Some Comments on Six Inequalities Associated With the Inefficiency of Ordinary Least Squares With One Regressor

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ABSTRACT

The study of the inefficiency of the ordinary least-squares estimator (OLSE) with one regressor by Watson (1951) required a lower bound for the efficiency defined as the ratio of the variance of the best linear unbiased estimator (BLUE) to the variance of the OLSE. Such a lower bound was provided by the Cassels inequality (1951), which we note is closely related to five other inequalities, including the well-known inequality usually attributed to Kantorovich (1948), but which was established already by Frucht (1943). The main purpose in this paper is to show how these six inequalities are related, with a historical perspective. We present some proofs and conclude that all six inequalities are essentially equivalent, in the sense that any one inequality implies the other five. We identify conditions for equality in each inequality and present the six continuous integral analogues. We end the paper with English translations of the seminal papers by Frucht (1943) and Schweitzer (1914), respect-
1. INTRODUCTION AND MISE-EN-SCÈNE

1.1. The Inefficiency of Ordinary Least Squares with One Regressor

Least-squares estimation is often used when the error covariance matrix may not be proportional to the identity matrix, e.g., when the errors may have different variances and/or are serially correlated. In such situations the ordinary least-squares estimator (OLSE) is usually not the best linear unbiased estimator (BLUE). Much of the early work arose in the context of serial correlation; cf. Anderson (1948) and Watson (1951). The first of these papers indicates when the OLSE and BLUE are the same; the second gives some answers to the question "how bad can least squares be?"—and so we need inequalities.

The study of the inefficiency of the OLSE with one regressor by Watson (1951) required a lower bound for the efficiency

$$ \phi = \frac{(x'x)^2}{x'Vx \cdot x'V^{-1}x}, $$

the ratio of the variance of the BLUE to the variance of the OLSE; here the $n \times 1$ nonnull vector $x$ comprises the values of the regressor, and $V$ is the $n \times n$ positive definite error covariance matrix. Let $V = P\Lambda P'$ denote a spectral decomposition of $V$, where the diagonal matrix $\Lambda = \text{diag}(\lambda_i)$, and let $u = P'x = \{u_i\}$. Then the efficiency

$$ \phi = \frac{\left(\sum_{i=1}^{n} u_i^2\right)^2}{\sum_{i=1}^{n} \lambda_i u_i^2 \cdot \sum_{i=1}^{n} \lambda_i^{-1} u_i^2}. $$

That $\phi < 1$ now also follows at once from the well-known Cauchy-Schwarz (Bouniakowsky) inequality.\(^1\)

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In 1950 the first author (Watson), in search of a lower bound for the efficiency $\phi$, asked the Cambridge mathematician John William Scott Cassels (1922–) to provide a “suitable” inequality. This Cassels inequality, which may be expressed as

$$\frac{\sum_{i=1}^{n} a_i^2 b_i w_i \cdot \sum_{i=1}^{n} a_i^2 b_i w_i}{(\sum_{i=1}^{n} a_i b_i w_i)^2} \leq \frac{(m + M)^2}{4mM},$$

where $a_i > 0$, $b_i > 0$, and $w_i \geq 0$ ($i = 1, \ldots, n$), with

$$m = \min_i \frac{a_i}{b_i} \quad \text{and} \quad M = \max_i \frac{a_i}{b_i},$$

appears as an appendix [13, 14] in Watson (1951, 1955). If we substitute $u_i^2 = a_i b_i w_i$ and $\lambda_i = a_i/b_i$ in (1.2), then it becomes the reciprocal of the left-hand side of (1.3) and so we obtain the efficiency inequality

$$\phi = \frac{(x'x)^2}{x'Vx \cdot x'V^{-1}x} = \frac{(\sum_{i=1}^{n} u_i^2)^2}{\sum_{i=1}^{n} \lambda_i u_i^2 \cdot \sum_{i=1}^{n} \lambda_i^{-1} u_i^2} \geq \frac{4mM}{(m + M)^2},$$

where the eigenvalues $\lambda_i$ of the error covariance matrix $V$ satisfy

$$0 < m \leq \lambda_i \leq M, \quad i = 1, \ldots, n.$$

When $\lambda_1 = M$ and $\lambda_n = m$, then equality holds on the right of (1.5) when $u_1 = u_n$ and $u_2 = \cdots = u_{n-1} = 0$; when $\lambda_1$ and $\lambda_n$ each have multiplicity one, then this condition is also necessary. Equality holds in the Cassels inequality (1.3) when $a_1 b_1 w_1 = a_n b_n w_n$, $w_2 = \cdots = w_{n-1} = 0$, and $a_1/b_1 = \max_i (a_i/b_i) = M$ and $a_n/b_n = \min_i (a_i/b_i) = m$ (and so $a_1 = \max_i a_i$, $a_n = \min_i a_i$, $b_1 = \min_i b_i$, and $b_n = \max_i b_i$).

Our main purpose in this paper is to show how the Cassels inequality (1.3) is associated with five closely related inequalities, which we have found in a search of the literature (from 1914 through 1959) and which we now introduce chronologically. Earlier comparisons of this type were made by Greub and Rheinboldt (1959), Diaz and Metcalf (1964), Mitrinović (1970,

\footnote{The first author recalls that in 1950 he asked Henry Ellis Daniels (1912–) who asked Cassels, as they were putting on their gowns before lecturing, for a “reverse” of the Cauchy-Schwarz inequality; Cassels just worked it out overnight.}

1.2. Five Inequalities Related to the Cassels Inequality

The earliest inequality related to the Cassels inequality that we have found is due to Schweitzer (1914),3 who showed that

\[
\frac{1}{n} \left( \lambda_1 + \cdots + \lambda_n \right) \cdot \frac{1}{n} \left( \frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_n} \right) \leq \frac{1}{2} \left( m + M \right) \cdot \frac{1}{2} \left( \frac{1}{m} + \frac{1}{M} \right)
\]

\[
= \frac{(m + M)^2}{4mM},\tag{1.7}
\]

where the \( \lambda_i \) satisfy the inequalities in (1.6). [It follows at once that the Schweitzer inequality (1.7) is the special case of the efficiency inequality (1.5) with all the \( u_i^2 = 1 \).]

In 1925 George Pólya (1887–1985) and Gábor Szegő (1895–1985) in (the first edition of) Vol. I of their well-known and influential problem book4 showed that

\[
\frac{\sum_{i=1}^n a_i^2 \cdot \sum_{i=1}^n b_i^2}{(\sum_{i=1}^n a_i b_i)^2} \leq \frac{(ab + AB)^2}{4abAB},\tag{1.8}
\]

where

\[
0 < a \leq a_i \leq A, \quad 0 < b \leq b_i \leq B \quad (i = 1, \ldots, n),\tag{1.9}
\]

3In Hungarian: An English translation is presented as Appendix B to this paper.

SIX INEQUALITIES AND INEFFECTIVENESS OF LEAST SQUARES

whereas Frucht (1943)\textsuperscript{5} and Kantorovich (1948) showed that

\[
\frac{\sum_{i=1}^{n} \lambda_i u_i^2 \cdot \sum_{i=1}^{n} \lambda_i^{-1} u_i^2}{(\sum_{i=1}^{n} u_i^2)^2} \leq \frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1 \lambda_n},
\]

(1.10)

where \(0 < \lambda_n \leq \lambda_i \leq \lambda_1\) fixed, \(i = 1, \ldots, n\). When \(\lambda_1 = M\) and \(\lambda_n = m\), then (1.10) coincides with the efficiency inequality (1.5). We will comment further on the relationship between the upper bounds in (1.7), (1.8), and (1.10) at the end of this section.

The inequality (1.10) is well known in the literature as the "Kantorovich inequality"\textsuperscript{6} and is undoubtedly the best known of our six inequalities. It is named after the Nobel Laureate and Academician Leonid Vital'evich Kantorovich (1912–1986) for the inequality he established in 1948 in a long survey article (in Russian) on "Functional Analysis and Applied Mathematics" ([30, pp. 142–144]; see also [31, pp. 106–107]). The inequality (1.10) had been established, however, five years earlier in 1943 by the graph theorist Roberto Frucht Wertheimer (1906–) in [18],\textsuperscript{7} and so we now name (1.10) the Frucht-Kantorovich inequality.

Another closely related inequality was established by Krasnosel'skii and Kreïn (1952) in a study of "iteration processes with minimal residuals" (in Russian):

\[
\frac{\sum_{i=1}^{n} \lambda_i^2 u_i^2 \cdot \sum_{i=1}^{n} u_i^2}{(\sum_{i=1}^{n} \lambda_i u_i^2)^2} \leq \frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1 \lambda_n},
\]

(1.11)

where, as in (1.10), \(0 < \lambda_n \leq \lambda_i \leq \lambda_1\) fixed, \(i = 1, \ldots, n\). A continuous version of (1.11) had been given already by Frucht (1943); see (A.4) in Appendix A below.

\textsuperscript{5}In Spanish: An English translation is presented in Appendix A to this paper.

\textsuperscript{6}The first usage of the term "Kantorovich inequality" seems to be by Greub and Rheinboldt (1959) and Newman (1959).

