Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics Letters 20 (2007) 829–834
Applied Mathematics Letters
www.elsevier.com/locate/aml

Limiting behavior of a global attractor for lattice nonclassical parabolic equations[☆]

Caidi Zhao[a,*], Shengfan Zhou[b]

[a] College of Mathematics and Information Science, Wenzhou University, 325035, PR China
[b] Department of Applied Mathematics, Shanghai Normal University, Shanghai 200234, PR China

Received 6 April 2006; accepted 28 June 2006

Abstract

We prove the upper semicontinuity of the global attractor corresponding to a class of lattice nonclassical parabolic equations.

Keywords: Lattice systems; Nonclassical parabolic equations; Global attractor; Upper semicontinuity

1. Introduction

In this letter, we discuss the limiting behavior of the following lattice nonclassical parabolic equations as $\nu \to 0$:

$$\dot{u}_i + (2u_i - u_{i+1} - u_{i-1}) + \nu(2\dot{u}_i - \dot{u}_{i+1} - \dot{u}_{i-1}) + \lambda_i u_i + f_i(u_i) = g_i, \quad i \in \mathbf{Z},\ t > 0, \tag{1.1}$$

where $\nu \in [0, 1/8]$. Lattice systems (1.1) can be regarded as a discrete analogue in spatial variables to the following nonclassical parabolic equation on $\mathbf{R}$:

$$u_t - \Delta u - \nu \Delta u_t + \lambda(x)u + f(u, x) = g. \tag{1.2}$$

When $\nu = 0$, (1.1) reduces to the following lattice systems:

$$\dot{u}_i + (2u_i - u_{i+1} - u_{i-1}) + \lambda_i u_i + f_i(u_i) = g_i, \quad i \in \mathbf{Z},\ t > 0, \tag{1.3}$$

which can be regarded as a discrete analogue to the following parabolic equation on $\mathbf{R}$:

$$u_t - \Delta u + \lambda(x)u + f(u, x) = g. \tag{1.4}$$

For $t = 0$, we specify the initial data

$$u_i(0) = u_{i,0}, \quad i \in \mathbf{Z}. \tag{1.5}$$

☆ This work was supported by the National Natural Science Foundation of China under Grants 10171072 and 10471086, and was supported by ZheJiang Province Natural Science Foundation under Grant M103043.

* Corresponding author.
E-mail address: zhaocaidi@yahoo.com.cn (C. Zhao).

0893-9659/$ - see front matter
doi:10.1016/j.aml.2006.06.019

When λ_i and f_i are independent of i, Bates et al. [1] studied the existence and upper semicontinuity of the global attractor for lattice systems (1.3) and (1.5). Later, Zhou [4] generalized the results of [1] to general lattice systems.

The main goal of this letter is to prove that $\mathcal{A}_\nu \to \mathcal{A}_0$ in the sense of the Hausdorff semidistance in ℓ^2 as $\nu \to 0$, where $\mathcal{A}_\nu$ and $\mathcal{A}_0$ are the global attractors corresponding to lattice systems (1.1) and (1.5), (1.3) and (1.5), respectively. For related research, one can refer to [3] for the singular limiting behavior of the global attractor for lattice FitzHugh–Nagumo systems.

2. Setting of the problem and preliminaries

Set

$$\ell^2 = \left\{ u = (u_i)_{i\in\mathbf{Z}} \mid u_i \in \mathbf{R},\ \sum_{i\in\mathbf{Z}} u_i^2 < +\infty \right\}, \tag{2.1}$$

and equip it with the following inner product and norm:

$$(u, v) = \sum_{i\in\mathbf{Z}} u_i v_i, \quad \|u\|^2 = (u, u),\ \forall u = (u_i)_{i\in\mathbf{Z}},\ v = (v_i)_{i\in\mathbf{Z}} \in \ell^2.$$

Obviously, $\ell^2 = (\ell^2, (\cdot,\cdot), \|\cdot\|)$ is a Hilbert space. Define some linear operators on ℓ^2 as follows:

$$(Au)_i = 2u_i - u_{i+1} - u_{i-1}, \qquad (Bu)_i = u_{i+1} - u_i, \qquad (B^*u)_i = u_{i-1} - u_i, \quad u = (u_i)_{i\in\mathbf{Z}} \in \ell^2.$$

Then B^* is the adjoint operator of B, and

$$A = B^*B = BB^*, \qquad A^* = A, \tag{2.2}$$

where A^* is the adjoint operator of A.

