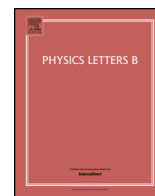


Contents lists available at [ScienceDirect](http://ScienceDirect)

## Physics Letters B

[www.elsevier.com/locate/physletb](http://www.elsevier.com/locate/physletb)Braneworld solutions for  $F(R)$  models with non-constant curvatureD. Bazeia<sup>a,b</sup>, A.S. Lobão Jr<sup>a</sup>, R. Menezes<sup>c,b</sup>, A.Yu. Petrov<sup>a,\*</sup>, A.J. da Silva<sup>d</sup><sup>a</sup> Departamento de Física, Universidade Federal da Paraíba, 58051-970 João Pessoa, PB, Brazil<sup>b</sup> Departamento de Física, Universidade Federal de Campina Grande, 58109-970 Campina Grande, PB, Brazil<sup>c</sup> Departamento de Ciências Exatas, Universidade Federal da Paraíba, 58297-000 Rio Tinto, PB, Brazil<sup>d</sup> Instituto de Física Universidade de São Paulo, 05314-970 São Paulo, SP, Brazil

## ARTICLE INFO

## Article history:

Received 29 November 2013

Received in revised form 16 December 2013

Accepted 7 January 2014

Available online 10 January 2014

Editor: M. Cvetič

## ABSTRACT

This work deals with braneworld scenarios with generalized gravity. We investigate models where the potential of the scalar field is polynomial or nonpolynomial. We obtain exact and approximated solutions for the scalar field, warp factor and energy density, in the complex scenario with no restriction on the scalar curvature. In particular, we describe the case where the brane may split, engendering internal structure, with the splitting caused by the same parameter that controls deviation from standard gravity.

© 2014 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/3.0/>). Funded by SCOAP<sup>3</sup>.

## 1. Introduction

The Randall–Sundrum (RS) theory proposes a description of the Universe with a five-dimensional (5D) spacetime of the anti de Sitter ( $AdS$ ) type, with a single extra spatial dimension of infinite extent. In this scenario, our Universe evolves as a brane embedded in a 5D bulk where gravity can propagate, but with the other fundamental interactions only propagating on the brane. The RS and other [2–10] braneworld scenarios are of current interest because they help us to better understand the cosmological constant and mass hierarchy problems [1–10].

The study of branes requires solving Einstein equations and modified Einstein equations in the case of modified gravity, in an  $AdS_5$  spacetime; see, e.g., Refs. [11–13]. Due to the interest in modified gravity (see, e.g., Ref. [14]), we have investigated braneworld scenarios under the substitution  $R \rightarrow F(R)$  in the action that describes the brane; see, e.g., [12,13]. An important tool for such calculations is the first-order formalism, which allows to reduce the order of the equations of motion and is important for generalized models, where the degree of complexity of the equations of motion is very high [15].

In the case of modified gravity, the investigations presented in Refs. [12,13] were based on the assumption that the scalar curvature is constant. In the current work, however, we abandon this restriction and solve the brane equations in the more general situation, for non-constant scalar curvature. Due to the complexity of the brane equations, we work with models described by  $F(R) = R + \alpha R^2$ , and we consider the case of a single real scalar

field. To obtain a larger family of solutions, we follow two distinct routes: firstly, we deal with the brane equations via an exact procedure; in the second case, we develop an approximation scheme, in which we consider  $\alpha$  small, working up to the first-order in  $\alpha$ . We study a single scalar field, but we work with models having polynomial interaction, as in the  $\phi^4$  model with spontaneous symmetry breaking, and nonpolynomial interaction, as in the sine-Gordon model.

A particularly interesting result of this work concerns scenarios where the brane may split, engendering internal structure. The mechanism used to describe the brane splitting is different from other descriptions, where thermal effects [7] and the presence of 2-kink solutions [8] play the role for the splitting, in scenarios with standard gravity.

## 2. Generalities

We start with a 5D action which describes a generalized  $F(R)$  brane, in which gravity is coupled to a real scalar field  $\phi$  in the form

$$S = \int d^5x \sqrt{|g|} \left( -\frac{1}{4} F(R) + \mathcal{L}(\phi, \nabla_a \phi) \right), \quad (1)$$

where  $\mathcal{L}(\phi, \nabla_a \phi)$  is the Lagrange density that accounts for the scalar field. It has the form

$$\mathcal{L} = \frac{1}{2} g_{ab} \nabla^a \phi \nabla^b \phi - V(\phi). \quad (2)$$

Here we are using  $4\pi G^{(5)} = 1$ ,  $g = \det(g_{ab})$ , and the signature of the metric is  $(+ - - -)$ . Also,  $V(\phi)$  is the potential, to be defined below. We take the spacetime coordinates and fields as

\* Corresponding author.

dimensionless quantities, and we use Latin indices for the bulk coordinates,  $a, b = 0, 1, 2, 3, 4$ , and Greek indices for the embedded  $(3+1)$ -dimensional space,  $\mu, \nu = 0, 1, 2, 3$ .

We study the case of a flat brane, with the line element

$$ds^2 = e^{2A} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2, \quad (3)$$

where  $e^{2A}$  is the warp factor,  $\eta_{\mu\nu}$  is the 4-dimensional Minkowski metric, and  $y = x^4$  is the extra dimension.

As usual, in the braneworld framework we suppose that both  $A$  and  $\phi$  are static and depend only on the extra dimension, that is, we set  $A = A(y)$  and  $\phi = \phi(y)$ . In this case, the equation of motion for the scalar field has the form

$$\phi'' + 4A'\phi' = V_\phi, \quad (4)$$

where prime denotes derivative with respect to the extra dimension, and  $V_\phi = dV/d\phi$ .

The energy density  $\rho$ , that is, the  $T_{00}$  component of the energy-momentum tensor is given by

$$\rho = -e^{2A(y)} \mathcal{L}. \quad (5)$$

For static fields the modified Einstein equations becomes

$$-\frac{2}{3}\phi'^2 = A''F_R - \frac{1}{3}A'F'_R + \frac{1}{3}F''_R, \quad (6a)$$

$$V(\phi) - \frac{1}{2}\phi'^2 = 2(A'' + A'^2)F_R - \frac{1}{4}F(R) - 2A'F'_R, \quad (6b)$$

where  $F_R = dF/dR$ ,  $F_{RR} = d^2F/dR^2$ , etc.

It was shown in [13] that if the scalar curvature is constant (so,  $R' = 0$ ), it is possible to obtain analytic solutions for the field equations, in the case of several scalar fields. Here, however, we shall study more general solutions, that is, we shall assume that the scalar curvature is a generic function of the extra dimension,  $R = R(y)$ . Therefore, from Eq. (6a) we have

$$\phi'^2 = -\frac{3}{2}A''F_R + \frac{1}{2}(A'R' - R'')F_{RR} - \frac{1}{2}F_{RRR}R'^2. \quad (7a)$$

Also, the potential can be found from (6b) and reads

$$V(\phi) = -\frac{1}{4}F(R) + \frac{1}{4}(5A'' + 8A'^2)F_R - \frac{1}{4}(7A'R' + R'')F_{RR} - \frac{1}{4}F_{RRR}R'^2. \quad (7b)$$

The scalar curvature is given in terms of the warp factor as

$$R = 8A'' + 20A'^2. \quad (8)$$

This can be used to rewrite Eqs. (7a) and (7b) as

$$\phi'^2 = -\frac{3}{2}A''F_R + 4(5A'^2A'' - 5A''^2 - 4A'A''' - A'''' )F_{RR} - 32F_{RRR}(5A'A'' + A''')^2, \quad (9a)$$

$$V(\phi) = -\frac{1}{4}F(R) + \frac{1}{4}(5A'' + 8A'^2)F_R - (70A'^2A'' + 24A'''A' + 10A''^2 + 2A'''' )F_{RR} - 16F_{RRR}(5A'A'' + A''')^2. \quad (9b)$$

It is not hard to check that in the simplest case where  $F(R) = R$ , these equations reduce to the standard result; see, e.g., [11]. However, in the general case these equations depend on the third and fourth derivative of the warp function, that is, on  $A'''$  and  $A''''$ .