\textsuperscript{7}We are very grateful to Josip E. Pečarić for drawing our attention (in September 1996) to this paper by Frucht (see also [40, pp. 125, 132]). According to Pečarić and Mond [42, p. 384] the Kantorovich inequality is originally due to Charles Hermite (1822–1901), but no reference is given.
Our sixth (and last) inequality is due to Greub and Rheinboldt (1959), who obtained this "weighted" version of the Pólya-Szegő inequality (1.8):

\[
\frac{\sum_{i=1}^{n}a_i^2w_i \cdot \sum_{i=1}^{n}b_i^2w_i}{(\sum_{i=1}^{n}a_i b_i w_i)^2} \leq \frac{(ab + AB)^2}{4abAB},
\]

(1.12)

where the \(a_i\) and \(b_i\) (and \(a, b, A, B\)) are as in (1.9) and the \(w_i \geq 0\) \((i = 1, \ldots, n)\).

In this paper we present some proofs and show which inequality implies what. We conclude that our six inequalities are all essentially equivalent, in the sense that any one inequality implies the other five. We identify conditions for equality in each inequality and note that continuous integral versions of unweighted discrete inequalities lead to corresponding weighted discrete versions (Hardy, Littlewood, and Pólya [27, p. 13]; Henrici [28]). We end the paper with English translations of the seminal papers by Frucht (1943) and Schweitzer (1914), respectively from the Spanish and Hungarian, and a fairly extensive bibliography.

### 1.3. The Upper Bounds

We end this introductory section with some comments on the upper bounds in our six inequalities.

The upper bound in the Frucht-Kantorovich inequality (1.10) and in the Krasnosel’skii-Krein inequality (1.11),

\[
\frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1 \lambda_n} = \frac{\lambda_1 + \lambda_n}{2} \cdot \frac{\lambda_1^{-1} + \lambda_n^{-1}}{2} = \left(\frac{1}{2} \left(\frac{\lambda_1 + \lambda_n}{\sqrt{\lambda_1 \lambda_n}}\right)\right)^2,
\]

(1.13)

is both the ratio of the arithmetic mean to the harmonic mean and the square of the ratio of the arithmetic mean to the geometric mean of \(\lambda_1\) and \(\lambda_n\). We may also express (1.13) as

\[
\frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1 \lambda_n} = \max_{i,j} \frac{(\lambda_i + \lambda_j)^2}{4\lambda_i \lambda_j}
\]

\[
= \frac{1}{4} \left(\sqrt{\frac{\lambda_1}{\lambda_n}} + \sqrt{\frac{\lambda_n}{\lambda_1}}\right)^2 = \frac{1}{4} \left(\sqrt{\kappa} + \frac{1}{\sqrt{\kappa}}\right)^2
\]

\[
= \frac{(\kappa + 1)^2}{4\kappa} \leq \frac{(\alpha + 1)^2}{4\alpha} \quad \text{for} \quad \kappa \leq \alpha.
\]

(1.14)
Here \( \kappa = \lambda_1/\lambda_n \) [when \( \lambda_1 \) is the largest eigenvalue of a positive definite matrix \( V \) and \( \lambda_n \) is its smallest eigenvalue, as in the efficiency inequality (1.5), then \( \kappa \) is known as the condition number of \( V \)]. Equality holds at the end of (1.14) if and only if \( \kappa = \alpha \).

If \( \lambda_1 \) and \( \lambda_n \) are not known, but we know that

\[
0 < m \leq \lambda_n \leq \lambda_1 \leq M
\]  

(1.15)

[cf. (1.6)], then, in view of (1.14), we can replace the upper bound \( (\lambda_1 + \lambda_n)^2 / 4\lambda_1 \lambda_n \) in the Frucht-Kantorovich and Krasnosel'skii-Kreǐn inequalities by the upper bound \( (m + M)^2 / 4mM \) (as in the Schweitzer and Cassels inequalities), since

\[
\frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1 \lambda_n} \leq \frac{(\alpha + 1)^2}{4\alpha} = \frac{(m + M)^2}{4mM},
\]  

(1.16)

with \( \alpha = M/m \). Equality holds in (1.16) if and only \( m = \lambda_n \) and \( M = \lambda_1 \).

It follows similarly that

\[
\frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1 \lambda_n} \leq \frac{(\beta + 1)^2}{4\beta} = \frac{(ab + AB)^2}{4abAB},
\]  

(1.17)

the upper bound in the Pólya-Szegő and Greub-Rheinboldt inequalities. Here \( \beta = AB/ab = (A/b)/(a/B) \). Since

\[
\frac{A}{b} = \max_i a_i / \min_i b_i \geq \max_i \frac{a_i}{b_i} = \lambda_1 \quad \text{and} \quad \frac{a}{B} = \min_i \frac{a_i}{b_i} \leq \min_i \frac{a_i}{b_i} = \lambda_n,
\]  

(1.18)

it follows that \( \kappa \leq \beta \). Equality holds in (1.17) if and only if

\[
\min_i \frac{a_i}{b_i} = \min_i \frac{a_i}{b_i} \quad \text{and} \quad \max_i \frac{a_i}{b_i} = \max_i \frac{a_i}{b_i}.
\]  

(1.19)

Beppo Levi, in his appendix to Frucht (1943) [cf. (A.7) in our Appendix A], comments on this condition in connection with the continuous version of the Frucht-Kantorovich inequality; cf. (2.20) below.

The two inequalities in (1.19), however, do not hold in general—but do hold when the \( a_i \)'s and \( b_i \)'s are "reversely ordered" as in the original proof by
Pólya and Szegö (1925); cf. (2.11) below. We see, therefore, that the Frucht-Kantorovich-Krasnosel'skiĭ-Kreĭn (and Schweitzer-Cassels) upper bound \((\lambda_1 + \lambda_n)^2 / 4\lambda_1\lambda_n\) is per se tighter (for two reasons) than the Pólya-Szegö–Greub-Rheinboldt upper bound \((aA + bB)^2 / 4abAB\).

2. THE SIX INEQUALITIES

We now present chronologically our six inequalities in some detail.

2.1. The Schweitzer Inequality (1914)

The oldest of our six inequalities,

\[
\frac{1}{n} \left( x_1 + \cdots + x_n \right) \cdot \frac{1}{n} \left( \frac{1}{x_1} + \cdots + \frac{1}{x_n} \right) \leq \frac{(m + M)^2}{4mM}, \tag{2.1}
\]

where

\[0 < m \leq x_i \leq M \quad (i = 1, \ldots, n),\]

was established by Schweitzer (1914), who used a limiting version of (2.1) to obtain its continuous analogue:

\[
\frac{1}{(b - a)^2} \int_a^b f(x) \, dx \cdot \int_a^b \frac{dx}{f(x)} \leq \frac{1}{2} (m + M) \cdot \frac{1}{2} \left( \frac{1}{m} + \frac{1}{M} \right); \tag{2.2}
\]

cf. (B.4) in Appendix B.

The complementary inequality

\[
1 \leq \frac{1}{n} \left( x_1 + \cdots + x_n \right) \cdot \frac{1}{n} \left( \frac{1}{x_1} + \cdots + \frac{1}{x_n} \right),
\]

which follows at once from the Cauchy-Schwarz inequality, is the well-known arithmetic–harmonic-mean inequality; see e.g. Mitrinović [39, pp. 27–28,

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8 Translated into English as our Appendix B. We believe that Pál Schweitzer died in 1941 (Mitrinović [39, p. 394]), but we do not know when he was born.
The arithmetic-geometric-harmonic-mean inequality follows at once from the equivalence of the geometric-harmonic-mean inequality with the well-known arithmetic-geometric-mean inequality.\(^9\)

Equality holds in the Schweitzer inequality \((2.1)\) only when \(n\) is even, and then if and only if

\[
x_1 = \cdots = x_{n/2} = m \quad \text{and} \quad x_{(n/2)+1} = \cdots = x_n = M.
\]

\(^{10}\)The proof of which by Cauchy (1821) \([15, \text{pp. 375--377}]\) is reprinted (in the original French) in Pflüg and Szegö \([44, \text{pp. 50--51; 45, p. 64}]\). Beckenbach and Bellman \([8]\) refer to the arithmetic-geometric-mean inequality as a result of "singular elegance" (p. 3) and present twelve proofs (pp. 4--19); Mitrinović \([39, \text{pp. 27--28}]\) observes that "it is likely that the Pythagoreans [fl. c. 6th cent. BC] knew of the inequality \(\sqrt{ab} \leq (a + b)/2\), but there is no doubt that it was proved by Euclid (fl. c. 300 BC) \([17, \text{Book V, Proposition 25, and Commentary, Vol. II, pp. 185--186}]\). The first, and one of the most beautiful proofs of the arithmetic-geometric mean inequality, was certainly the one given by Cauchy."
with \( \lambda_i = x_i, u_i = 1, \lambda_1 = M, \) and \( \lambda_n = m, \) and so the Frucht-Kantorovich inequality is a "weighted" version of the Schweitzer inequality.

Surprisingly, the Schweitzer inequality also implies the Frucht-Kantorovich inequality. As observed by Makai (1961) (cf. Alpargu [1, §2.6.2]), the continuous Schweitzer inequality (2.2) may be used to obtain the (discrete) Frucht-Kantorovich inequality: To see this, we put \( a = 0, b = \Sigma_{i=1}^n u_i^2 \) in (2.2) and

\[
f(x) = \begin{cases} 
\lambda_1 & \text{for } 0 \leq x < u_1^2, \\
\lambda_i & \text{for } \Sigma_{j=1}^{i-1} u_j^2 \leq x < \Sigma_{j=1}^i u_j^2 \quad (i = 2, \ldots, n),
\end{cases}
\]

where \( 0 < m < \lambda_i < M \) (\( i = 1, \ldots, n \)), and (2.5) follows.

