ given by (7), we derive the following one-parameter family for the simultaneous approximation of all simple zeros of a polynomial P:

$$\hat{z}_i = z_i - \frac{(\alpha+1)W_i}{\alpha(1+G_{1,i}) \pm \sqrt{(1+G_{1,i})^2 + 2(\alpha+1)W_iG_{2,i}}}, \qquad (i \in I_n). \tag{9}$$

Now we present some special cases of the iteration formula (9).

For $\alpha = 0$, the family (9) gives *Ostrowski-like method*

$$\hat{z}_i = z_i - \frac{W_i}{\sqrt{(1+G_{1,i})^2 + 2W_iG_{2,i}}}, \qquad (i \in I_n). \tag{10}$$

As in the case of other considered methods, the name comes from the fact that the method (10) can be obtained by applying the Ostrowski method [7] to the function $h_i(z)$.

Setting $\alpha = 1$ in (9), we obtain *Euler-like method*

$$\hat{z}_i = z_i - \frac{2W_i}{1+G_{1,i} \pm \sqrt{(1+G_{1,i})^2 + 4W_iG_{2,i}}}, \qquad (i \in I_n). \tag{11}$$

If we let $\alpha = 1/(n-1)$, where n is the polynomial degree, (9) becomes *Laguerre-like method*

$$\hat{z}_i = z_i - \frac{nW_i}{1+G_{1,i} \pm \sqrt{((n-1)(1+G_{1,i}))^2 + 2n(n-1)W_iG_{2,i}}}, \qquad (i \in I_n). \tag{12}$$

The case $\alpha = -1$ is not obvious at first sight and it requires a limiting operation in (9). After short calculation we find that $\alpha = -1$ gives

$$\hat{z}_i = z_i - \frac{W_i(1+G_{1,i})}{(1+G_{1,i})^2 + W_iG_{2,i}}, \qquad (i \in I_n). \tag{13}$$

This formula can be derived directly by applying the classical Halley's formula to the function $h_i(z)$ so that (13) will be referred to as *Halley-like method.* Let us note that Ellis and Watson [8] derived the iteration formula (13) using a different approach.

Letting $\alpha \to \infty$ in (9), we get

$$\hat{z}_i = z_i - \frac{W_i}{1+G_{1,i}} = z_i - \frac{W_i}{1 + \sum_{j\neq i} \frac{W_j}{z_i - z_j}}, \qquad (i \in I_n). \tag{14}$$

This is the third-order iteration method proposed for the first time by Börsch-Supan [9]. Let us note that this method can be directly obtained by applying Newton's method to the function $h_i(z)$.

3. CLASS OF METHODS FOR MULTIPLE ZEROS

Let us assume that the polynomial P has the zero $\zeta_1, \dots, \zeta_\nu$ $(\nu \le n)$ with the known multiplicities $m_1, \dots, m_\nu$ $(m_1 + \cdots + m_\nu = n)$, respectively. The iteration formula (1) can not be applied for the construction of fast algorithms in the case of multiple zeros. For this reason, we use the modified Hansen-Patrick's formula (2) for multiple zeros and apply it to the Weierstrass' correction

$$W_i(z) = \frac{P(z)}{\prod_{j \ne i}(z - z_j)^{m_j}}. \tag{15}$$

Note that Sakurai and Petković [10] applied this idea for simple zeros.

With the abbreviations

$$W_i = \frac{P(z_i)}{\prod_{j \ne i}(z_i - z_j)^{m_j}}, \qquad \delta_{\lambda,i} = \frac{P^{(\lambda)}(z_i)}{P(z_i)},$$
$$S_{\lambda,i} = \sum_{j \ne i} \frac{m_j}{(z_i - z_j)^\lambda} \ (\lambda = 1, 2), \qquad I_\nu = \{1, \dots, \nu\},$$

from (15) we find

$$\left.\frac{(W_i(z))'}{W_i(z)}\right|_{z=z_i} = \delta_{1,i} - S_{1,i}, \tag{16}$$

$$\left.\frac{(W_i(z))''}{(W_i(z))'}\right|_{z=z_i} = \delta_{1,i} - S_{1,i} + \frac{\delta_{2,i} - \delta_{1,i}^2 + S_{2,i}}{\delta_{1,i} - S_{1,i}}. \tag{17}$$