To find explicit solutions, we follow [12] and choose the simplest nontrivial polynomial function  $F(R) = R + \alpha R^2$ , where  $\alpha$  is a real parameter. In this case we can write (9) as

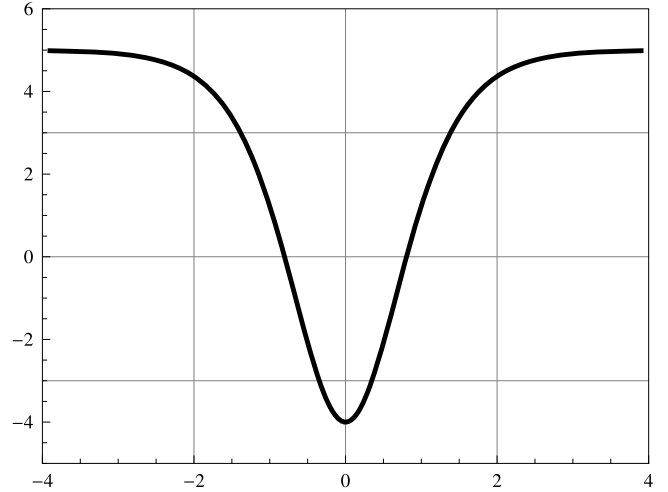


Fig. 1. The scalar curvature, depicted from Eq. (15) for  $k = 1$  and  $B = 1/2$ .

$$\phi'^2 = -\frac{3}{2}A'' - 4\alpha(5A''A'^2 + 16A''^2 + 8A'A''' + 2A'''' ), \quad (10a)$$

$$V(\phi) = -3A'^2 - \frac{3}{4}A'' - 2\alpha(10A'^4 + 69A'^2A'' + 24A'A''' + 8A''^2 + 2A'''' ). \quad (10b)$$

Also, the energy density can be written in terms of the warp function as

$$\rho = -e^{2A} \left[ \frac{3}{2}A'' + 3A'^2 + 4\alpha(5A'^4 + 37A'^2A'' + 16A'A''' + 12A''^2 + 2A'''' ) \right], \quad (11)$$

or

$$\rho = -\frac{d}{dy} \left[ e^{2A} \left( \frac{3}{2}A' + 4\alpha \left( \frac{13}{3}A'^3 + 2A''' + 12A'A'' \right) \right) \right] + \frac{44\alpha}{3}e^{2A}A'^4. \quad (12)$$

If  $\alpha = 0$ , the energy density becomes  $\rho = -(3/2)(e^{2A}A)'$ . It is a total derivative, so the energy vanishes in the standard situation. If  $\alpha \neq 0$ , the first term in Eq. (12) does not contribute to the energy, so we can write

$$E = \frac{44}{3}\alpha \int dy e^{2A}A'^4, \quad (13)$$

and the sign of  $\alpha$  controls the sign of the energy.

### 3. Exact procedure

Let us start studying a model that engenders exact results. To do this, we choose the following warp function [6]

$$A(y) = B \ln[\operatorname{sech}(ky)], \quad (14)$$

where  $B$  and  $k$  are positive parameters. This allows us to write  $R$  as

$$R(y) = 4Bk^2[5B - (5B + 2)\operatorname{sech}^2(ky)]. \quad (15)$$

The scalar curvature (15) is depicted in Fig. 1. In the limit  $y \rightarrow \pm\infty$ ,  $R$  tends to a constant value,  $20B^2k^2$ , and at  $y = 0$ , the scalar curvature becomes  $-8Bk^2$ .

For the warp function (14), the expressions (10) reduce to

$$\frac{\phi'^2}{Bk^2} = \frac{3}{2}S_k^2 + 4\alpha k^2 S_k^2 [5B^2 + 16B + 8 - (5B^2 + 32B + 12)S_k^2], \quad (16a)$$

$$\frac{V(y)}{Bk^2} = -3B + 3\left(B + \frac{1}{4}\right)S_k^2 - 2\alpha k^2 [10B^3 - (20B^3 + 69B^2 + 48B + 8)S_k^2 + (10B^3 + 69B^2 + 56B + 12)S_k^4]. \quad (16b)$$

Here we are using  $S_k = S(ky) = \text{sech}(ky)$ , for simplicity. As one knows, solutions  $\phi = \phi(y)$  of the above Eq. (16a) must go to some constant value  $\bar{\phi}$  asymptotically. Thus, if we want our model to make physical sense, the potential (16b) should go to a vacuum value for  $\bar{\phi}$ . In the general case, the asymptotic values of the potential and its derivative with respect to the field are, respectively,

$$\Lambda_5 \equiv V(\phi \rightarrow \pm\bar{\phi}) = -B^2k^2(3 + 20\alpha B^2k^2) \quad (17)$$

and  $V_\phi(\phi \rightarrow \pm\bar{\phi}) \rightarrow 0$ .

If we use Eq. (16a), we can infer the range of values for the parameter  $\alpha$ : since  $\phi$  and its derivative are real, one must have  $\phi'^2 > 0$  and this implies the following condition

$$\frac{-3}{8k^2(8 + 16B + 5B^2)} = \alpha_1 \leq \alpha \leq \alpha_2 = \frac{3}{32k^2(1 + 4B)}. \quad (18)$$

Furthermore, the energy density can be written as

$$\begin{aligned} \frac{\rho}{Bk^2} = & -3BS_k^{2B} + 3\left(B + \frac{1}{2}\right)S_k^{2B+2} \\ & - 4\alpha k^2 S_k^{2B} [5B^3 - (10B^3 + 37B^2 + 32B + 8)S_k^2 \\ & + (5B^3 + 37B^2 + 44B + 12)S_k^4], \end{aligned} \quad (19)$$

and for  $\alpha = 0$  it becomes

$$\rho(y) = -3B^2k^2 S_k^{2B} + 3Bk^2 \left(B + \frac{1}{2}\right) S_k^{2B+2}. \quad (20)$$

We see that the energy density (19) displays the presence of an inflection point at  $y = 0$ , such that

$$\alpha_s = \frac{3 + 9B}{8k^2(16 + 60B + 49B^2)}. \quad (21)$$

This means that at  $\alpha = \alpha_s$  the solution of Eq. (16a) begins to split, inducing internal structure to the brane. Moreover, the energy density behaves asymptotically as

$$\rho(y) = -2^{2B} B^2 k^2 (3 + 20\alpha k^2 B^2) e^{-2kBy} + \dots \quad (22)$$

In Fig. 2, in the upper panel we depict the allowed region of  $\alpha$  for distinct values of  $B$  and  $k = 1$ . The two gray regions follow in accordance with Eq. (18). The solid line is obtained for  $\alpha = \alpha_1$ , the dashed line for  $\alpha = \alpha_s$ , and the dotted line for  $\alpha = \alpha_2$ . In the light gray region the energy density has a maximum at  $y = 0$  and for  $\alpha$  in the darker region the energy density has a local minimum at  $y = 0$ . The dashed line represents the appearance of the inflection point, where the brane starts to split. The darker gray region identifies the region where the brane splits, engendering internal structure. To further illustrate this situation, in the lower panel in Fig. 2 we depict the energy density (19) for  $\alpha = -1/46$ ,  $\alpha = 0$ ,  $\alpha = 15/932$  and  $\alpha = 1/32$ . The case  $\alpha = 1/32$  nicely illustrates the brane splitting.