The discrete Schweitzer inequality also implies the (discrete) Frucht-Kantorovich inequality, as was shown by Henrici (1961).\(^{11}\) To see this, we put \( \lambda_i = x_i \) and \( u_i^2 / \Sigma u_i^2 = w_i \) in (2.5), and let

\[
\text{LSI} = \frac{1}{n} \sum x_i \cdot \frac{1}{n} \sum \frac{1}{x_i} \quad \text{and} \quad \text{LKI} = \sum w_j x_j \cdot \sum w_j \frac{1}{x_j}, \quad (2.6)
\]

so that LSI denotes the left-hand side of the Schweitzer inequality (2.1) and LKI the left-hand side of the Frucht-Kantorovich inequality (2.19). It suffices to prove that LSI = LKI for all \( w_i \) rational with \( \Sigma_{i=1}^n w_i = 1 \). We choose \( n \) to be "very large" so that each \( x_i \) occurs "many times," and write

\[
x_{(1)} < x_{(2)} < \cdots < x_{(d)}
\]

for the \( d \) distinct \( x \)'s with multiplicities \( m_1, m_2, \ldots, m_d \) and \( \Sigma_{j=1}^d m_j = n \). Then

\[
\text{LSI} = \frac{\Sigma m_j x_{(j)}}{\Sigma m_j} \cdot \frac{\Sigma m_j / x_{(j)}}{\Sigma m_j} = \sum w_j x_{(j)} \cdot \sum w_j \frac{1}{x_{(j)}} = \text{LKI} \quad (2.7)
\]

with \( w_j = m_j / \Sigma m_j \), and so the Frucht-Kantorovich inequality is essentially equivalent to the Schweitzer inequality.

\(^{11}\)A similar technique had already been proposed already in 1934 by Hardy, Littlewood, and Pólya [27, p. 13].
2.2. The Pólya-Szegő Inequality (1925)

Our second oldest inequality,

\[
\frac{\sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2}{(\sum_{i=1}^{n} a_i b_i)^2} \leq \frac{(ab + AB)^2}{4abAB},
\]

(2.8)

where

\[
0 < a \leq a_i \leq A, \quad 0 < b \leq b_i \leq B \quad (i = 1, \ldots, n),
\]

(2.9)

was proved in 1925 by Pólya and Szegő [44, pp. 57, 213–214; 45, pp. 71–72, 253–255], who also showed that equality holds in (2.8) if and only if

\[
p = n + \frac{a}{A} \quad \text{and} \quad q = n + \frac{b}{B}
\]

are integers and if \(p\) of the numbers \(a_1, \ldots, a_n\) are equal to \(a\) and \(q\) of these numbers are equal to \(A\), and if the corresponding numbers \(b_i\) are equal to \(B\) and \(b\) respectively.

If we put \(a_i^2 = 1/b_i^2 = x_i\), \(b = 1/A\), \(B = 1/a\), \(M = A^2\), and \(m = a^2\) in the Pólya-Szegő inequality (2.8), then it becomes the Schweitzer inequality (2.1).

The continuous version of the special case of (2.8) with \(a = b\) and \(A = B\),

\[
\frac{\int f^2(x) \, dx}{\left[ \int f(x) g(x) \, dx \right]^2} \leq \frac{(a^2 + A^2)^2}{4a^2A^2},
\]

(2.10)

was already posed in 1914 as a “Problem” [33] by József Kürschák (1864–1933). The continuous version of (2.8) without assuming that \(a = b\) and \(A = B\), and with the upper bound in (2.10) replaced by the upper bound in (2.8), was also given by Pólya and Szegő in 1925, op. cit.

According to Cargo (1972, p. 41) the Pólya-Szegő inequality (2.8) was re-proved by Gheorghiu (1933) “by considering the center of gravity of certain weighted points on a parabola.”

12 In the same journal and volume as Schweitzer [51], just over a hundred pages later! As far as we know, there was no published solution per se to this “Problem” [33].
2.2.1. The Original Proof by Pólya and Szegö (1925). The original proof of (2.8) by Pólya and Szegö (1925) [44, pp. 213–214; 45, pp. 253–254] is of interest. We may, without loss of generality, suppose that $a_1 > \cdots > a_n$; then to maximize the left-hand side of (2.8) we must have that the critical $b_i$'s be reversely ordered,¹³ i.e., that

$$b_1 \leq \cdots \leq b_n.$$  \hfill (2.11)

Pólya and Szegö then continue by defining nonnegative numbers $u_i$ and $v_i$ for $i = 1, \ldots, n - 1$ and $n > 2$ such that

$$a_i^2 = u_i a_1^2 + v_i a_n^2 \quad \text{and} \quad b_i^2 = u_i b_1^2 + v_i b_n^2. \quad \hfill (2.12)$$

Since $a_i b_i > u_i a_1 b_1 + v_i a_n b_n$ the left-hand side of (2.8):

$$\frac{\sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2}{(\sum_{i=1}^{n} a_i b_i)^2} \leq \frac{(Ua_1^2 + V a_n^2) (Ub_1^2 + V b_n^2)}{(Ua_1 b_1 + V a_n b_n)^2},$$

where $U = \sum_{i=1}^{n} u_i$ and $V = \sum_{i=1}^{n} v_i$. This reduces the problem to that with $n = 2$, which is solvable by elementary methods, leading to

$$\frac{\sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2}{(\sum_{i=1}^{n} a_i b_i)^2} \leq \frac{(a_1 b_1 + a_n b_n)^2}{4 a_1 a_n b_1 b_n}, \quad \hfill (2.13)$$

where, since the $a_i$'s and $b_i$'s are here reversely ordered,

$$a_1 = \max_i a_i, \quad a_n = \min_i a_i, \quad b_1 = \min_i b_i, \quad b_n = \max_i b_i. \quad \hfill (2.14)$$

If we now assume, as in (2.9), that

$$0 < a \leq a_i \leq A, \quad 0 < b \leq b_i \leq B \quad (i = 1, \ldots, n), \quad \hfill (2.15)$$

¹³For if $b_k > b_m$ with $k < m$, then we could interchange $b_k$ and $b_m$: $b_k^2 + b_m^2 = b_m^2 + b_k^2$ and $a_k b_k + a_m b_m \geq a_k b_m + a_m b_k$. 

then [cf. (1.14)]

$$\frac{(a_1b_1 + a_nb_n)^2}{4a_1a_nb_1b_n} \leq \frac{(ab + AB)^2}{4abAB}, \quad (2.16)$$

and (2.8) follows.

The inequality (2.13) is tighter than (2.8) when at least one of the four inequalities

$$a \leq a_n, \quad a_1 \leq A, \quad b \leq b_1, \quad b_n \leq B \quad (2.17)$$

is strict.

When the $a_i$’s and $b_i$’s satisfy (2.14), or are reversely ordered, then the upper bound in (2.13) coincides with the upper bound in the Frucht-Kantorovich inequality, with

$$\lambda_1 = \max_i \frac{a_i}{b_i} = \frac{\max_i a_i}{\min_i b_i} \quad \text{and} \quad \lambda_n = \min_i \frac{a_i}{b_i} = \frac{\min_i a_i}{\max_i b_i}, \quad (2.18)$$

as noted by Levi; cf. (1.19) and (A.7).

2.3. The Frucht-Kantorovich Inequality (1943, 1948)

Frucht (1943) and Kantorovich (1948) showed that

$$\frac{\sum_{i=1}^{n} \lambda_i u_i^2 \cdot \sum_{i=1}^{n} \lambda_i^{-1} u_i^2}{\left(\sum_{i=1}^{n} u_i^2\right)^2} \leq \frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1 \lambda_n}, \quad (2.19)$$

where $0 < \lambda_n \leq \lambda_i \leq \lambda_1$ fixed, $i = 1, \ldots, n$.

As noted in our introduction, the inequality (2.19) is well known as the "Kantorovich inequality," but we will now call it the "Frucht-Kantorovich inequality."

---

14 Translated into English as our Appendix A.
In his appendix to Frucht (1943), Levi obtained the continuous analogue:

\[
1 < \frac{\int_a^b f(t) g(t) \, dt \, \int_a^b \frac{f(t)}{g(t)} \, dt}{\left[ \int_a^b f(t) \, dt \right]^2} \leq \frac{(m_g + M_g)^2}{4m_g M_g}; \quad (2.20)
\]

cf. (A.6) in Appendix A (and [1, p. 8]), where \(0 < m_g \leq g(t) \leq M_g\) for \(0 < a < t < b\), while \(f(t), g(t), \) and \(1/g(t)\) are integrable functions on \([a, b]\).

2.3.1. Proofs of the Frucht-Kantorovich Inequality. Frucht's "barycentric method"\(^{15}\) in proving (2.19) was rediscovered by Watson (1987), who gave a general geometric method for finding this and other inequalities, which we now sketch. Let \((\lambda_i, \mu_i)\) denote \(n\) points \((i = 1, \ldots, n)\) in the plane \(R^2\), and let \(w_i \geq 0, \sum_{i=1}^n w_i = 1\). Then the point \((L, M)\) with \(L = \sum_{i=1}^n \lambda_i w_i\) and \(M = \sum_{i=1}^n \mu_i w_i\) lies in the convex closure of the \(n\) points \((\lambda_i, \mu_i)\). So if we seek the extremes of the product \(LM\), we must seek the rectangular hyperbolae cutting this convex set \(LM = k\) with the smallest and largest values of \(k\). In the special case when \(\mu_i = 1/\lambda_i\), the points \((\lambda_i, \mu_i)\) lie on \(xy = 1\), so that the convex closure is above this curve but below the chord joining \((\lambda_1, 1/\lambda_1)\) to \((\lambda_n, 1/\lambda_n)\), where \(\lambda_1 \geq \cdots \geq \lambda_n\). The hyperbola \(LM = 1\) is the lower bound. The hyperbola with this chord as tangent gives the upper bound. Then only an elementary computation is needed to obtain (2.19).