Substituting f'/f and f''/f' by W_i'/W_i and W_i''/W_i' in (2), we obtain

$$\hat{z}_i = z_i - \frac{m_i(m_i\alpha + 1)}{m_i\alpha \frac{W_i'}{W_i} \pm \sqrt{m_i(m_i\alpha - \alpha + 1)\left(\frac{W_i'}{W_i}\right)^2 - m_i(m_i\alpha + 1)\frac{W_i'}{W_i}\frac{W_i''}{W_i'}}}, \qquad (i \in I_\nu),$$

that is,

$$\hat{z}_i = z_i - \frac{m_i(m_i\alpha + 1)}{m_i\alpha(\delta_{1,i} - S_{1,i}) \pm \sqrt{m_i(m_i\alpha + 1)\left(\delta_{1,i}^2 - \delta_{2,i} - S_{2,i}\right) - m_i\alpha(\delta_{1,i} - S_{1,i})^2}}, \qquad (i \in I_\nu). \tag{18}$$

This is the family of iteration methods for the simultaneous determination of all simple or multiple zeros of the polynomial P.

Let us consider now some special cases which originate from (18).

For $\alpha = 0$ we obtain

$$\begin{aligned} \hat{z}_i &= z_i - \frac{\sqrt{m_i}}{\sqrt{\delta_{1,i}^2 - \delta_{2,i} - S_{2,i}}} \\ &= z_i - \frac{\sqrt{m_i}}{\sqrt{(P'(z_i)^2 - P(z_i)P''(z_i))/(P(z_i)^2) - \sum_{j \ne i} m_j/((z_i - z_j)^2)}}, \qquad (i \in I_\nu). \end{aligned} \tag{19}$$

This method resembles Gargantini's square-root method realized in complex circular arithmetic [11]. The iteration method (19) and its modifications were studied in details by Petković and Stefanović [12].

Letting $\alpha = -1/m_i$ for each particular $i \in I_\nu$, after a limiting operation in (18) we find

$$\hat{z}_i = z_i - \frac{2m_i(\delta_{1,i} - S_{1,i})}{(\delta_{1,i} - S_{1,i})^2 - m_i\left(\delta_{2,i} - \delta_{1,i}^2 + S_{2,i}\right)}, \qquad (i \in I_\nu), \tag{20}$$

which is the simultaneous method of Halley's type. Indeed, substituting (16) and (17) instead of f'/f and f''/f' in Halley's method for multiple zeros (3), we obtain (20). In the special case when all zeros are simple ($m_1 = m_2 = \cdots = m_n = 1$), the iteration formula (20) reduces to the method proposed by Sakurai *et al.* [13].

Taking $\alpha = 1/(n-m_i)$ in (18) for each particular $i \in I_\nu$, we obtain the counterpart of Laguerre's method for the simultaneous determination of multiple zeros

$$\hat{z}_i = z_i - \frac{n}{(\delta_{1,i} - S_{1,i})\left(1 \pm \sqrt{\frac{n-m_i}{m_i} - n\left(1 + \frac{S_{2,i}-\delta_{1,i}^2+S_{2,i}}{(\delta_{1,i}-S_{1,i})^2}\right)}\right)}, \qquad (i \in I_\nu). \tag{21}$$

We prove in Section 4 that all iteration methods presented in this section have the order of convergence *four* for finite parameter α. But letting $\alpha \to \infty$ in (18) we find

$$\hat{z}_i = z_i - \frac{m_i}{\delta_{1,i} - S_{1,i}} = z_i - \frac{m_i}{(P'(z_i))/(P(z_i)) - \sum_{j\neq i} m_j/(z_i - z_j)}, \qquad (i \in I_\nu), \tag{22}$$

which resembles Gargantini's interval arithmetic method for the simultaneous inclusion of multiple zeros presented in [14]. The iteration method (22) can be also regarded as a modification of the known Maehly-Ehrlich-Aberth method for multiple zeros which possesses cubic convergence (see [15–18]). Obviously, this method can be derived substituting W_i'/W_i instead of f'/f in Schröder's formula (4).