The energy of the brane, given by Eq. (13), is

$$E = 11\alpha k^3 \sqrt{\pi} \frac{B^4 \Gamma(B)}{\Gamma(5/2 + B)}. \quad (23)$$

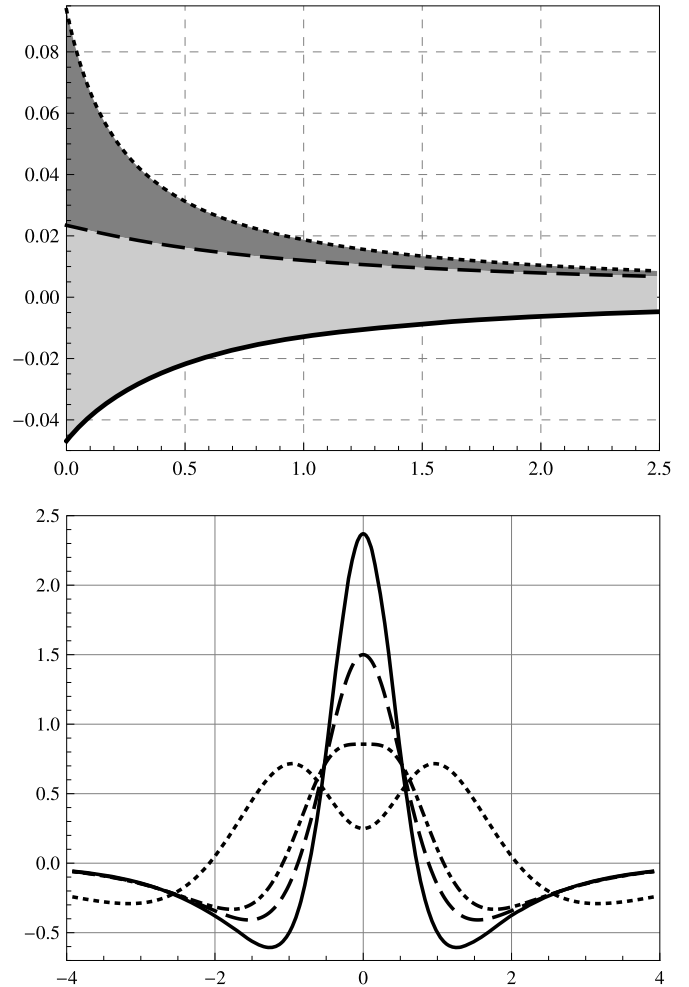


Fig. 2. Upper panel: the region bounded by  $\alpha$  in Eq. (18), for  $k = 1$  and  $\alpha = -1/46$  (solid line),  $\alpha = 15/932$  (dashed line) and  $\alpha = 1/32$  (dotted line). Lower panel: the energy density (19) obtained for  $k = 1$  and  $B = 1/2$ , and for  $\alpha = -1/46$  (solid line),  $\alpha = 0$  (dashed line),  $\alpha = 15/932$  (dotted-dashed line) and  $\alpha = 1/32$  (dotted line).

We can solve Eq. (16a), to find a function  $\phi = f(y)$  that may be inverted to give  $y = f^{-1}(\phi)$ , which allows us to write the potential in the usual way  $V = V(\phi)$ . The general solution of Eq. (16a) looks like

$$\begin{aligned} \phi(y) = & \frac{\sqrt{6B}}{2} \sqrt{1 - \frac{\alpha}{\alpha_1}} F(ky, P) + \frac{\sqrt{3B}}{2} \left[ 1 - \frac{2\alpha_1 - \alpha_2}{\alpha_1 \alpha_2} \alpha \right. \\ & \left. + \frac{\alpha_1 - \alpha}{\alpha_1} \cosh(2ky) \right]^{1/2} \tanh(ky), \end{aligned} \quad (24)$$

where

$$F(y, P) = - \int_0^y \left[ \frac{\sinh^2(x)}{\sqrt{P + \sinh^2(x)}} \right] dx, \quad (25)$$

and

$$P = \frac{\alpha \alpha_1 - \alpha_1 \alpha_2}{\alpha \alpha_2 - \alpha_1 \alpha_2}. \quad (26)$$

The general result leads to some interesting particular cases. The first is the limiting case for  $\alpha = \alpha_1$  in Eq. (18); in this case  $P \gg 1$ , and so we can write the solution  $\phi$  and the potential as, respectively

$$\phi(y) = \sqrt{\frac{3B(6+B)(2+5B)}{16+2B(16+5B)}} \tanh(ky), \quad (27a)$$

$$V(\phi) = \frac{3Bk^2}{4} \left( \frac{12+24B+5B^2}{8+16B+5B^2} \right) - k^2(1+2B)\phi^2 + k^2 \frac{(1+2B)(8+16B+5B^2)}{3B(6+B)(2+5B)} \phi^4. \quad (27b)$$

The second case is the standard case, where  $\alpha = 0$  and  $P = 1$ ; here we obtain

$$\phi(y) = \sqrt{6B} \arctan \left[ \tanh \left( \frac{ky}{2} \right) \right], \quad (28a)$$

$$V(\phi) = -3B^2k^2 + 3Bk^2 \left( B + \frac{1}{4} \right) \cos^2 \left( \sqrt{\frac{2}{3B}} \phi \right), \quad (28b)$$

which reproduces the result obtained in [6].

The third case concerns the value  $\alpha = \alpha_2$ , here we have  $P = 0$ . Also,

$$\phi(y) = \sqrt{\frac{3B(6+B)(2+5B)}{8+32B}} [1 - \operatorname{sech}(ky)] \operatorname{sign}(y), \quad (29a)$$

$$V(\phi) = -\frac{3B^2k^2}{8} \left( \frac{8+32B+5B^2}{1+4B} \right) + \frac{3Bk^2}{16} (6+B)(2+5B) \times \left[ 1 - \left( \frac{3B(6+B)(2+5B)}{8+32B} \right)^{-1/2} |\phi| \right]^2 - \frac{3Bk^2}{16} \frac{(6+B)(1+2B)(2+5B)}{1+4B} \times \left[ 1 - \left( \frac{3B(6+B)(2+5B)}{8+32B} \right)^{-1/2} |\phi| \right]^4. \quad (29b)$$

Note that each one of the particular solutions obtained above satisfies the following conditions:  $\phi(y \rightarrow \pm\infty) \rightarrow \pm\bar{\phi}$ , where

$$\bar{\phi}_1 = \sqrt{\frac{3B(6+B)(2+5B)}{16+2B(16+5B)}},$$

$$\bar{\phi}_0 = \frac{\pi}{4} \sqrt{6B},$$

$$\bar{\phi}_2 = \sqrt{\frac{3B(6+B)(2+5B)}{8+32B}}$$

with  $\bar{\phi}_1 < \bar{\phi}_0 < \bar{\phi}_2$ , for  $\alpha = \alpha_1$ ,  $\alpha = 0$  and  $\alpha = \alpha_2$ , respectively. Furthermore, for each one of the three cases we have  $V(\bar{\phi}) = v$  and  $V_\phi(\bar{\phi}) = 0$ , where

$$v_1 = -3B^2k^2 \left( \frac{16+32B+5B^2}{16+32B+10B^2} \right),$$

$$v_0 = -3B^2k^2,$$

$$v_2 = -3B^2k^2 \left( \frac{8+32B+5B^2}{8+32B} \right),$$

where  $v_1 < v_0 < v_2$ , which agrees with the previous result obtained in Eq. (17).

In the upper panel in Fig. 3 we depict the three cases where solutions are obtained exactly. We depict the solutions for  $k = 1$  and  $B = 1/2$ , for  $\alpha = -1/46$  (line solid),  $\alpha = 0$  (dashed line) and  $\alpha = 1/32$  (dotted line). We see that the defect changes, behaving as a 2-kink when  $\alpha > \alpha_s (= 15/932)$ . In the lower panel in Fig. 3 we depict the potential, for the same values of the parameters  $k$ ,  $B$  and  $\alpha$ .