This method would also give the discrete version of the inequality

\[
1 \leq \frac{(t_2 - t_1) \int_{t_1}^{t_2} f^2(t) \, dt}{\left[ \int_{t_1}^{t_2} f(t) \, dt \right]^2} \leq \frac{(m_f + M_f)^2}{4m_f M_f}; \quad (2.21)
\]

obtained by Frucht (1943); cf. (A.4) in Appendix A [see also the Krasnosel'ski-Kreĭ inequality (1.11) and (2.33)]. To see this we consider the convex closure of the points \((\lambda_1, \lambda_1^2), \ldots, (\lambda_n, \lambda_n^2)\) and seek the maximum of the ratio \(\sum_{i=1}^n \lambda_i^2 w_i/(\sum_{i=1}^n \lambda_i w_i)^2 = y/x^2\), where \((x, y)\) falls in the convex set. Clearly we seek the parabola \(y = kx^2\) with maximum \(k\). And this will have the chord joining \((\lambda_1, \lambda_1^2)\) and \((\lambda_n, \lambda_n^2)\) as a tangent. The continuous version then follows as a limit.

\(^{15}\)As noted by Beckenbach (1943), this method had already been used by Gheorghiu (1933) "to obtain a sharpened form of Cauchy's inequality and also an analogously sharpened form of the Hőlder-Jensen inequality."

There are many other proofs of the Frucht-Kantorovich inequality in the literature. Five of these, which are presented in Alpargu (1996, Chapter 1), are by Kantorovich (1948), Anderson (1971, p. 569), Styan (1983), Bühler (1987), and Pták (1995); see also Marshall and Olkin (1964). (The proofs by Styan (1983) and Bühler (1987) are also given in Alpargu and Styan [2].)

2.3.2. Three Footnotes. In a footnote, Frucht (1943, p. 44),\textsuperscript{16} observed that the Pólya-Szegő inequality (2.8) was "una acotación análoga" [an analogous bound] to the Frucht-Kantorovich inequality (2.19), and Beckenbach (1943), in his review of Frucht (1943), stated that (2.19) "is included in" the Pólya-Szegő inequality; a similar statement was made by Mäkeläinen (1970, p. 88). Kantorovich (1948), also in a footnote ([30, p. 143]; see also [31, p. 106]), stated that (2.19) is "a special case" of the Pólya-Szegő inequality, but George E. Forsythe, who edited the 1952 English translation of Kantorovich (1948), observed, again in a footnote [31, p. 106], that "it is not clear to me that Kantorovich's inequality really is a special case" of the Pólya-Szegő inequality, which observation Greub and Rheinboldt (1959) found to be "well justified."

In view of the proofs by Makai (1961) and Henrici (1961) just presented, we must agree with Frucht, Beckenbach, Kantorovich, and Mäkeläinen. If, however, we substitute

$$a_i = \lambda_i^{1/2} u_i \quad \text{and} \quad b_i = \lambda_i^{-1/2} u_i$$  \hspace{1cm} (2.22)

in the left-hand side of the Pólya-Szegő inequality (2.8), then it becomes the left-hand side of the Frucht-Kantorovich inequality (2.19), but the upper bound in (2.19) only "reduces"\textsuperscript{17} to the upper bound in (2.8) when

$$\max_i \frac{a_i}{b_i} = \max_i \frac{a_i}{\min_i b_i} \quad \text{and} \quad \min_i \frac{a_i}{b_i} = \min_i \frac{a_i}{\max_i b_i} ;$$  \hspace{1cm} (2.23)

cf. (2.18). The condition (2.23), which does hold when the $a_i$'s and $b_i$'s are reversely ordered (as in the original proof by Pólya and Szegő just presented) does not, however, hold in general, even when the $a_i$ and $b_i$ are defined as in (2.22), and so we should agree with Forsythe, Greub, and Rheinboldt (and Levi).

\textsuperscript{16}Footnote 22 in Appendix A.

\textsuperscript{17}As observed by Levi in Frucht (1943), cf. (A.7) in Appendix A.
2.3.3. A Vector-Matrix Formulation. The Frucht-Kantorovich inequality (2.19) is often expressed in the vector-matrix form [cf. (1.5) above]

\[
\frac{t'At \cdot t'A^{-1}t}{(t' t)^2} \leq \frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1 \lambda_n},
\]

(2.24)

where \( t \) is a real \( n \times 1 \) nonnull vector and \( A \) is a real \( n \times n \) symmetric positive definite matrix, with \( \lambda_1 \) and \( \lambda_n \), respectively, its (fixed) largest and smallest (necessarily positive) eigenvalues. Watson (1987) gives an analogue to (2.24) when \( A \) is singular and so a generalized inverse \( A^- \) is used instead of the inverse \( A^{-1} \) [see also Baksalary and Puntanen (1991), Pečarić, Puntanen, and Styan (1996), and Alpargu (1996, p. 61) for further extensions in this direction]. Watson (1987) also obtains the maximum of \( t'At \cdot t'A^{-1}t/(t' t)^2 \) when \( A \) has some negative eigenvalues.

Equality holds in (2.24) when the vector

\[
t = \frac{1}{\sqrt{2}} (p_1 \pm p_n),
\]

(2.25)

where \( p_1 \) and \( p_n \) are orthonormal eigenvectors of \( A \) corresponding, respectively, to \( \lambda_1 \) and \( \lambda_n \). When \( \lambda_1 \) and \( \lambda_n \) both have multiplicity 1, this condition is also necessary. When \( \lambda_1 \) and \( \lambda_n \), however, have multiplicities \( f \geq 1 \) and \( h \geq 1 \), respectively, so that

\[
\lambda_1 = \cdots = \lambda_f > \lambda_{f+1} \geq \cdots \geq \lambda_{n-h} > \lambda_{n-h+1} = \cdots = \lambda_n,
\]

(2.26)

say, then for equality in (2.24) we need

\[
t = \frac{1}{\sqrt{2}} (P_1 a_1 \pm P_n a_n),
\]

where \( P_1 \) and \( P_n \) are matrices, respectively \( n \times f \) and \( n \times h \), whose columns are orthonormal eigenvectors of \( A \) corresponding, respectively, to \( \lambda_1 \) and \( \lambda_n \). The vectors \( a_1 \) and \( a_n \) are arbitrary except that \( a_1' a_1 = a_n' a_n = 1 \).

Equality holds in the Frucht-Kantorovich inequality (2.19) whenever

\[
u_1 = u_n \quad \text{and} \quad u_2 = \cdots = u_{n-1} = 0;
\]

when \( \lambda_1 \) and \( \lambda_n \) both have multiplicity 1, then this condition is also necessary. When \( \lambda_1 \) and \( \lambda_n \), however, have multiplicities \( f \geq 1 \) and \( h \geq 1 \),
respectively, as in (2.26), then for equality in (2.19) we need
\[ u_{f+1} = \cdots = u_{n-h} = 0 \quad \text{and} \quad u_1 + \cdots + u_f = u_{n-h+1} + \cdots + u_n. \]

2.4. The Cassels Inequality (1951)
Our fourth inequality,
\[ \frac{\sum_{i=1}^{n} a_i^2 w_i \cdot \sum_{i=1}^{n} b_i^2 w_i}{(\sum_{i=1}^{n} a_i b_i w_i)^2} \leq \frac{(m + M)^2}{4 m M}, \tag{2.27} \]
where \( a_i > 0, b_i > 0, \) and \( w_i \geq 0 \) \((i = 1, \ldots, n)\), and
\[ m = \min_i \frac{a_i}{b_i} \quad \text{and} \quad M = \max_i \frac{a_i}{b_i}, \tag{2.28} \]
was proved in 1951 by Cassels ([13]; see also [14]).
Equality holds in (2.27) when \( w_1 = 1/a_1 b_1, w_n = 1/a_n b_n, w_2 = \cdots = w_{n-1} = 0, m = a_n/b_1, \) and \( M = a_1/b_n. \)

If, in (2.27), we put the weights \( w_i = 1 \), we obtain the "unweighted" Cassels inequality:
\[ \frac{\sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2}{(\sum_{i=1}^{n} a_i b_i)^2} \leq \frac{(m + M)^2}{4 m M}, \tag{2.29} \]
which per se is tighter than the Pólya-Szegő inequality (2.8); cf. (2.23) and (1.17).