REMARK 1. Note that both formulas (9) and (18) contain a $\pm$ sign in front of the square root. As mentioned in [1], we have to choose the sign in such a way that the denominators of the corrections $\hat{z}_i - z_i$ are nonzero. In the case of the family (9) this discontinuity appears at $\alpha = 1 + 2W_iG_{2,i}/(1 + G_{1,i})^2$. But, in practice, such an accident is almost impossible. This being the case then we choose the other sign. Furthermore, considering the expressions (25) and (30) we can observe that the choice of the sign "+" provides that the main parts of the iteration formulas (9) and (18) are cubically convergent methods; for the family (9) we have Börsch-Supan's method (14), while for (18) the iteration method (22) appears.

4. ORDER OF CONVERGENCE

Now we will prove that the iteration methods of the families (9) and (18) have fourth-order convergence for any fixed and finite parameter α. In our convergence analysis, we will use the notation introduced above. Besides, let $\hat{\varepsilon}_i = \hat{z}_i - \zeta_i$ and $\varepsilon_i = z_i - \zeta_i$ be the errors in the current and previous iteration, respectively. For any two complex numbers α and β which are of the same order in magnitude, we will write $\alpha = O_M(\beta)$. In our analysis, we will suppose that the errors $\varepsilon_1, \dots, \varepsilon_n$ are of the same order in magnitude, that is $\varepsilon_i = O_M(\varepsilon_j)$ for any pair $i, j \in I_\nu$ ($\nu \le n$). Furthermore, let $\varepsilon \in \{\varepsilon_1, \dots, \varepsilon_\nu\}$ be the error with the maximal magnitude (that is $|\varepsilon| \ge |\varepsilon_i|$ $(i = 1, \dots, \nu)$ but still $\varepsilon = O_M(\varepsilon_i)$ for any $i \in I_\nu$.

First, we consider the convergence speed of the family (9) for simple zeros.

THEOREM 1. *If the approximations $z_1, \dots, z_n$ are sufficiently close to the zeros of P, then the family of zero finding methods (9) has the order of convergence four.*

PROOF. Let us introduce the abbreviations

$$\Sigma_i = \sum_{j\neq i} \frac{W_j}{\zeta_i - z_j}, \qquad t_i = \frac{W_iG_{2,i}}{(1 + G_{1,i})^2}.$$

Since

$$W_j = (z_j - \zeta_j) \prod_{k\neq j} \frac{z_j - \zeta_k}{z_j - z_k},$$

we have the estimates

$$W_i = O_M(\varepsilon_i) = O_M(\varepsilon), \qquad G_{1,i} = O_M(\varepsilon), \qquad G_{2,i} = O_M(\varepsilon),$$
$$\Sigma_i = O_M(\varepsilon), \qquad t_i = O_M(\varepsilon^2). \tag{23}$$

Let z be a complex number such that $|z| \ll 1$. Then we have the developments

$$\sqrt{1+z} = 1 + \frac{z}{2} - \frac{z^2}{8} + \cdots \quad \text{and} \quad (1+z)^{-1} = 1 - z + z^2 - z^3 + \cdots . \tag{24}$$

Starting from (9) and using developments (24), we find

$$\hat{z}_i = z_i - \frac{(\alpha+1)W_i}{\alpha(1+G_{1,i}) + (1+G_{1,i})\sqrt{1+2(\alpha+1)t_i}} = z_i - \frac{(\alpha+1)W_i}{(1+G_{1,i})\left(\alpha+1+(\alpha+1)t_i + O_M\left(t_i^2\right)\right)},$$

wherefrom

$$\hat{z}_i = z_i - \frac{W_i}{(1+G_{1,i})\left(1+t_i+O_M\left(t_i^2\right)\right)} = z_i - \frac{W_i}{1+G_{1,i}}\left(1 - \frac{W_iG_{2,i}}{(1+G_{1,i})^2} + O_M\left(t_i^2\right)\right). \tag{25}$$