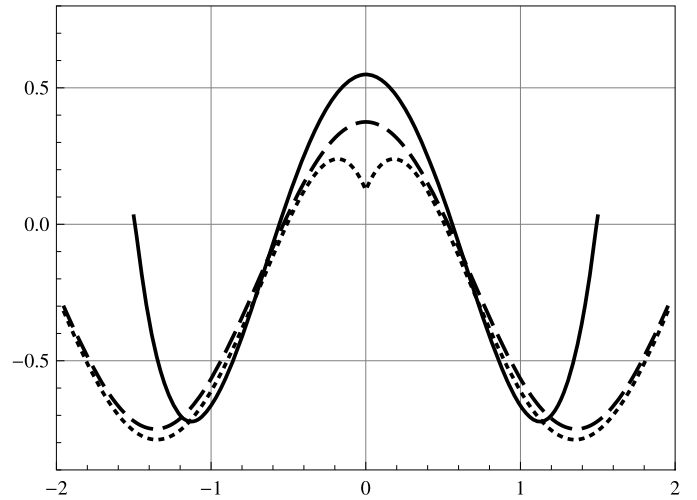
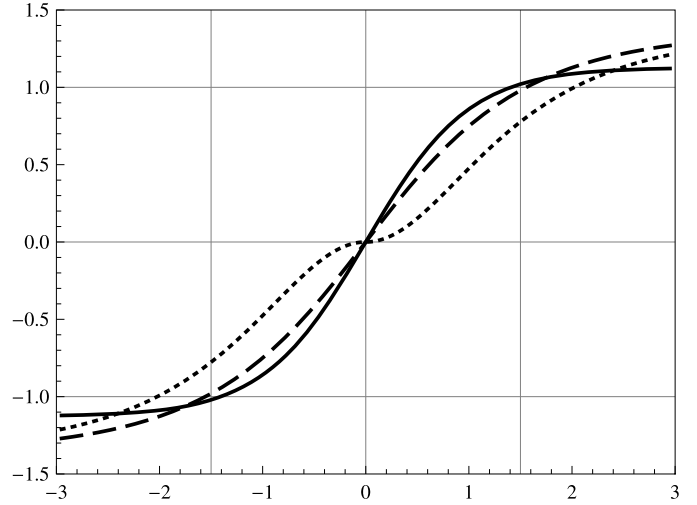


Fig. 3. Upper panel: the three specific solutions (27a), (28a) and (29a) for  $k = 1$ ,  $B = 1/2$  and  $\alpha = -1/46$  (solid line),  $\alpha = 0$  (dashed line) and  $\alpha = 1/32$  (dotted line). Lower panel: the three specific potentials (27b), (28b) and (29b), for the same values of the parameters  $k$ ,  $B$  and  $\alpha$ .

#### 4. Perturbative procedure

In this section we follow the perturbative procedure introduced in [16] to investigate specific models. For this, we suppose that the field is expanded in the form  $\phi(y) = \phi_0(y) + \alpha\phi_\alpha(y)$ . Furthermore, the potential and the warp function are also written as  $V(\phi) = V_0(\phi) + \alpha V_\alpha(\phi)$  and  $A(y) = A_0(y) + \alpha A_\alpha(y)$ ;  $\alpha$  is now a small parameter and  $\phi_0$  and  $A_0$  are the solution for  $\alpha = 0$ , such that

$$\phi'_0 = \frac{1}{2} W_{\phi_0}, \quad \text{and} \quad A'_0 = -\frac{1}{3} W(\phi_0), \quad (30)$$

where  $W_{\phi_0} = W_\phi|_{\phi=\phi_0}$ , etc. Also,

$$V_0(\phi) = \frac{1}{8} W_\phi^2 - \frac{1}{3} W^2. \quad (31)$$

Using Eqs. (10a) and (10b) we can write, up to the first order in  $\alpha$ ,

$$\phi'_\alpha(\phi_0) = -\frac{3}{2} \frac{A''_\alpha}{W_{\phi_0}} + \frac{4}{3} W_{\phi_0} \left( \frac{5}{18} W^2 - \frac{4}{3} W_{\phi_0}^2 - \frac{4}{3} W W_{\phi_0\phi_0} + W_{\phi_0\phi_0}^2 + \frac{1}{2} W_{\phi_0} W_{\phi_0\phi_0\phi_0} \right) \quad (32a)$$

and

$$V_\alpha(\phi_0) = -W_{\phi_0} \left( \frac{1}{4} W_{\phi_0\phi_0} - \frac{2}{3} W \right) \phi_\alpha + 2W A'_\alpha - \frac{3}{4} A''_\alpha - \frac{20}{81} W^4 + \frac{23}{9} W^2 W_{\phi_0}^2 - \frac{8}{3} W W_{\phi_0}^2 W_{\phi_0\phi_0} - \frac{4}{9} W_{\phi_0}^4 + \frac{2}{3} W_{\phi_0}^2 W_{\phi_0\phi_0}^2 + \frac{1}{3} W_{\phi_0}^3 W_{\phi_0\phi_0\phi_0}. \quad (32b)$$

We assume that  $A'_\alpha$  is a function of the field  $\phi_0$ , with the specific form

$$A'_\alpha = \beta W(\phi_0) + \gamma W^3(\phi_0), \quad (33)$$

where  $\beta$  and  $\gamma$  are real parameters. This allows us to write the previous relations as

$$\phi'_\alpha = \frac{4}{3} W_{\phi_0} \left[ -\frac{9}{16} \beta + \left( \frac{5}{18} - \frac{27}{16} \gamma \right) W^2 - \frac{4}{3} W_{\phi_0}^2 - \frac{4}{3} W W_{\phi_0\phi_0} + W_{\phi_0\phi_0}^2 + \frac{1}{2} W_{\phi_0} W_{\phi_0\phi_0\phi_0} \right], \quad (34a)$$

and

$$V_\alpha(\phi_0) = -W_{\phi_0} \left( \frac{1}{4} W_{\phi_0\phi_0} - \frac{2}{3} W \right) \phi_\alpha + 2\beta W^2 - \frac{3\beta}{8} W_{\phi_0}^2 + 2 \left( \gamma - \frac{10}{81} \right) W^4 + \left( \frac{23}{9} - \frac{9\gamma}{8} \right) W^2 W_{\phi_0}^2 - \frac{8}{3} W W_{\phi_0}^2 W_{\phi_0\phi_0} - \frac{4}{9} W_{\phi_0}^4 + \frac{2}{3} W_{\phi_0}^2 W_{\phi_0\phi_0}^2 + \frac{1}{3} W_{\phi_0}^3 W_{\phi_0\phi_0\phi_0}. \quad (34b)$$

We can thus reconstruct the potential

$$V(\phi) = \frac{1}{8} W_\phi^2 - \frac{1}{3} W^2 + \alpha \left[ W_\phi \left( \frac{2}{3} W - \frac{1}{4} W_{\phi\phi} \right) \phi_\alpha + 2\beta W^2 - \frac{3\beta}{8} W_\phi^2 + 2 \left( \gamma - \frac{10}{81} \right) W^4 + \left( \frac{23}{9} - \frac{9\gamma}{8} \right) W^2 W_\phi^2 - \frac{8}{3} W W_\phi^2 W_{\phi\phi} - \frac{4}{9} W_\phi^4 + \frac{2}{3} W_\phi^2 W_{\phi\phi}^2 + \frac{1}{3} W_\phi^3 W_{\phi\phi\phi} \right]. \quad (35)$$

So we can solve Eq. (34a) to obtain  $\phi_\alpha(\phi_0)$  and therefore the potential  $V_\alpha(\phi_0)$  in (34b). Furthermore, for the choice (33) the warp function is obtained as

$$A(\phi_0) = -\frac{2}{3} (1 - 3\alpha\beta) \int \frac{W}{W_{\phi_0}} d\phi_0 + 2\alpha\gamma \int \frac{W^3}{W_{\phi_0}} d\phi_0, \quad (36)$$

and the constant of integration is obtained from the condition  $A(0) = 0$ .

To see how the above procedure works, let us now illustrate the results with two very distinct examples, one with polynomial potential, and the other with nonpolynomial potential.