An integral analogue of the Cassels inequality (2.27) is
\[ \frac{\int_c^d f^2(x) h^2(x) \, dx \cdot \int_c^d g^2(x) h^2(x) \, dx}{\left[ \int_c^d f(x) g(x) h^2(x) \, dx \right]^2} \leq \frac{(m + M)^2}{4 m M}, \]
where \( f(x), g(x), \) and \( h(x) \) are continuous positive functions on the interval \([c, d]\) with \( 0 < m \leq f(x)/g(x) \leq M \) and \( \int_c^d h^2(x) \, dx < \infty. \)

2.4.1. Two Proofs of the Cassels Inequality. The original proof by Cassels (1951) is of interest. We begin with the assertion that
\[ \frac{(1 + k \omega)(1 + k^{-1} \omega)}{(1 + \omega)^2} \leq \frac{(1 + k)(1 + k^{-1})}{4}, \quad k > 0, \quad \omega > 0, \tag{2.30} \]
which being a form of (2.27) for \( n = 2 \), shows that it holds for \( n = 2 \). To prove that the maximum of (2.27) is attained when no more than two \( w_t \)'s are nonzero, Cassels then notes that if, e.g., \( w_1, w_2, w_3 \neq 0 \) led to an extremum \( M \) of \( XY/Z^2 \), then we would have the three linear equations:

\[
a_k^2 X + b_k^2 Y - 2Ma_k b_k Z = 0 \quad (k = 1, 2, 3). \tag{2.31}
\]

Nontrivial solutions exist if and only if the three vectors \([a_k^2, b_k^2, a_k b_k]\) are linearly dependent. But this will be so only if, for some \( i \neq j \) \((i, j = 1, 2, 3)\), \( a_i = \gamma a_j, b_i = \gamma b_j \). And if that were true we could, e.g., drop the \( a_i, b_i \) terms and so deal with the same problem with one fewer variable. If only one \( w_i \neq 0 \), then \( M = 1 \), the lower bound. So we need only examine all pairs \( w_i \neq 0, w_j \neq 0 \). The result (2.27) then quickly follows.

We may also prove the Cassels Inequality (2.27) using the barycentric method of Frucht (1943) and Watson (1987). We substitute \( w_i = u_i/b_i^2 \) in the left-hand side of (2.27), which may then be expressed as the ratio

\[
\frac{N}{D^2}, \quad \text{where} \quad N = \sum_{i=1}^{n} \left( \frac{a_i}{b_i} \right)^2 u_i \quad \text{and} \quad D = \sum_{i=1}^{n} \left( \frac{a_i}{b_i} \right) u_i, \tag{2.32}
\]

assuming, without loss of generality, that \( \sum_{i=1}^{n} u_i = 1 \). But the point with coordinates \((D, N)\) must lie within the convex closure of the \( n \) points \((a_i/b_i, a_i^2/b_i^2)\). The value of \( N/D^2 \) at points on the parabola is unity. If \( m = \min_i a_i/b_i \) and \( M = \max_i a_i/b_i \) [cf. (1.4)], then the minimum must lie on the chord joining the point \((m, m^2)\) and \((M, M^2)\). Some easy calculus then leads to (2.27).

2.5. The Krasnosel'skiĭ-Kreĭn Inequality (1952)

Our fifth inequality,

\[
\frac{\sum_{i=1}^{n} \lambda_i^2 u_i^2}{\left( \sum_{i=1}^{n} \lambda_i u_i^2 \right)^2} \leq \frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1 \lambda_n}, \tag{2.33}
\]

where, as in (1.10), \( 0 < \lambda_n \leq \lambda_i \leq \lambda_1 \) fixed \((i = 1, \ldots, n)\), was proved in 1952 by Krasnosel'skiĭ and Kreĭn [32, pp. 323–325]. A continuous version of
(2.33) had been given already by Frucht (1943); cf. (2.21) above and (A.4) in Appendix A below.

Equality holds in (2.33) when \( u_1 = 1/\sqrt{\lambda_1}, \; u_n = 1/\sqrt{\lambda_n}, \) and \( u_2 = \cdots = u_{n-1} = 0. \)

We may express the Krasnosel’skii-Kreĭn inequality in vector-matrix form as

\[
\frac{t'A^2 t \cdot t'}{(t'At)^2} \leq \frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1 \lambda_n},
\]

where (again) \( \lambda_1 \) and \( \lambda_n \) are the largest and smallest (fixed) eigenvalues of the \( n \times n \) positive definite matrix \( A \), and \( t \) is an \( n \times 1 \) nonnull vector.

The “Krasnosel’skii-Kreĭn inequality” (2.34), however, is just an alternative version of the Frucht-Kantorovich inequality (2.24). Since \( A \) is positive definite, we may define a symmetric positive definite square root \( A^{1/2} \) and substitute \( t = A^{-1/2}u \) and then \( u = t \) in (2.34) to realize (2.24).

Indeed, as shown by Schopf (1960) (see also Householder [29, p. 83]),

\[
\frac{x'A^{v+1} x \cdot x'A^{v-1} x}{(x'A^v x)^2} \leq \frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1 \lambda_n},
\]

where \( v \) is an integer. [To establish (2.35) we put \( t = A^{v/2}x = (A^{1/2})^v x \) in (2.24).] Moreover Schopf (1960) showed that (2.35) remains valid for any \( v \) when \( A \) is complex Hermitian positive definite and \( x \) is complex (with \( x' \) its conjugate transpose).

Clearly \( v = 0 \) in (2.35) yields the Frucht-Kantorovich inequality (2.24), while \( v = 1 \) yields the Krasnosel’skii-Kreĭn inequality (2.33).

Equality holds in the Krasnosel’skii-Kreĭn inequality (2.34) when

\[
t = \frac{1}{\sqrt{\lambda_1}} p_1 \pm \frac{1}{\sqrt{\lambda_n}} p_n
\]

[cf. (2.25)], where \( p_1 \) and \( p_n \) are orthonormal eigenvectors of \( A \) corresponding, respectively, to \( \lambda_1 \) and \( \lambda_n \).
The Krasnosel’ski-Kreǐn inequality (2.33) is, however, equivalent to the Cassels inequality (2.27)—for if we substitute

\[
\lambda_i = \frac{a_i}{b_i}; \quad u_i^2 = b_i^2 w_i; \quad \lambda_1 = \max_i \frac{a_i}{b_i} = M, \quad \lambda_n = \min_i \frac{a_i}{b_i} = m
\]

(2.36)

in (2.33), it becomes (2.27) [and the reverse substitution takes us back to (2.33)].

2.6. The Greub-Rheinboldt Inequality (1959)

Our sixth and last inequality,

\[
\frac{\sum_{i=1}^{n} a_i^2 w_i \cdot \sum_{i=1}^{n} b_i^2 w_i}{(\sum_{i=1}^{n} a_i b_i w_i)^2} \leq \frac{(ab + AB)^2}{4abAB},
\]

(2.37)

where

\[
0 < a \leq a_i \leq A, \quad 0 < b \leq b_i \leq B \quad (i = 1, \ldots, n),
\]

was established by Greub and Rheinboldt (1959).

Equality holds in (2.37) when \( w_1 = 1/a_1b_1, w_n = 1/a_nb_n, w_2 = \ldots = w_{n-1} = 0, m = a_n/b_1, M = a_1/b_n, \) with \( a_1 = A, a_n = a, b_1 = B, \) and \( b_n = b. \)

The Greub-Rheinboldt inequality (2.37) is a weaker version of the Cassels inequality (2.27) in that the upper bound in (2.27) is usually tighter than the upper bound in (2.37). When the \( a_i \)'s and \( b_i \)'s are reversely ordered, then the two upper bounds coincide, as do the two inequalities.

We note that the Greub-Rheinboldt inequality (2.37) is a weighted version of the Pólya-Szegő inequality (2.8) in the same sense that the Frucht-Kantorovich inequality (2.19) is a weighted version of the Schweitzer inequality (2.1).
An integral analogue\(^{18}\) of the Greub-Rheinboldt inequality is

\[
\frac{\int_c^d f^2(x)h^2(x) \, dx \cdot \int_c^d g^2(x)h^2(x) \, dx}{\left[ \int_c^d f(x)g(x)h^2(x) \, dx \right]^2} \leq \frac{(ab + AB)^2}{4abAB},
\]

where \(f(x), g(x),\) and \(h(x)\) are continuous positive functions on the interval \([c, d]\) with \(0 < a \leq f(x) \leq A, 0 < b \leq g(x) \leq B,\) and \(\int_c^d h^2(x) \, dx < \infty.\)

3. THE SIX INEQUALITIES ARE ESSENTIALLY EQUIVALENT

To see that our six inequalities are essentially equivalent, we start with the oldest—the Schweitzer inequality (2.1). Then in view of the proofs by Makai (1961) and Henrici (1961) [cf. (2.7)], it follows that

Schweitzer inequality (2.1) \(\Rightarrow\) Frucht-Kantorovich inequality (2.19).

With the inequality (2.35) given by Schopf [49] we saw that

Frucht-Kantorovich inequality (2.24) \(\Rightarrow\)

Krasnosel'skii-Kreĭn inequality (2.34).

But in view of the substitution (2.36) we found that

Krasnosel'skii-Kreĭn inequality (2.33) \(\Rightarrow\) Cassels inequality (2.27).

Because the upper bound in the Cassels inequality is in general tighter than

\(^{18}\)Mitrinović [39, p. 60] observed that such an integral analogue was "known" but did not give it; cf. [1, p. 24].
the upper bound in the Greub-Rheinboldt inequality, we have [cf. (1.17)] that

Cassels inequality (2.27) \(\Rightarrow\) Greub-Rheinboldt inequality (2.37),

which being a weighted version of the Pólya-Szegő inequality means that

Greub-Rheinboldt inequality (2.37) \(\Rightarrow\) Pólya-Szegő inequality (2.8),

which being a weighted version of the Schweitzer inequality brings us full circle with

Pólya-Szegő inequality (2.8) \(\Rightarrow\) Schweitzer inequality (2.1).