Setting $z := \zeta_i$ in (7) (with $h_i(\zeta_i) = 0$) we obtain $W_i = \varepsilon_i(1+\Sigma_i)$. By virtue of this (25) becomes

$$\hat{z}_i = z_i - \frac{\varepsilon_i(1+\Sigma_i)}{1+G_{1,i}}\left(1 - \frac{\varepsilon_i(1+\Sigma_i)G_{2,i}}{(1+G_{1,i})^2} + O_M\left(t_i^2\right)\right),$$

wherefrom, taking into account the estimates (23),

$$\hat{\varepsilon}_i := \hat{z}_i - \zeta_i = \varepsilon_i - \frac{\varepsilon_i(1+\Sigma_i)\left[(1+G_{1,i})^2 - \varepsilon_i(1+\Sigma_i)G_{2,i}\right]}{(1+G_{1,i})^3} + O_M\left(\varepsilon^5\right).$$

After short rearrangement we find

$$\hat{\varepsilon}_i = \frac{\varepsilon_i(X_i+Y_i+Z_i)}{(1+G_{1,i})^3} + O_M\left(\varepsilon^5\right), \tag{26}$$

where

$$X_i = G_{1,i} - \Sigma_i + \varepsilon_i G_{2,i}, \qquad Y_i = G_{1,i}\left[2(G_{1,i}-\Sigma_i) + G_{1,i}^2 - G_{1,i}\Sigma_i\right], \qquad Z_i = \varepsilon_i G_{2,i}\Sigma_i(2+\Sigma_i).$$

Since

$$G_{1,i} - \Sigma_i = \sum_{j\neq i} \frac{W_j}{z_i - z_j} - \sum_{j\neq i} \frac{W_j}{\zeta_i - z_j} = -\varepsilon_i \sum_{j\neq i} \frac{W_j}{(\zeta_i - z_j)(z_i - z_j)}, \tag{27}$$

we have

$$G_{1,i} - \Sigma_i + \varepsilon_i G_{2,i} = \varepsilon_i^2 \sum_{j\neq i} \frac{W_j}{(z_i - z_j)^2(\zeta_i - z_j)}. \tag{28}$$

According to (23), (27), and (28), we estimate

$$X_i = \varepsilon_i^2 O_M(\varepsilon), \qquad Y_i = O_M\left(\varepsilon^3\right), \qquad Z_i = \varepsilon_i O_M\left(\varepsilon^2\right). \tag{29}$$

The denominator $(1+G_{1,i})^3$ tends to 1 when the errors $\varepsilon_1, \dots, \varepsilon_n$ tend to 0. Having in mind this fact and the estimates (29), we find from (26) $\hat{\varepsilon}_i = \varepsilon_i O_M(\varepsilon^3) = O(\varepsilon_i^4)$, which completes the proof of the theorem. ∎

In the following theorem, we consider the convergence rate of the family of iteration functions (18) for multiple zeros.

THEOREM 2. *If the approximations $z_1, \dots, z_\nu$ are sufficiently close to the zeros of P, then the family of zero finding methods (18) has the order of convergence four.*

PROOF. Let us introduce

$$V_i = \frac{\delta_{1,i}^2 - \delta_{2,i} - S_{2,i}}{(\delta_{1,i} - S_{1,i})^2}.$$

The family (18) can be written in the form

$$\hat{z}_i = z_i - \frac{m_i(m_i\alpha + 1)}{(\delta_{1,i} - S_{1,i})\left(m_i\alpha + \sqrt{1 + (m_i\alpha + 1)(m_iV_i - 1)}\right)}, \qquad (i \in I_\nu). \tag{30}$$