#### 4.1. Polynomial potential

In this first example, we choose the following function

$$W(\phi) = 2\phi - \frac{2}{3} \phi^3. \quad (37)$$

In the absence of gravity, this choice represents the well-known  $\phi^4$  model, with spontaneous symmetry breaking. With this, we can write the warp function as

$$A(\phi_0) = -\frac{1}{3} \left( \frac{1}{3} + \frac{32}{9} \alpha\gamma - \alpha\beta \right) \phi_0^2 + \frac{38}{27} \alpha\gamma \phi_0^4 - \frac{32}{81} \alpha\gamma \phi_0^6 + \frac{1}{27} \alpha\gamma \phi_0^8 + \frac{2}{3} \left( \frac{1}{3} - \frac{16}{9} \alpha\gamma - \alpha\beta \right) \ln(1 - \phi_0^2), \quad (38)$$

which obeys  $A(0) = 0$ . For the scalar field, we get its first-order corrections as

$$\phi_\alpha(\phi_0) = -\left( \frac{224}{9} + \frac{3\beta}{2} \right) \phi_0 - \left( 6\gamma - \frac{3056}{81} \right) \phi_0^3 - \left( \frac{416}{81} - \frac{12\gamma}{5} \right) \phi_0^5 - \left( \frac{2\gamma}{7} - \frac{80}{1701} \right) \phi_0^7. \quad (39a)$$

Moreover, the first-order corrections to the potential become

$$V_\alpha(\phi_0) = -\frac{160}{9} - \frac{3\beta}{2} + \left( \frac{3056}{27} - 18\gamma + 4\beta \right) \phi_0^2 + \left( 52\gamma + \frac{3\beta}{2} - \frac{38336}{243} \right) \phi_0^4 + \left( \frac{53600}{729} - \frac{632\gamma}{15} - \frac{4\beta}{9} \right) \phi_0^6 + \left( \frac{52\gamma}{3} - \frac{992}{81} \right) \phi_0^8 + \left( \frac{4624}{15309} - \frac{2854\gamma}{945} \right) \phi_0^{10} + \left( \frac{80\gamma}{567} - \frac{320}{45927} \right) \phi_0^{12}, \quad (39b)$$

and so the potential (35) becomes

$$V(\phi) = \frac{1}{2} - \frac{7}{3} \phi^2 + \frac{25}{18} \phi^4 - \frac{4}{27} \phi^6 + \alpha \left[ -\frac{160}{9} - \frac{3\beta}{2} + \left( \frac{3056}{27} - 18\gamma + 4\beta \right) \phi^2 + \left( 52\gamma + \frac{3\beta}{2} - \frac{38336}{243} \right) \phi^4 + \left( \frac{53600}{729} - \frac{632\gamma}{15} - \frac{4\beta}{9} \right) \phi^6 + \left( \frac{52\gamma}{3} - \frac{992}{81} \right) \phi^8 + \left( \frac{4624}{15309} - \frac{2854\gamma}{945} \right) \phi^{10} + \left( \frac{80\gamma}{567} - \frac{320}{45927} \right) \phi^{12} \right]. \quad (40)$$

The model obtained with  $\alpha = 0$  has a solution given by

$$\phi_0(y) = \tanh(y). \quad (41)$$

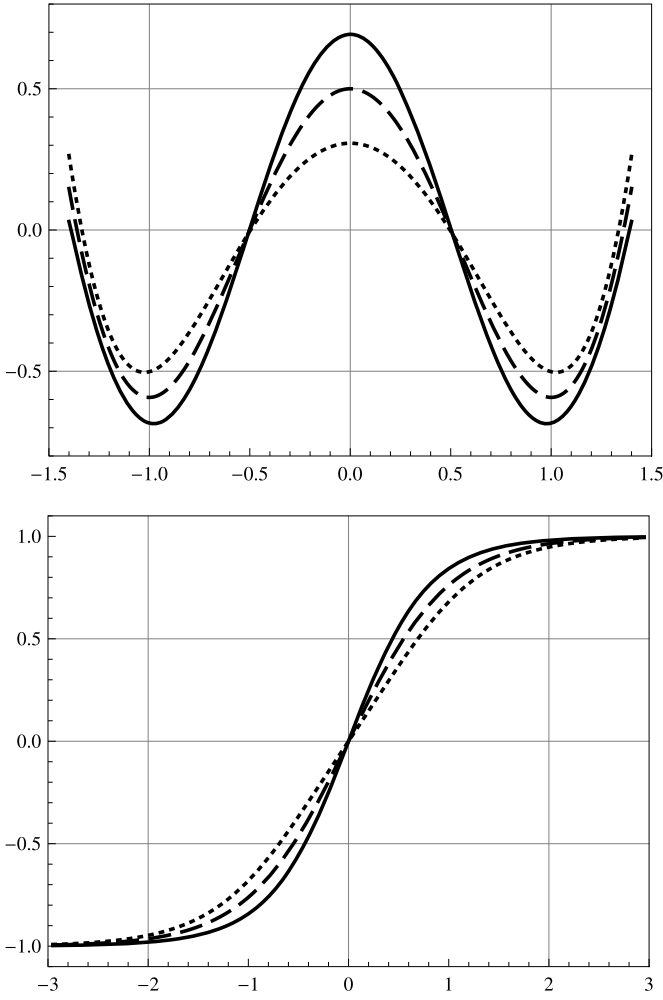
Using Eq. (39a), we can write the complete solution  $\phi(y)$  as

$$\phi(y) = \tanh(y) - \alpha \left[ \left( \frac{224}{9} + \frac{3\beta}{2} \right) \tanh(y) + \left( 6\gamma - \frac{3056}{81} \right) \tanh^3(y) + \left( \frac{416}{81} - \frac{12\gamma}{5} \right) \tanh^5(y) + \left( \frac{2\gamma}{7} - \frac{80}{1701} \right) \tanh^7(y) \right]. \quad (42)$$

It obeys  $\phi(y \rightarrow \pm\infty) \rightarrow \pm\bar{\phi}$ , where

$$\bar{\phi} \equiv \left[ 1 + \alpha \left( \frac{13184}{1701} - \frac{136\gamma}{35} - \frac{3\beta}{2} \right) \right]. \quad (43)$$

We see that  $\phi$  is shifted asymptotically by some value, which depends on  $\alpha$ , as shown in the expression above. We also see that if  $\beta = (16/25515) \times (8240 - 4131\gamma)$  the field  $\phi$  converges to the same asymptotic value of the field  $\phi_0$ , i.e.,  $\phi(y \rightarrow \pm\infty) = \phi_0(y \rightarrow$



**Fig. 4.** Upper panel: the potential (40) for  $\gamma = 1.6$ ,  $\beta = 1$  with  $\alpha = -0.01$  (solid line),  $\alpha = 0$  (dashed line) and  $\alpha = 0.01$  (dotted line). Lower panel: the kinklike solution (42) for the same values of parameters.

$\pm\infty$ ). In Fig. 4, in the upper and lower panels we depict the potential and kinklike solutions, respectively, for some small values of  $\alpha$ . Furthermore, we have

$$V(\phi \rightarrow \bar{\phi}) = -\frac{16}{27} + \frac{32\alpha}{6561}(1296\gamma + 729\beta - 160). \quad (44)$$

Here we note that the minima of the potential are displaced up or down from the standard situation with  $\alpha = 0$  by the value

$$\Delta V_{\min} = \frac{32\alpha}{6561}(1296\gamma + 729\beta - 160).$$

Similarly, the maxima are also displaced up or down by

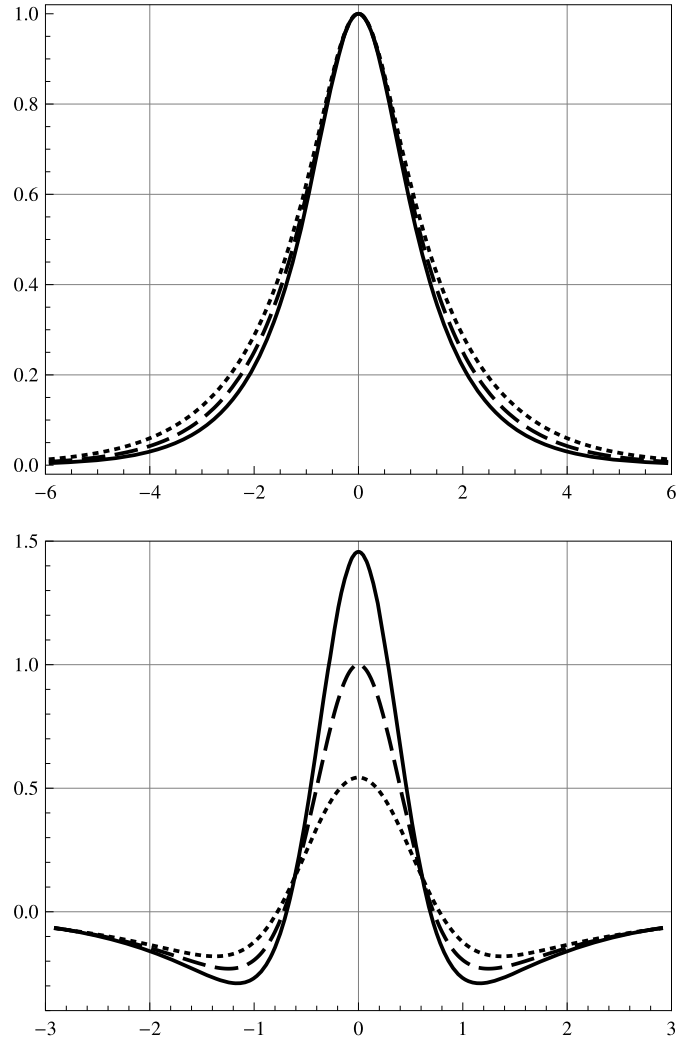
$$\Delta V_{\max} = -\alpha \left( \frac{160}{9} + \frac{3\beta}{2} \right).$$

We can also note that  $V_\phi(\phi \rightarrow \bar{\phi}) = 0$ .