APPENDIX A. ON SOME INEQUALITIES: Observation concerning the solution proposed by Ing. Ernesto M. Saleme of Problem No. 21 [Math. Notae 2:197–199 (1942)], by Roberto Frucht Wertheimer (1943) [18] [with an untitled appendix by Beppo Levi and with the proof and generalization by Ernesto M. Saleme [47] of (A.1) and the proof [of (A.1)] by Abraham H. Bender [9]]\(^{19}\)

We would like to draw attention to the fact that the "barycentric" method used by Ing. Saleme to prove\(^{20}\) the two inequalities in Problem No. 21 [47] also admits another interesting generalization, which leads to a more general inequality than these two inequalities:

\[
\sum_{i=1}^{n} x_i^2 \geq \frac{k^2}{n}, \quad \sum_{i=1}^{n} \frac{1}{x_i} \geq \frac{n^2}{k} \tag{A.1}
\]

(where \(x_1, x_2, \ldots, x_n\) are positive numbers such that \(\sum_{i=1}^{n} x_i = k\)).

We consider the \(n\) points with coordinates \(x_i\) and \(y_i = 1/x_i\), which lie on the equilateral hyperbola \(y = 1/x\). Assuming that a positive weight \(m_i\) is


\(^{20}\) See the part of this appendix starting just before (A.8).
applied at the point $P(x_1, y_1)$, the coordinates of the center of gravity of these $n$ weights are

$$X_G = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}, \quad Y_G = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i / x_i}{\sum_{i=1}^{n} m_i}.$$  

Since $xy = 1$ for all points $(x, y)$ on the hyperbola, it follows that the product $X_G Y_G$ (= area of the rectangle $O'G'GC''$ in Fig. 1) must be greater than or equal to 1:

$$\frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \cdot \frac{\sum_{i=1}^{n} m_i / x_i}{\sum_{i=1}^{n} m_i} \geq 1.$$

If we now take unit weights $m_i = 1$, we obtain

$$\frac{\sum_{i=1}^{n} x_i}{n} \cdot \frac{\sum_{i=1}^{n} 1 / x_i}{n} \geq 1,$$

which yields the second inequality in (A.1):

$$\sum_{i=1}^{n} \frac{1}{x_i} \geq \frac{n^2}{k}.$$

In order to obtain the first inequality in (A.1) it suffices to take $m_i = x_i$, which yields the inequality

$$\frac{\sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} x_i} \cdot \frac{n}{\sum_{i=1}^{n} x_i} \geq 1,$$

which is

$$\sum_{i=1}^{n} x_i^2 \geq \frac{k^2}{n}.$$
FIG. 1.
In this way, we see that the two inequalities [in (A.1)] of Problem No. 21 appear as special cases of the more general inequality

\[ 1 \leq \frac{\sum_{i=1}^{n} m_i x_i \sum_{i=1}^{n} m_i / x_i}{(\sum_{i=1}^{n} m_i)^2}. \]  

(A.2)

But the same considerations that have led us to this inequality allow us to find an upper bound for the ratio in (A.2). To ease the notation, let us suppose that \( x_1 > x_2 > \cdots > x_n > 0 \). Then independently of the numerical values of the positive weights \( m_i \) concentrated on the points \( P_1, P_2, \ldots, P_n \), their center of gravity \( G \) must lie inside the segment of the hyperbola bounded by the chord joining the two extreme points:

\[ P_1 = \left( x_1, \frac{1}{x_1} \right) \quad \text{and} \quad P_n = \left( x_n, \frac{1}{x_n} \right) \]

and the corresponding arc of the hyperbola \( xy = 1 \). Hence the equilateral hyperbola with equation \( xy = X_G Y_G \) passing through \( G \) must lie between \( xy = 1 \) and the hyperbola \( xy = X_T Y_T \) touching the chord \( P_1 P_n \) at the point \( T = (X_T, Y_T) \). In other words, we must have \( 1 \leq X_G Y_G \leq X_T Y_T \), which gives the desired bound; it remains to calculate the product \( X_T Y_T \), which is easy if we use the well-known theorem.\(^{21}\) The point of contact of a tangent to a hyperbola divides the segment of the tangent cut by the asymptotes in equal parts.

We see immediately that the line \( P_1 P_n \) cuts the asymptotes (coordinate axes) at the points \((x_1 + x_n, 0)\) and \((0, (x_1 + x_n)/x_1 x_n)\), and so we obtain for the coordinates \((X_T, Y_T)\) of the midpoint \( T \) the values

\[ X_T = \frac{x_1 + x_n}{2}, \quad Y_T = \frac{x_1 + x_n}{2 x_1 x_n}, \]

with product

\[ X_T Y_T = \frac{(x_1 + x_n)^2}{4 x_1 x_n}, \]

as desired.

---

This result generalizes immediately on observing that the ordering of the points \(x_1, x_2, \ldots, x_n\) is immaterial. We relax, therefore, the condition \(x_1 \geq x_2 \geq \cdots \geq x_n\) and obtain the inequalities

\[
1 \leq \frac{\sum_{i=1}^{n} m_i x_i \sum_{i=1}^{n} m_i / x_i}{(\sum_{i=1}^{n} m_i)^2} \leq \frac{(X_m + X_M)^2}{4X_m X_M},
\]

subject only to the condition that \(0 < X_m \leq x_i \leq X_M\) and \(0 < m_i\) \((i = 1, 2, \ldots, n)\).\(^{22}\)

As an application of the inequalities (A.3) we will show that the following inequalities hold:

\[
1 \leq \frac{(t_2 - t_1) \int_{t_1}^{t_2} f^2(t) \, dt}{\left[ \int_{t_1}^{t_2} f(t) \, dt \right]^2} \leq \frac{(m_f + M_f)^2}{4 m_f M_f},
\]

where the continuous function \(f(t)\) satisfies \(0 < m_f \leq f(t) \leq M_f\) on the interval \(t_1 \leq t \leq t_2\).

In fact if we partition the interval \([t_1, t_2]\) into subintervals at the points \(\tau_i\) with widths \(\Delta \tau_i\), then we can approximate the integrals \(\int_{t_1}^{t_2} f(t) \, dt\) and \(\int_{t_1}^{t_2} f^2(t) \, dt\) by the sums \(\sum_i f(\tau_i) \Delta \tau_i\) and \(\sum_i f^2(\tau_i) \Delta \tau_i\) and see that it suffices to set

\[
m_i = f(\tau_i) \Delta \tau_i \quad \text{and} \quad x_i = f(\tau_i)
\]

in (A.2) to obtain, passing to the limit as \(\Delta \tau_i \to 0\), the corresponding inequalities (A.4) for integrals.

**Example.** Let \(t_1 = 0, t_2 = 1, m_f = 1, M_f = 2\). Then it follows that

\[
1 \leq \frac{\int_{0}^{1} f^2(t) \, dt}{\left[ \int_{0}^{1} f(t) \, dt \right]^2} \leq \frac{9}{8}
\]

\(^{22}\)An analogous bound, but with a completely different proof, can be found in the book by G. Pólya and G. Szegő: *Aufgaben und Lehrsätze aus der Analysis I* (Berlin, 1925), p. 57, Problem No. 92. (See also Problem No. 93, in the same book, for the corresponding bound for integrals.) (Cf. pp. 57, 213–214 in [44] and pp. 71–72, 253–255 in [45].)
for any continuous function satisfying the condition $1 \leq f(t) \leq 2$ on the interval $0 \leq t \leq 1$.

We conclude by observing that there are no continuous functions $f(t)$ that yield equality on the right of (A.5), i.e.,

$$\frac{\int_0^1 f^2(t) \, dt}{\left[ \int_0^1 f(t) \, dt \right]^2} = \frac{9}{8};$$

but for any $\varepsilon > 0$ we can find a continuous function $f(t)$ such that

$$\frac{\int_0^1 f^2(t) \, dt}{\left[ \int_0^1 f(t) \, dt \right]^2} > \frac{9}{8} - \varepsilon.$$

We leave the proof to the reader.

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[Untitled Appendix]

The formula (A.2) admits a further generalization since for the choices in (A.4) we may substitute the more general

$$m_i = f(\tau_i) \Delta \tau_i, \quad x_i = g(\tau_i), \quad f(t), g(t) \text{ positive functions}.$$

Denoting by $m_g$ and $M_g$ numbers such that

$$m_g \leq g(t) \leq M_g \quad \text{for} \quad a \leq t \leq b,$$

the inequalities (A.2) become, by passing to the limit as $\Delta \tau_i \to 0$,

$$1 \leq \frac{\int_a^b f(t) g(t) \, dt \int_a^b f(t)/g(t) \, dt}{\left[ \int_a^b f(t) \, dt \right]^2} \leq \frac{(m_g + M_g)^2}{4m_g M_g}.$$  \hfill (A.6)

If in (A.6) we make the further substitutions

$$f(t) g(t) = F^2(t) \quad \text{and} \quad \frac{f(t)}{g(t)} = G^2(t),$$
then

\[ f(t) = F(t)G(t), \]

and the first inequality in (A.6) becomes the well-known Schwarz inequality

\[
\left( \int_a^b F(t)G(t) \, dt \right)^2 \leq \int_a^b F^2(t) \, dt \int_a^b G^2(t) \, dt.
\]