In our proof, we will use the abbreviations

$$A_i = \sum_{j \neq i} \frac{m_j\varepsilon_j}{(z_i - \zeta_j)(z_i - z_j)}, \qquad B_i = \sum_{j \neq i} \frac{m_j(2z_i - z_j - \zeta_j)\varepsilon_j}{(z_i - \zeta_j)^2(z_i - z_j)^2}$$

and the identities

$$\delta_{1,i} = \sum_{j=1}^{n} \frac{m_j}{z_i - \zeta_j}, \qquad \delta_{1,i}^2 - \delta_{2,i} = \sum_{j=1}^{n} \frac{m_j}{(z_i - \zeta_j)^2},$$

whence

$$\delta_{1,i} - S_{1,i} = \frac{1}{\varepsilon_i}(m_i - A_i\varepsilon_i), \qquad \delta_{1,i}^2 - \delta_{2,i} - S_{2,i} = \frac{1}{\varepsilon_i^2}\left(m_i - B_i\varepsilon_i^2\right).$$

According to this, we find

$$m_iV_i - 1 = m_i\frac{1/(\varepsilon_i^2)\left(m_i - B_i\varepsilon_i^2\right)}{1/(\varepsilon_i^2)\left(m_i - A_i\varepsilon_i\right)^2} - 1 = \frac{\varepsilon_i\left(2m_iA_i - A_i^2\varepsilon_i - m_iB_i\right)}{\left(m_i - A_i\varepsilon_i\right)^2} = G_i\varepsilon_i,$$

where

$$G_i = \frac{2m_iA_i - A_i^2\varepsilon_i - m_iB_i\varepsilon_i}{(m_i - A_i\varepsilon_i)^2}.$$

We have the following estimates:

$$A_i = O_M(\varepsilon), \qquad B_i = O_M(\varepsilon), \qquad G_i\varepsilon_i = O_M(\varepsilon\varepsilon_i) = O_M\left(\varepsilon^2\right). \tag{31}$$

Using (31) and developments (24), we obtain

$$\sqrt{1 + (m_i\alpha + 1)(m_iV_i - 1)} = \sqrt{1 + (m_i\alpha + 1)G_i\varepsilon_i} = 1 + \frac{(m_i\alpha + 1)G_i\varepsilon_i}{2} + O_M\left(\varepsilon^4\right).$$

According to this, we find from (30)

$$\begin{aligned}
\hat{\varepsilon}_i := \hat{z}_i - \zeta_i &= \varepsilon_i - \frac{m_i(m_i\alpha + 1)}{(\delta_{1,i} - S_{1,i})\left[m_i\alpha + 1 + ((m_i\alpha + 1)G_i\varepsilon_i)/2 + O_M(\varepsilon^4)\right]} \\
&= \varepsilon_i - \frac{m_i\varepsilon_i}{(m_i - A_i\varepsilon_i)\left[1 + (G_i\varepsilon_i)/2 + O_M(\varepsilon^4)\right]} \\
&= \frac{\varepsilon_i^2\left((m_iG_i)/2 - A_i\right)}{(m_i - A_i\varepsilon_i)\left[1 + (G_i\varepsilon_i)/2 + O_M(\varepsilon^4)\right]} + O_M\left(\varepsilon^5\right).
\end{aligned}$$

Since the denominator $(m_i - A_i\varepsilon_i)\left[1 + (G_i\varepsilon_i)/2 + O_M(\varepsilon^4)\right]$ is bounded and tends to m_i as $\varepsilon \to 0$, to prove the theorem it is sufficient to show that $m_iG_i/2 - A_i = O_M(\varepsilon^2)$. Using the estimates (31) we get

$$\frac{m_iG_i}{2} - A_i = \frac{3m_iA_i^2\varepsilon_i - m_i^2B_i\varepsilon_i - 2A_i^3\varepsilon_i^2}{2(m - A_i\varepsilon_i)^2} = O_M\left(\varepsilon^2\right).$$

Therefore, we have proved that

$$\hat{\varepsilon}_i = \varepsilon_i^2 O_M\left(\varepsilon^2\right) = O_M\left(\varepsilon_i^4\right),$$

which means that the order of convergence of the family of iteration method (18) is four. ∎