With the solution (41) the warp function becomes

$$A(y) = \frac{1}{3} \left( \frac{1}{3} - \frac{16}{9}\alpha\gamma - \alpha\beta \right) (S^2 + 4 \ln S) + \frac{4}{9}\alpha\gamma S^4 + \frac{20}{81}\alpha\gamma S^6 + \frac{\alpha\gamma}{27} S^8, \quad (45)$$

where  $S = S(y) = \text{sech}(y)$ . This shows that, to make  $e^{2A} > 0$  we must impose that  $\alpha$  is bigger (lesser) than  $3(16\gamma + 9\beta)^{-1}$  for  $\gamma$



**Fig. 5.** Upper panel: warp factor given by (45) for  $\gamma = 1.6$  and  $\beta = 1$  with  $\alpha = -0.01$  (solid line),  $\alpha = 0$  (dashed line) and  $\alpha = 0.01$  (dotted line). Lower panel: energy density (46) for the same values of parameters.

lesser (bigger) than  $-9\beta/16$ . In Fig. 5, in the upper panel we depict the warp factor  $e^{2A}$  for some values of the parameters, and in the lower panel we depict the energy density (46) for the same values of parameters.

Using Eqs. (40) and (42) we can find the energy density as

$$\rho(y) = \frac{e^{\frac{2}{9}S^2}}{27e^{\frac{2}{9}}} S^{8/9} (4S^6 + 39S^4 - 16) + \frac{\alpha S^{8/9}}{e^{\frac{2}{9}(1-S^2)}} \left[ \sum_{i=0}^7 a_{2i} S^{2i} + \ln(S) \left( b_2 + \sum_{j=1}^2 b_{2j+2} S^{2j+2} \right) \right], \quad (46)$$

where

$$\begin{aligned} a_0 &= \frac{14176\gamma}{2187} + \frac{256\beta}{81} - \frac{5120}{6561}, & a_{10} &= \frac{3008}{5103} - \frac{128048\gamma}{25515}, \\ a_2 &= \frac{448256}{15309} - \frac{356624\gamma}{25515} - \frac{427}{81}, & a_{12} &= \frac{4918\gamma}{15309} - \frac{320}{45927}, \\ a_4 &= \frac{10432}{189} - \frac{188906\gamma}{25515} - \frac{10\beta}{27}, & a_{14} &= \frac{8\gamma}{729}, \end{aligned}$$

$$a_6 = -\frac{4144160}{45927} - \frac{142804\gamma}{76545} - \frac{34\beta}{81}, \quad b_2 = \frac{128}{81} \left( \frac{16\gamma}{9} + \beta \right),$$

$$a_8 = -\frac{543968}{15309} + \frac{58244\gamma}{5103} - \frac{8\beta}{81}, \quad -\frac{16}{39}b_4 = 4b_6 = b_2.$$

Asymptotically, we have

$$\rho(y) = -\frac{2^{8/9}}{e^{2/9}} \left( \frac{16}{27} - \alpha[a_0 + b_2(-y + \ln 2)] \right) e^{-8y/9} + \dots \quad (47)$$

Now, if we use Eq. (8), we find the scalar curvature in the form

$$R(y) = \frac{16}{81} (20 - 42S^4 - 5S^6) - \frac{16}{243} \alpha [640\gamma + 360\beta - 3(752 + 171\beta\gamma)S^4 - 10(32\gamma + 9\beta)S^6 + 1332\gamma S^8 + 564\gamma S^{10} + 40\gamma S^{12}]. \quad (48)$$

We see that asymptotically, the scalar curvature goes to the constant value

$$R \rightarrow \frac{320}{81} - \frac{640}{243} \alpha (16\gamma + 9\beta). \quad (49)$$

At the origin it becomes

$$R(0) = -\frac{16}{3} + \frac{16}{81} \alpha [752 - 90\beta + \gamma(171\beta - 752)]. \quad (50)$$

#### 4.2. Nonpolynomial potential

Another relevant example is the sine-Gordon model, which is defined by the following function

$$W(\phi) = 2a \sin(b\phi), \quad (51)$$

where  $a$  and  $b$  are real parameters. The unperturbed solution is given by

$$\phi_0(y) = \frac{1}{b} \arcsin[\tanh(ab^2 y)]. \quad (52)$$

Using Eqs. (34) we can write the first-order contributions to the solution and the potential as

$$\phi_\alpha(\phi_0) = \left( \frac{40a^2}{27} + \frac{8a^2b^4}{3} - 9a^2\gamma - \frac{3\beta}{2} \right) \phi_0 - \frac{a^2}{b} \left( \frac{20}{27} + \frac{64b^2}{9} + 4b^4 - \frac{9\gamma}{2} \right) \sin(2b\phi_0), \quad (53a)$$

$$V_\alpha(\phi_0) = c_1 \sin^2(b\phi_0) + c_2 \sin^4(b\phi_0) + c_3 \phi_0 \sin(2b\phi_0), \quad (53b)$$

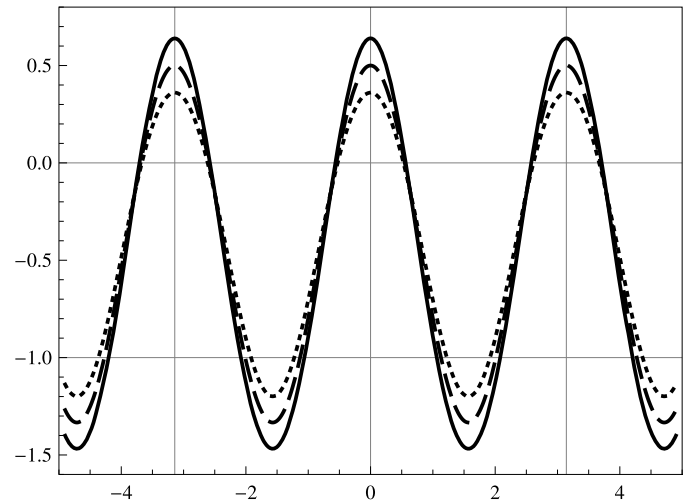
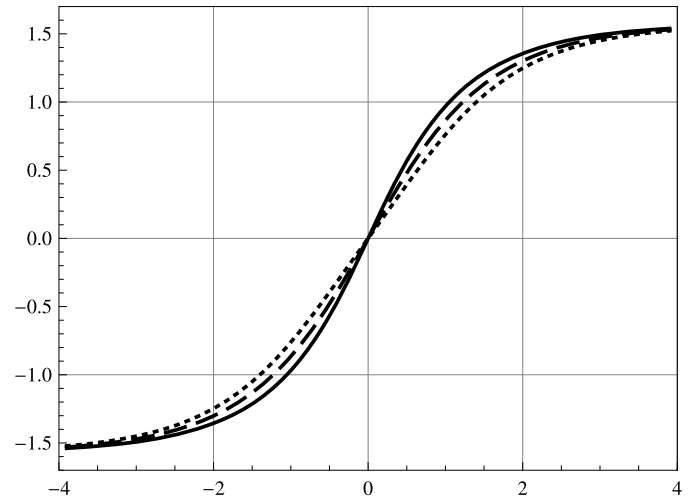
where

$$c_1 = a^4 \left[ -\frac{320}{81} + \frac{224b^4}{9} + \frac{46b^6}{3} + b^2 \left( \frac{50}{27} - \frac{45\gamma}{4} \right) + 22\gamma \right] + a^2 \left( 8 + \frac{3b^2}{2} \right) \beta,$$

$$c_2 = \frac{17a^4}{216} (384b^4 + 216b^6 + b^2(40 - 243\gamma) - 216\gamma),$$

$$c_3 = \frac{a^2b}{324} [2a^2(320 - 1152b^3 + 576b^4 - 864b^5 + 216b^6 - b^2(120 - 729\gamma) - 1944\gamma) - 81(8 + 6b + 3b^2)\beta].$$