The upper bound given by the right-hand side of (A.6), which involves the maximum and minimum of the ratio \( F : G \), reduces to the upper bound found in the cited Problem No. 93 in the book by Pólya and Szegő when this maximum and minimum are replaced by the corresponding ratios:

\[
\max \frac{F}{G} \quad \text{and} \quad \min \frac{F}{G}.
\]  

\[ \text{Problem No. 21 (With solution by Abraham H. Bender, and with solution and generalization by Ernesto M. Saleme).}^{24} \quad \text{Given positive numbers } x_1, x_2, \ldots, x_n \text{ such that } \sum_{i=1}^n x_i = k, \text{ establish the inequalities [cf. (A.1) above]}
\]

\[
\sum_{i=1}^n x_i^2 \geq \frac{k^2}{n}, \quad \sum_{i=1}^n \frac{1}{x_i} \geq \frac{n^2}{k}. \]  

\[ \text{Solution No. 1 by Mr. Abraham H. Bender, student in the Faculty of Mathematical Sciences [Facultad de C. Matemáticas] of Rosario (Argentina).}
\]

\[ \text{Part I.} \quad \text{Since } (a - b)^2 \geq 0, \text{ we see that}
\]

\[
a^2 + b^2 \geq 2ab,
\]

\[ \text{Just signed "B. L." in the original paper.} \]

\[ \text{Mathematicae Notae: Boletín del Instituto de Matemática (Rosario) 2:35, 195–199 (1942).} \]

The original "Problema N° 21" was posed (anonymously, but presumably by Beppo Levi) on p. 35; the solutions [9, 47] appear on pp. 195–199.
and applying this inequality to all the terms $x_1, x_2, x_3, \ldots, x_n$, taken two by two, we have

\[
x_1^2 + x_2^2 \geq 2x_1x_2,
\]

\[
x_1^2 + x_3^2 \geq 2x_1x_3,
\]

\[\vdots\]

\[
x_{n-1}^2 + x_n^2 \geq 2x_{n-1}x_n,
\]

and adding, we obtain

\[
(n - 1)\left( x_1^2 + x_2^2 + \cdots + x_n^2 \right) \geq 2 \sum x_i x_j,
\]

(A.9)

where the sum on the right-hand side is taken over all combinations of $i, j$.

Adding the sum $x_1^2 + x_2^2 + \cdots + x_n^2$ to each side of (A.9) yields

\[
n\left( x_1^2 + x_2^2 + \cdots + x_n^2 \right) \geq x_1^2 + x_2^2 + \cdots + x_n^2 + 2 \sum x_i x_j.
\]

(A.10)

But the right-hand side of (A.10) is precisely the expansion of $(\sum x_i)^2 = k^2$ [recall the condition $\sum x_i = k$], which upon substitution yields [the first inequality in (A.8)]

\[
\sum_{i=1}^{n} x_i^2 \geq \frac{k^2}{n}.
\]

Part II. To establish the second inequality [in (A.8)], we expand

\[
k \sum_{i=1}^{n} \frac{1}{x_i} = (x_1 + x_2 + \cdots + x_n) \left( \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \right)
\]

\[
= \frac{x_1}{x_1} + \frac{x_1}{x_2} + \cdots + \frac{x_1}{x_n} + \frac{x_2}{x_1} + \cdots + \frac{x_2}{x_n} + \cdots + \frac{x_n}{x_1} + \cdots + \frac{x_n}{x_n}
\]

[we recall that the $x_i$ are required to satisfy the condition $\sum x_i = k$]. We now group the fractions with equal numerator and denominator, as well as the pairs of reciprocal fractions, to obtain

\[
k \sum_{i=1}^{n} \frac{1}{x_i} = \sum_{i=1}^{n} \frac{x_i}{x_i} + \sum_{i<j} \left( \frac{x_i}{x_j} + \frac{x_j}{x_i} \right),
\]

(A.11)
where the last sum is over the

\[ \binom{n}{2} = \frac{n(n-1)}{2} \]

combinations of the subscripts \( i < j \).

Dividing both sides of the inequality \( x_i^2 + x_j^2 \geq 2x_ix_j \) by \( x_ix_j \) yields

\[ \frac{x_i}{x_j} + \frac{x_j}{x_i} \geq 2, \]

and so from (A.11) we see that

\[ k \sum_{i=1}^{n} \frac{1}{x_i} \geq n + 2 \cdot \frac{n(n-1)}{2} = n^2, \]

and hence we have established [the second inequality in (A.8)]

\[ \sum_{i=1}^{n} \frac{1}{x_i} \geq \frac{n^2}{k}. \]

**Solution No. 2 and Generalization** by Engineer Ernesto M. Saleme of Tucumán (Argentina).

**Part I.** In a coordinate system we identify points \( x_i, y_i = x_i^2 \), which lie on the parabola \( y = x^2 \). Assuming that a unit weight is applied at each of these points, the coordinates of their center of gravity are

\[ X_g = \frac{\sum x_i}{n} = \frac{k}{n}, \quad Y_g = \frac{\sum x_i^2}{n}. \]

This center of gravity must lie in the interior of the parabola. Hence the horizontal line \( y = Y_g \) cuts the curve \( y = x^2 \) at a point with coordinate \( X' \geq X_g \). As a consequence \( Y_g = (X')^2 \geq X_g^2 \), from which we obtain [the first inequality in (A.8)]

\[ \sum x_i^2 \geq \left( \frac{k}{n} \right)^2 \cdot n = \frac{k^2}{n}. \]
Part II. Similarly, we identify points $x_i, y_i = 1/x_i$, which lie on the hyperbola $y = 1/x$. The coordinates of the barycenter are

$$X_g = \frac{k}{n}, \quad Y_g = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{x_i} = \frac{1}{X'}.$$ 

Given the form of the hyperbola $y = 1/x$, we must have $X' \leq X_g$, and as a consequence we have established [the second inequality in (A.8)]

$$\sum_{i=1}^{n} \frac{1}{x_i} \geq n \cdot \frac{1}{X_g} = \frac{n^2}{k}.$$ 

Generalization. The method used above allows us to establish inequalities of the same type when the terms in the sums are values of a monotonic function $f(n)$, whose curvature has constant sign (i.e., a concave or convex function).

Indeed, if $f(x_i)$ is increasing and concave downwards (convex towards the positive $y$-axis) the horizontal line $y = Y_g$ will cut the curve at the point with $x$-coordinate $X' \leq X_g$ and we will have

$$X_g = \frac{k}{n}, \quad Y_g = \frac{1}{n} \sum_{i=1}^{n} f(x_i) = f(X') \leq f(X_g)$$

and as a consequence

$$\sum_{i=1}^{n} f(x_i) \leq n \cdot f\left(\frac{k}{n}\right). \quad (A.12)$$

If, however, $f(x_i)$ is increasing and concave upwards (convex towards the positive $x$-axis) then similarly we have $X' \geq X_g$ and hence

$$Y_g = \frac{1}{n} \sum_{i=1}^{n} f(x_i) = f(X') \geq f(X_g)$$
from which we obtain

$$\sum_{i=1}^{n} f(x_i) \geq n \cdot f\left(\frac{k}{n}\right).$$

Applying this method, we can, for example, find an upper bound for the product $x_1 x_2 \cdots x_n$, where the $x_i$ satisfy the condition $\sum x_i = k$. Indeed, taking logarithms, we have

$$\log x_1 x_2 \cdots x_n = \sum_{i=1}^{n} \log x_i.$$

The curve $y = \log x$ is increasing and concave downwards, and hence, applying (A.12), we obtain

$$\sum_{i=1}^{n} \log x_i \leq n \log \frac{k}{n} = \log \left(\frac{k}{n}\right)^n.$$

Taking antilogarithms, we have

$$x_1 x_2 \cdots x_n \leq \left(\frac{k}{n}\right)^n. \quad (A.13)$$

As an example, since

$$n! = 2 \times 3 \times 4 \times \cdots \times n \quad \text{and} \quad 2 + 3 + 4 + \cdots + n = \frac{(n + 2)(n - 1)}{2},$$

applying (A.13) yields the inequality

$$n! \leq \left(\frac{(n + 2)(n - 1)}{2(n - 1)}\right)^{n-1} = \left(\frac{n + 2}{2}\right)^{n-1},$$

which, on the other hand, can be easily established by a direct argument.

---

25 In view of the condition $\sum x_i = k$, the inequality (A.13) is just a special case of the geometric-arithmetic-mean inequality; cf. footnote 10.
APPENDIX B. AN INEQUALITY ABOUT THE ARITHMETIC MEAN
by Pál Schweitzer (1914)\textsuperscript{36}

We will prove the following theorem: If any natural numbers fall between two positive bounds, then the product of the arithmetic mean of these numbers and the arithmetic mean of the reciprocals of these numbers cannot exceed the product of the arithmetic mean of the two bounds and the arithmetic mean of the reciprocals of the two bounds

\[ z = \frac{1}{n} (t_1 + \cdots + t_n) \cdot \frac{1}{n} \left( \frac{1}{t_1} + \cdots + \frac{1}{t_n} \right) \leq \frac{1}{2} (m + M) \cdot \frac{1}{2} \left( \frac{1}{m} + \frac{1}{M} \right), \]

where \( 0 < m \leq t_i \leq M \) \((i = 1, \ldots, n)\).