5. NUMERICAL RESULTS

To demonstrate the convergence speed and the behavior of some methods belonging to our families (9) and (18), we have tested these methods in the examples of algebraic polynomials. FORTRAN 77 with quad-precision arithmetic has been employed. We have used several values for the parameter α and chosen common starting approximations $z_1^{(0)}, \dots, z_n^{(0)}$ for each method. The accuracy of approximations has been estimated by the maximal error

$$\varepsilon^{(m)} = \max_i \left| z_i^{(m)} - \zeta_i \right|,$$

where $m = 0, 1, \dots$ is the iteration index. Stopping criterion has been given by the inequality

$$E^{(m)} = \max_{i \le i \le n} \left| P\left(z_i^{(m)}\right) \right| < \tau,$$

where τ is a given tolerance.

EXAMPLE 1. The polynomial of degree $n = 9$

$$P(z) = z^9 + 3z^8 - 3z^7 - 9z^6 + 3z^5 + 9z^4 + 99z^3 + 297z^2 - 100z - 300$$

with the zeros -3, ± 1, $\pm 2i$, $\pm 2 \pm i$ has been solved. We have terminated the iteration process when the stopping criterion $E^{(m)} = \max_{1 \le i \le 9} |P(z_i^{(m)})| < \tau = 10^{-12}$ was satisfied.

All tested methods have started with Aberth's initial approximations

$$z_k^{(0)} = -\frac{a_1}{n} + r_0 \exp(i\theta_k), \quad i = \sqrt{-1}, \qquad \theta_k = \frac{\pi}{n}\left(2k - \frac{3}{2}\right) \quad (k = 1, \dots, n) \tag{32}$$

(see [15]). First, we have taken very crude initial approximations equidistantly spaced on the circle with radius $r_0 = 100$. The results are as follows.

Ostrowski-like method (10),	$\alpha = 0$	$E^{(m)} < 10^{-12}$ after 15 iterations
Euler-like method (11),	$\alpha = 1$	$E^{(m)} < 10^{-12}$ after 18 iterations
Laguerre-like method (12),	$\alpha = \dfrac{1}{(n-1)} = 0.125$	$E^{(m)} < 10^{-12}$after 15 iterations
Halley-like method (13),	$\alpha = -1$	$E^{(m)} < 10^{-12}$ after 17 iterations
Large parameter method,	$\alpha = 1000$	$E^{(m)} < 10^{-12}$ after 23 iterations

In the next experiment, we have chosen better approximations (but still relatively rough) lying on the circle with radius $r_0 = 4$. The results are displayed below.

Ostrowski-like method (10),	$\alpha = 0$	$E^{(m)} < 10^{-12}$ after 8 iterations
Euler-like method (11),	$\alpha = 1$	$E^{(m)} < 10^{-12}$ after 6 iterations
Laguerre-like method (12),	$\alpha = \dfrac{1}{(n-1)} = 0.125$	$E^{(m)} < 10^{-12}$ after 6 iterations
Halley-like method (13),	$\alpha = -1$	$E^{(m)} < 10^{-12}$ after 7 iterations
Large parameter method,	$\alpha = 1000$	$E^{(m)} < 10^{-12}$ after 8 iterations

Finally, for the set of good starting values ($\varepsilon^{(0)} = 0.36$)

$$\{-3.3+0.2i, -1.2-0.3i, 0.2+1.7i, -1.8+1.3i, -1.8-0.7i, 2.3+1.2i, 1.8-0.7i, 1.2+0.3i, 0.2-2.3i\}$$

all tested methods have shown very fast convergence and reached the accuracy $\varepsilon^{(m)} < 10^{-15}$ after three iterations. The results of the two iterations, expressed by the maximal errors $\varepsilon^{(m)}$ $(m = 1, 2)$, are as follows:

Ostrowski-like method (10),	$\alpha = 0$	$\varepsilon^{(1)} = 3.40(-2)\varepsilon^{(2)} = 4.73(-7)$
Euler-like method (11),	$\alpha = 1$	$\varepsilon^{(1)} = 4.16(-2)\varepsilon^{(2)} = 9.74(-7)$
Laguerre-like method (12),	$\alpha = \dfrac{1}{(n-1)} = 0.125$	$\varepsilon^{(1)} = 3.51(-2)\varepsilon^{(2)} = 5.29(-7)$
Halley-like method (13),	$\alpha = -1$	$\varepsilon^{(1)} = 2.86(-2)\varepsilon^{(2)} = 1.86(-7)$
Large parameter method,	$\alpha = 1000$	$\varepsilon^{(1)} = 6.28(-2)\varepsilon^{(2)} = 3.42(-6)$

In the above list $A(-h)$ means $A \times 10^{-h}$. Let us note that in the third iteration these methods have produced approximations with at least 24 exact decimal digits.

EXAMPLE 2. Methods from the class (9) obtained for the same parameters as in Example 1 have been tested in the example of the monic polynomial P of degree $n = 25$ given by

$$\begin{aligned}
P(z) = z^{25} &+ (0.752 + 0.729i)z^{24} + (-0.879 - 0.331i)z^{23} + (0.381 - 0.918i)z^{22} \\
&+ (0.781 - 0.845i)z^{21} + (-0.046 - 0.917i)z^{20} + (0.673 + 0.886i)z^{19} \\
&+ (0.678 + 0.769i)z^{18} + (-0.529 - 0.874i)z^{17} + (0.288 + 0.095i)z^{16} \\
&+ (-0.018 + 0.799i)z^{15} + (-0.957 + 0.386i)z^{14} + (0.675 - 0.872i)z^{13} \\
&+ (0.433 - 0.562i)z^{12} + (-0.760 + 0.128i)z^{11} + (-0.693 - 0.882i)z^{10} \\
&+ (0.770 - 0.467i)z^{9} + (-0.119 + 0.277i)z^{8} + (0.274 - 0.569i)z^{7} \\
&+ (-0.028 - 0.238i)z^{6} + (0.387 + 0.457i)z^{5} + (-0.855 - 0.186i)z^{4} \\
&+ (0.223 - 0.048i)z^{3} + (0.317 + 0.650i)z^{2} \\
&+ (-0.573 + 0.801i)z + (0.129 - 0.237i).
\end{aligned}$$

The coefficients $a_k \in \mathbb{C}$ of P (except the leading coefficient) were chosen by the random generator as $\mathrm{Re}(a_k) = \texttt{random(x)}$, $\mathrm{Im}(a_k) = \texttt{random(x)}$, where $\texttt{random(x)} \in (-1, 1)$. Using Corollary 6.4k from [19, p. 457], we find that all zeros of the above polynomial lie in the annulus $\{z : r = 0.3054 < |z| < 2.0947 = R\}$.

For comparison, we have also tested the well-known Weierstrass' method (often called Durand-Kerner or Dochev's method)

$$z_i^{(m+1)} = z_i^{(m)} - \frac{P\left(z_i^{(m)}\right)}{\prod_{\substack{j=1 \\ j\neq i}}^{n} \left(z_i^{(m)} - z_j^{(m)}\right)}, \qquad (i \in I_n;\ m = 0, 1, \dots) \tag{33}$$

(see [3]–[6]). This method is one of the most efficient methods for the simultaneous approximation of all zeros of a polynomial and possesses almost global convergence (a conjecture that is not proved yet). All tested methods have started with Aberth's initial approximations given by (32). In this example, the stopping criterion was given by $E^{(m)} = \max_{1\le i\le 25} |P(z_i^{(m)})| < \tau = 10^{-7}$.

We performed three experiments taking $r_0 = 1.2$, 10, and 100 in (32). The first value is equal to the arithmetic mean of the radii $r = 0.3054$ and $R = 2.0947$ of the inclusion annulus given above. The values $r_0 = 10$ and $r_0 = 100$ have been chosen to demonstrate the influence of r_0 to the convergence speed of the tested methods but also to show very good convergence in the situation when the initial approximations are very crude. Table 1 gives the number of iterative steps for the considered iteration procedures and Weierstrass' method (33). From this table, we see that the fourth-order methods (9) require less than half of iterations produced by the second-order

Table 1. The number of iterations for various initial approximations and $\tau = 10^{-7}$.