Also, using the solution (52) we obtain



**Fig. 6.** Upper panel: solution (54) for  $a = b = \beta = 1$  and  $\gamma = 143/486$ , for  $\alpha = -0.01$  (solid line),  $\alpha = 0$  (dashed line) and  $\alpha = 0.01$  (dotted line). Lower panel: potential (56) for the same values of  $a, b, \beta, \gamma$  and  $\alpha$ .

$$\phi(y) = \frac{1}{b} \left[ 1 + \alpha \left( \frac{40a^2}{27} + \frac{8a^2b^4}{3} - 9a^2\gamma - \frac{3\beta}{2} \right) \right] \times \arcsin[\tanh(ab^2 y)] - \frac{2\alpha a^2}{b} \left( \frac{20}{27} + \frac{64b^2}{9} + 4b^4 - \frac{9\gamma}{2} \right) \frac{\tanh(ab^2 y)}{\cosh(ab^2 y)}. \quad (54)$$

The asymptotic value of  $\phi$  is  $\phi(y \rightarrow \pm\infty) = \pm\bar{\phi}$ , where

$$\bar{\phi} = \frac{\pi}{2b} + \frac{\alpha\pi}{108b} (80a^2 + 144a^2b^4 - 486a^2\gamma - 81\beta). \quad (55)$$

We see that if  $\beta = (2a^2/81) \times (40 + 72b^4 - 243\lambda)$  we have that  $\phi(y \rightarrow \pm\infty) = \phi_0(y \rightarrow \pm\infty)$ . In Fig. 6, in the upper panel we depict the solution (54) for  $a = b = \beta = 1$ ,  $\gamma = 143/486$  and some values of  $\alpha$ . Also, the potential has the form

$$V(\phi) = \frac{a^2b^2}{2} - \frac{1}{6}a^2(8 + 3b^2)\sin^2(b\phi) + \alpha [c_1 \sin^2(b\phi) + c_2 \sin^4(b\phi) + c_3 \phi \sin(2b\phi)], \quad (56)$$

which is depicted in Fig. 6, in the lower panel. We see that

$$V(\phi \rightarrow \bar{\phi}) = -\frac{4a^2}{3} + 8\alpha a^2 \left[ \beta + 4a^2 \left( \lambda - \frac{10}{81} \right) \right], \quad (57)$$

and  $V_\phi(\phi \rightarrow \bar{\phi}) = 0$ .

Furthermore, the warp function is

$$A(\phi_0) = \frac{2a^2\alpha\gamma}{b^2} \cos(2b\phi_0) + \frac{2}{3b^2} [1 - 3\alpha(\beta + 4\gamma a^2)] \\ \times \ln[\cos(b\phi_0)] - \frac{2a^2\alpha\gamma}{b^2}, \quad (58)$$

which obeys  $A(0) = 0$ . Using (52) we have

$$A(y) = -\frac{4\alpha\gamma a^2}{b^2} + \frac{4\alpha\gamma a^2}{b^2} S^2 \\ + \frac{2}{3b^2} [1 - 3\alpha(\beta + 4\gamma a^2)] \ln(S), \quad (59)$$

where  $S = \text{sech}(ab^2 y)$ .

The energy density is

$$\rho(y) = \frac{1}{3} a^2 [2 + 3b^2 - 2 \text{sech}^{-1}(2ab^2 y)] S^{\frac{6b^2+4}{3b^2}} \\ - \frac{a^2\alpha}{162b^2} S^{\frac{4}{3b^2}} \left[ d_1 S^4 + d_2 S \tanh(ab^2 y) \right. \\ \left. \times \arcsin[\tanh(ab^2 y)] + \sum_{i=1}^2 (g_i + h_i \ln(S)) S^{2i-2} \right], \quad (60)$$

where

$$d_1 = 18a^2(40b^4 + 384b^6 + 216b^8 - 96\gamma - 144b^2\gamma - 243b^4\gamma),$$

$$d_2 = b^2(8 + 3b^2)(81\beta - 2a^2(40 + 72b^4 - 243\gamma)),$$

$$g_1 = 640a^2b^2 - 1728a^2\gamma - 5184a^2b^2\gamma - 1296b^2\beta,$$

$$g_2 = -3456a^2\gamma - 864\beta,$$

$$h_1 = a^2[27\gamma(8 + 9b^2)^2 - 8b^2(40 + 9b^2(5 + 24b^2 + 15b^4))] \\ + 81b^2\beta(8 + 3b^2),$$

$$h_2 = 216(4 + 3b^2)(4a^2\gamma + \beta).$$

In Fig. 7, in the upper panel we depict the warp factor  $e^{2A(y)}$ , for some values of the parameters. For this model we must have  $\alpha < [3(4a^2\gamma + \beta)]^{-1}$ , to make  $e^{2A(y)} > 0$ . In Fig. 7, in the lower panel we depict the behavior of the energy density (60), for the same values of parameters.

The scalar curvature becomes

$$R(y) = \frac{16a^2}{9} [5 - (5 + 3b^2)S^2] - \frac{160}{3} \alpha a^2 (4a^2\gamma + \beta) \\ - \frac{64}{3} \alpha \gamma a^4 (10 + 9b^2) S^4 \\ + \frac{16a^2\alpha}{3} (80a^2\gamma + 36a^2b^2\gamma + 10\beta + 3b^2\beta) S^2. \quad (61)$$

It goes asymptotically to the constant value  $R(y \rightarrow \pm\infty) = (80a^2/9) \times [1 - 6\alpha(4a^2\gamma + \beta)]$ . Also, at the origin it gives  $R(0) = (16a^2b^2/3) \times (3\alpha\beta - 1)$ .

## 5. Comments and conclusions

In this work we succeeded to find exact and approximated solutions for the warp factor, the scalar field  $\phi$  and energy density

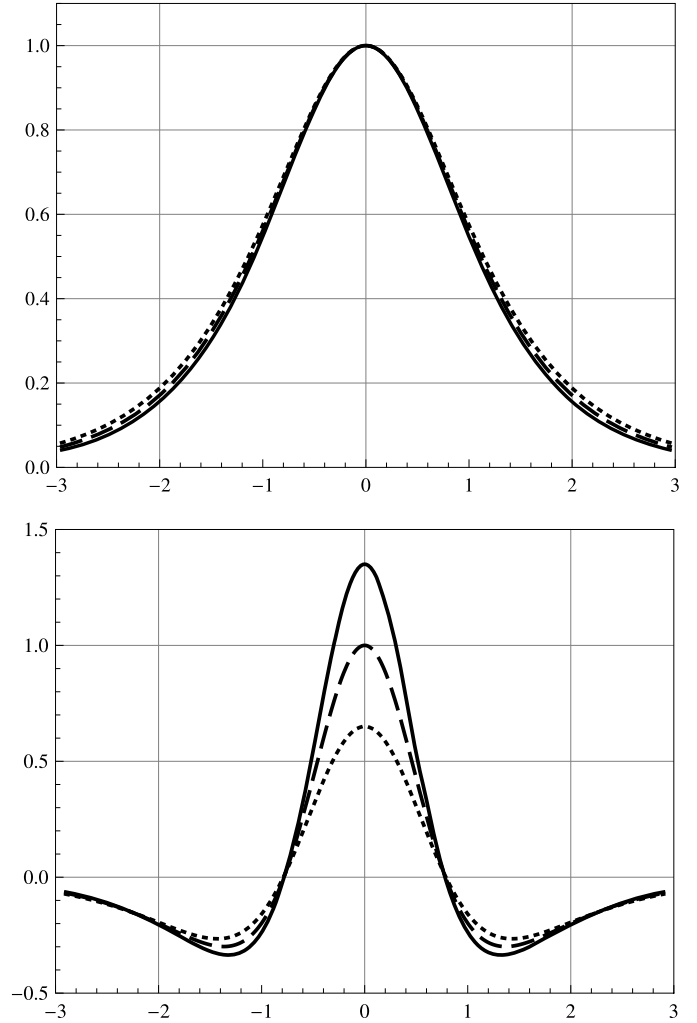


Fig. 7. Upper panel: warp factor (59) for  $a = b = \beta = 1$  and  $\gamma = 143/486$ , for  $\alpha = -0.01$  (solid line),  $\alpha = 0$  (dashed line) and  $\alpha = 0.01$  (dotted line). Lower panel: energy density (60) for the same values of  $a, b, \beta, \gamma$  and  $\alpha$ .