To prove (B.1), let us consider for the moment that all the \( t \)'s, with the exception of \( t_i \), are fixed, and find at which point \( t_i \) in the interval \((m, M)\) the function

\[ z = f(t_i) = \frac{1}{n^2} (t_i + A) \left( \frac{1}{t_i} + B \right) \]

attains its maximum value. Differentiating this function, we obtain

\[ \frac{dz}{dt_i} = \frac{1}{n^2} \left( B - \frac{A}{t_i^2} \right), \]

which vanishes only at \( t_i = \sqrt{A/B} \) and \( z \), therefore, has a minimum or maximum in the interval \((m, M)\) at either \( t_i = m \) or \( t_i = M \) according as

\[ \frac{1}{n^2} (M + A) \left( \frac{1}{M} + B \right) \geq \frac{1}{n^2} (m + A) \left( \frac{1}{m} + B \right) \]

\textsuperscript{36}In Hungarian: Egy egyenlőtlenség az aritmetikai középértékkről, \textit{Mathematikai és Physikai Lapok (Budapest)} 23:257–261 (1914). English translation by Levente T. Tolnai and Robert Vermes, with some slight editing by George P. H. Styan. (An earlier version of this translation appeared as Appendix A in \[1\].) We believe that Pál Schweitzer dies in 1941 (cf. Mitrović \[39, p. 394\]), but we do not know when he was born.
or equivalently

\[ A \cdot \frac{m + M}{mM} \geq B. \]

For any A and B, i.e., for any \( t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n \), if we want \( z \) to be at its maximum, then we choose the value of \( t_i \) as \( M \) or \( m \) according to

\[ A \cdot \frac{m + M}{mM} \geq B. \]

We can apply this argument to every \( t_i \) to obtain the maximum value of \( z \) as the following:

\[
Z_{\text{max}} = \frac{1}{n^2} \left[ aM + (n - a)m \right] \left[ \frac{a}{M} + \frac{n - a}{m} \right],
\]

where \( a \) and \( n - a \) count the numbers of \( t \)'s equal to \( M \) and \( m \), respectively. We may then write

\[
Z_{\text{max}} \leq \frac{1}{2}(m + M) \cdot \frac{1}{2} \left( \frac{1}{m} + \frac{1}{M} \right) - \frac{1}{n^2} \left( \frac{n}{2} - a \right)^2 \frac{(m - M)^2}{mM},
\]

or

\[
Z_{\text{max}} \leq \frac{1}{2}(m + M) \cdot \frac{1}{2} \left( \frac{1}{m} + \frac{1}{M} \right),
\]

which proves the required inequality (B.1). Equality holds in (B.1) if and only if \( n \) is even and \( a = n/2 \), i.e., an equal number of \( t \)'s are equal to \( m \) and to \( M \).

Our inequality (B.1) can be used to establish an upper bound for the integral of reciprocal functions. Consider the numbers \( t_1, \ldots, t_n \) as the values of the positive function \( t = f(x) \) corresponding to the equally spaced values \( x_1, \ldots, x_n \). The left-hand side of (B.1) then becomes

\[
\frac{1}{n} \left[ f(x_1) + f(x_2) + \cdots + f(x_n) \right] \cdot \frac{1}{n} \left[ \frac{1}{f(x_1)} + \frac{1}{f(x_2)} + \cdots + \frac{1}{f(x_n)} \right],
\]
which may be considered an approximation to

$$
\frac{1}{b-a} \int_a^b f(x) \, dx \cdot \frac{1}{b-a} \int_a^b \frac{dx}{f(x)}. \quad (B.2)
$$

If we now take limits in (B.2), we obtain

$$
\lim_{n \to \infty} \frac{1}{n^2} [ f(x_1) + \cdots + f(x_n) ] \left( \frac{1}{f(x_1)} + \cdots + \frac{1}{f(x_n)} \right) \leq \frac{1}{2} (m + M) \cdot \frac{1}{2} \left( \frac{1}{m} + \frac{1}{M} \right), \quad (B.3)
$$

where $m$ and $M$ denote a lower and upper bound, respectively, for $f(x)$. Replacing the terms on the left-hand side of (B.3) by integrals, we obtain

$$
\frac{1}{(b-a)^2} \int_a^b f(x) \, dx \cdot \int_a^b \frac{dx}{f(x)} \leq \frac{1}{2} (m + M) \cdot \frac{1}{2} \left( \frac{1}{m} + \frac{1}{M} \right) \quad (B.4)
$$

and hence

$$
\int_a^b \frac{dx}{f(x)} \leq \frac{(b-a)^2}{\int_a^b f(x) \, dx} \cdot \frac{1}{2} (m + M) \cdot \frac{1}{2} \left( \frac{1}{m} + \frac{1}{M} \right).
$$

If we also take into account that

$$
\int_a^b \frac{dx}{f(x)} \geq \frac{(b-a)^2}{\int_a^b f(x) \, dx},
$$

which comes from the Cauchy-Schwarz inequality

$$
\int_a^b \varphi^2 \, dx \cdot \int_a^b \psi^2 \, dx \geq \left( \int_a^b \varphi \psi \right)^2 \quad (\varphi > 0, \ \psi > 0)
$$

on setting $\varphi = \sqrt{f(x)}$ and $\psi = 1/\sqrt{f(x)}$, then we obtain

$$
\frac{(b-a)^2}{\int_a^b f(x) \, dx} \leq \int_a^b \frac{dx}{f(x)} \leq \frac{(b-a)^2}{\int_a^b f(x) \, dx} \cdot \frac{1}{2} (m + M) \cdot \frac{1}{2} \left( \frac{1}{m} + \frac{1}{M} \right).
$$
Since
\[
\frac{1}{2}(m + M) \cdot \frac{1}{2} \left( \frac{1}{m} + \frac{1}{M} \right) = \left( \frac{1}{2} \frac{(m + M)}{\sqrt{mM}} \right)^2
\]
is the square of the ratio of the arithmetic mean to the geometric mean of \( m \) and \( M \), it will be near 1 when \( M \) does not differ much from \( m \). Then the integral \( \int_a^b \frac{dx}{f(x)} \) will be squeezed between two tight bounds.

Our inequality (B.3) is also useful in approximating integrals of reciprocal functions. If we take the arithmetic mean of the lower and upper bounds,
\[
\frac{1}{2} \cdot \frac{(b - a)^2}{\int_a^b f(x) \, dx} \left( \frac{(m + M)^2}{4mM} + 1 \right),
\]
the error \( \delta \) we make is smaller in absolute value than half of the difference between the bounds, i.e.,
\[
|\delta| < \frac{(b - a)^2}{\int_a^b f(x) \, dx} \cdot \frac{1}{4} \cdot \left( \frac{m^2 + M^2}{2mM} - 1 \right). \tag{B.5}
\]

This inequality can be used to approximate logarithms. By setting \( f(x) = x \) in (B.5) and simplifying, we find by taking logarithms
\[
\log x + \frac{2}{2x + 1} \left( 1 + \frac{1}{8x(x + 1)} \right)
\]
instead of \( \log x + 1 \), while the absolute value of the error is
\[
|\delta| < \frac{1}{4x(x + 1)(2x + 1)}. \tag{B.5}
\]

If we calculate the logarithm in this way, the error we make starting at \( x = 10 \) is smaller than
\[
\frac{1}{4 \times 10 \times 11 \times 21} = \frac{1}{9240}.
\]
and starting at $x = 20$ is smaller than

$$\frac{1}{4 \times 20 \times 21 \times 41} = \frac{1}{68880}.$$  

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The original version [58] of this paper was by the first author (only) and was presented at the Fourth International Workshop on Matrix Methods for Statistics (Montréal, July 1995), and it was this presentation and the associated discussion that led to the M.Sc. thesis by the second author [1] and the current involvement of the third author and the papers [2] and [3]; this paper is based, in part, on [58] and [3], and on Chapter 2 of [1].

Much of the biographical information was obtained by visiting the excellent “MacTutor History of Mathematics Archive” web site: http://www-groups.dcs.st-and.ac.uk/history/Mathematicians (run by John J. O'Connor and Edmund F. Robertson in St. Andrews, Scotland, U.K.).

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\(^{27}\) English translation by Graciela Prieri of [18] and of [9] and [47] are combined as Appendix A of this paper.

\(^{28}\) Where biographies and pictures of Bouniakowsky (Bunyakovsky [sic]), Cauchy, Euclid, Hardy, Hermite, Kantorovich, Littlewood, Pólya, Pythagoras, Schwarz, Szegő et al., may be found (and downloaded); a biography of Kürschák is also available but, as of 14 February 1997, without a picture.
Harran University, ŞanlıUrfa, Turkey (to the second author), and by a research grant from the Natural Sciences and Engineering Research Council of Canada (to the third author).

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47 Ernesto M. Saleme, Problema N° 21.—Siendo \( x_1, x_2, x_3, \ldots, x_n \) números positivos con la condición \( \sum x_i = k \), demostrar las desigualdades \( \sum_{i=1}^{n} x_i^2 \geq k^2/n \), \( \sum_{i=1}^{n} 1/x_i \geq n^2/k \). 2° Solución y Generalización (in Spanish), *Math. Notae* 2:197–199 (1942). (Translated into English in Appendix A of this paper together with [9] and [18].)


51 Pál Schweitzer, Egy egyenlőtlenségl az aritmetikai középtértékrol (in Hungarian: "An inequality about the arithmetic mean"), *Math. Phys. Lapok (Budapest)* 23:257–261 (1914). (Translated into English as Appendix B of this paper and Appendix A of [1].)


53 Geoffrey Stuart Watson, Serial Correlation in Regression Analysis, Ph.D. Thesis, Dept. of Experimental Statistics, North Carolina State College, Raleigh; Univ. of North Carolina Mimeograph Ser., No. 49, 1951. (Includes an appendix by J. W. S. Cassels [13]. Published in part as [54] and [59].)


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