	$\alpha = 0$	$\alpha = 1$	$\alpha = -1$	$\alpha = \frac{1}{(n-1)}$	$\alpha = 1000$	Method (33)
$r_0 = 1.2$	8	8	5	11	7	13
$r_0 = 10$	24	28	24	22	36	65
$r_0 = 100$	40	56	49	39	62	124

methods (33) if the parameter α in (9) is not too large. This means that the convergence behavior of the proposed methods (9) is at least as good as the behavior of Weierstrass' method (33).

EXAMPLE 3. The polynomial

$$\begin{aligned}P(z) =& z^{13} + (-11+4i)z^{12} + (46-44i)z^{11} + (-74+204i)z^{10} + (-105-516i)z^9 \\ &+ (787+616i)z^8 + (-1564+392i)z^7 + (724-2344i)z^6 + (2351+2616i)z^5 \\ &+ (-4389+980i)z^4 + (430-5248i)z^3 + (4662+540i)z^2 + (-135+2700i)z - 675\end{aligned}$$

has been taken to illustrate numerically simultaneous methods from the family (18) for finding multiple zeros. The factorization of P is

$$P(z) = (z+1)^2(z-3)^3\left(z^2-2z+5\right)^2(z+i)^4.$$

Thus, the polynomial P has five zeros $\zeta_1 = -1,\ \zeta_2 = 3,\ \zeta_3 = 1+2i,\ \zeta_4 = 1-2i,\ \zeta_5 = -i$ of the multiplicity $m_1 = 2,\ m_2 = 3,\ m_3 = 2,\ m_4 = 2,\ m_5 = 4$, respectively.

The initial approximations have been selected to be

$$\underline{\{-1.3+0.2i, 3.2+0.3, 1.3+2.2, 1.3-2.2, 0.2-1.3\}}.$$

The results obtained in the first two iterations are displayed below.

Ostrowski-like method (19),	$\alpha = 0$	$E^{(1)} = 9.31(-3)\varepsilon^{(2)} = 9.53(-9)$
Halley-like method (20),	$\alpha = \dfrac{-1}{m_i}$	$E^{(1)} = 8.89(-3)\varepsilon^{(2)} = 5.89(-9)$
Laguerre-like method (21),	$\alpha = \dfrac{1}{(13-m_i)}$	$E^{(1)} = 9.40(-3)\varepsilon^{(2)} = 4.43(-9)$
Large parameter method,	$\alpha = 1000$	$E^{(1)} = 3.45(-2)E^{(2)} = 3.72(-6)$

We have tested several other polynomials with the degree in the range [5, 25] using initial approximations of different precision with respect to the zeros. The computational results of these tests have been consistent with those presented above.

In our experiments, we have used various values of the parameter α. We did not find a specific value of α giving a method from our family which is asymptotically best for all P. All tested methods have shown almost the same behavior for a wide range of values of the parameter α and very fast convergence for good initial approximations.

Computational results point to good convergence properties of the presented methods in the case of very crude initial approximations. This is, generally speaking, an appropriate feature and important advantage of iterative processes. Also, numerical examples indicated that the choice of relatively large values of the parameter α can produce the inferior behavior of the corresponding methods from the families (9) and (18). This fact substantiates the convergence analysis in Sections 3 and 4 which asserts that for very large α the order of convergence of the methods from the families (9) and (18) approaches to three.

In their paper [1], Hansen and Patrick found that Laguerre's method for finding a single zero is superior to other methods for $|z|$ large. But, in the case of simultaneous methods from the families (9) and (18), such a superior behavior of Laguerre-like methods (12) and (21) does not occur. A theoretical consideration of the behavior of simultaneous methods is very complicated so that we have not been able to give a deep behavior analysis of the methods belonging to the families (9) and (18) and make the ranking of these methods. A number of numerical examples show only that one of the considered methods is the fastest for some polynomials, while the another one is the fastest for some other polynomials. Actually, the convergence behavior strongly depends on the structure of the polynomial solved and initial approximations.

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