in models of  $F(R)$  brane with a non-constant curvature. We used  $F(R) = R + \alpha R^2$  to solve the equations of motion, and we studied models with the potential for the scalar field engendering polynomial or nonpolynomial interactions. For several distinct examples, we showed that the warp factor is indeed a well-behaved function, the scalar field displays kinklike behavior, and the profile of the energy density appears as expected.

Interestingly, we have found a situation where brane splitting behavior may appear, induced by the parameter  $\alpha$ , which controls the way the generalized gravity enters the game. This effect is different from the brane splitting behavior found in [7,8], where thermal or 2-kink effects may induce the splitting, in models with standard gravity. In this work, the splitting is directly related to the parameter that controls deviation from standard gravity.

We have investigated how the addition of higher order power in the curvature may contribute to the splitting of the brane. In the case with  $F(R) = R + \alpha R^n$ , for  $n = 3, 4, \dots$ , we have checked up to  $n = 5$ , that the brane splitting effect works as in the case studied before, for  $n = 2$ . The results suggest that the brane splitting is a generic effect, for the above polynomial modification of the standard gravity. Further details of the calculations will be given elsewhere.



## Acknowledgements

The authors would like to thank CAPES and CNPq for partial financial support. The work by A.Yu.P. has been supported by the CNPq project 303438/2012-6.

## References

- [1] L. Randall, R. Sundrum, *Phys. Rev. Lett.* **83** (1999) 4690.
- [2] N. Arkani-Hamed, S. Dimopoulos, G. Dvali, *Phys. Lett. B* **429** (1998);  
I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, G. Dvali, *Phys. Lett. B* **436** (1998) 257.
- [3] W.D. Goldberger, M.B. Wise, *Phys. Rev. Lett.* **83** (1999) 4922.
- [4] N. Arkani-Hamed, S. Dimopoulos, G. Dvali, N. Kaloper, *Phys. Rev. Lett.* **84** (2000) 586.
- [5] O. DeWolfe, D.Z. Freedman, S. Gubser, A. Karch, *Phys. Rev. D* **62** (2000) 046008;  
C. Csaki, J. Erlich, T. Hollowood, Y. Shirman, *Nucl. Phys. B* **581** (2000) 309;  
C. Csaki, J. Erlich, G. Grojean, T. Hollowood, *Nucl. Phys. B* **584** (2000) 359.
- [6] M. Gremm, *Phys. Lett. B* **478** (2000) 434.
- [7] A. Campos, *Phys. Rev. Lett.* **88** (2002) 141602.
- [8] D. Bazeia, C. Furtado, A.R. Gomes, *J. Cosmol. Astropart. Phys.* **0402** (2004) 002.
- [9] G. Dvali, G. Gabadadze, M. Porrati, *Phys. Lett. B* **485** (2000) 208;  
A. Karch, L. Randall, *J. High Energy Phys.* **0105** (2001) 008;  
M. Porrati, *Phys. Lett. B* **498** (2001) 92;  
F.A. Brito, M. Cvetič, S.-C. Yoon, *Phys. Rev. D* **64** (2001) 064021;  
M. Cvetič, N.D. Lambert, *Phys. Lett. B* **540** (2002) 301;  
A. Melfo, N. Pantoja, A. Skirzewski, *Phys. Rev. D* **67** (2003) 105003;  
D. Bazeia, F.A. Brito, J.R. Nascimento, *Phys. Rev. D* **68** (2003) 085007;  
O. Castillo-Felisola, A. Melfo, N. Pantoja, A. Ramirez, *Phys. Rev. D* **70** (2004) 104029;  
K. Takahashi, T. Shiromizu, *Phys. Rev. D* **70** (2004) 103507;  
A. Celi, et al., *Phys. Rev. D* **71** (2005) 045009;  
A. Celi, J. High Energy Phys. **0702** (2007) 078;  
A. Ceresole, G. Dall'Agata, *J. High Energy Phys.* **0703** (2007) 110.
- [10] C. Csaki, TASI lectures on extra dimensions and branes, arXiv:hep-ph/0404096.
- [11] D. Bazeia, A.R. Gomes, *J. High Energy Phys.* **0405** (2004) 012;  
D. Bazeia, F. Brito, L. Losano, *J. High Energy Phys.* **0611** (2006) 064;  
V.I. Afonso, D. Bazeia, L. Losano, *Phys. Lett. B* **634** (2006) 526.
- [12] V.I. Afonso, D. Bazeia, R. Menezes, A.Y. Petrov, *Phys. Lett. B* **658** (2007) 71.
- [13] D. Bazeia, R. Menezes, A.Yu. Petrov, A.J. da Silva, *Phys. Lett. B* **726** (2013) 523.
- [14] S.M. Carroll, et al., *Phys. Rev. D* **71** (2005) 063513;  
G. Cognola, et al., *J. Cosmol. Astropart. Phys.* **0502** (2005) 010;  
M. Amarzguoui, O. Elgaroy, D.F. Mota, T. Multamaki, *Astron. Astrophys.* **454** (2006) 707;  
S. Capozziello, S. Norijji, S.D. Odintsov, A. Troisi, *Phys. Lett. B* **639** (2006) 135;  
S. Norijji, S.D. Odintsov, *Phys. Rev. D* **74** (2006) 086005;  
L. Amendola, D. Polarski, S. Tsujikawa, *Phys. Rev. Lett.* **98** (2007) 131302;  
S. Norijji, S.Y.-S. Song, W. Hu, I. Sawicki, *Phys. Rev. D* **75** (2007) 044004;  
L. Amendola, R. Gannouji, D. Polarski, S. Tsujikawa, *Phys. Rev. D* **75** (2007) 083504;  
I. Sawicki, W. Hu, *Phys. Rev. D* **75** (2007) 127502;  
M. Cvetič, M. Robnik, *Phys. Rev. D* **77** (2008) 124003;  
D. Bazeia, A.R. Gomes, L. Losano, *Int. J. Mod. Phys. A* **24** (2009) 1135;  
Y.-X. Liu, Z.-H. Zhao, S.-W. Wei, Y.-S. Duan, *J. Cosmol. Astropart. Phys.* **02** (2009) 003;  
C.A.S. Almeida, M.M. Ferreira Jr., A.R. Gomes, R. Casana, *Phys. Rev. D* **79** (2009) 125022;  
Y. Zhong, Y.-X. Liu, K. Yang, *Phys. Lett. B* **699** (2011) 398;  
Y.-X. Liu, Y. Zhong, Z.-H. Li, *J. High Energy Phys.* **1106** (2011) 135;  
A. Ahmed, B. Grzadkowski, *J. High Energy Phys.* **1301** (2013) 177.
- [15] M. Cvetič, S. Griffies, S.-J. Rey, *Nucl. Phys. B* **381** (1992) 301;  
D. Bazeia, L. Losano, R. Menezes, *Phys. Lett. B* **668** (2008) 246;  
D. Bazeia, A.R. Gomes, L. Losano, R. Menezes, *Phys. Lett. B* **671** (2009) 402.
- [16] C.A.G. Almeida, D. Bazeia, L. Losano, R. Menezes, *Phys. Rev. D* **88** (2013) 025007.