We now make some assumptions on functions f_i, λ_i (see also in [4]).

($\mathbf{A_1}$) $f_i(0) = 0,\ f_i(u_i)u_i \geq 0,\ \forall\, i \in \mathbf{Z},\ \forall\, u_i \in \mathbf{R}$.

($\mathbf{A_2}$) There exists a continuous function $\alpha(r) : \mathbf{R}_+ \longmapsto \mathbf{R}_+$ such that

$$\sup_{i\in\mathbf{Z}} \max_{u_i\in[-r,r]} |f_i'(u_i)| \leq \alpha(r), \quad \forall r \in \mathbf{R}_+.$$

($\mathbf{A_3}$) There exist two positive constants λ_0 and $\hat{\lambda}_0$ such that

$$0 < \lambda_0 \leq \lambda_i \leq \hat{\lambda}_0 < +\infty, \quad \forall i \in \mathbf{Z}.$$

With assumptions ($\mathbf{A_1}$)–($\mathbf{A_2}$), [4] showed that f is locally Lipschitz continuous from ℓ^2 to ℓ^2:

$$\|f(u) - f(v)\|^2 \leq \alpha^2(r)\|u - v\|^2, \quad \forall u,\ v \in \ell^2 \text{ with } \|u\|,\ \|v\| \leq r. \tag{2.3}$$

Problem (1.3) and (1.5) can be expressed as the following first-order lattice system with initial data:

$$\dot{u} + Au + \lambda u + f(u) = g,\ t > 0, \qquad u(0) = (u_{i,0})_{i\in\mathbf{Z}} = u_0, \tag{2.4}$$

where $u = (u_i)_{i\in\mathbf{Z}}$, $Au = ((Au)_i)_{i\in\mathbf{Z}}$, $\lambda u = (\lambda_i u_i)_{i\in\mathbf{Z}}$, $f(u) = (f_i(u_i))_{i\in\mathbf{Z}}$, $g = (g_i)_{i\in\mathbf{Z}}$.

Proposition 2.1 (*[4]*). *If* ($\mathbf{A_1}$)–($\mathbf{A_3}$) *hold and* $g \in \ell^2$*, then problem* (2.4) *possesses a unique solution* $u \in C([0, +\infty), \ell^2) \cap C^1((0, +\infty), \ell^2)$ *and the solution operators*

$$S_0(t) : u_0 \in \ell^2 \longmapsto S_0(t)u_0 = u(t) \in \ell^2, \quad t \geq 0, \tag{2.5}$$

form a continuous semigroup $\{S_0(t)\}_{t\geq 0}$ *on* ℓ^2*. Moreover,* $\{S_0(t)\}_{t\geq 0}$ *has a global attractor* $\mathcal{A}_0$ *in* ℓ^2*.*

3. Existence of a global attractor

In this section, we verify the existence of a global attractor for the semigroup $\{S_\nu(t)\}_{t\geq 0}$ corresponding to problem (1.1) and (1.5).

Lemma 3.1. *There exists $(I+\nu A)^{-1} \in \mathcal{L}(\ell^2)$ such that $(I+\nu A)^{-1}(I+\nu A) = I$, where I is the identity operator on ℓ^2 and $\mathcal{L}(\ell^2)$ is the set of bounded linear operators from ℓ^2 to ℓ^2. Moreover, $(I+\nu A)$ and $(I+\nu A)^{-1}$ are both positive and self-adjoint operators on ℓ^2.*

Proof. By the definition of A,

$$\|Au\|^2 = \sum_{i\in\mathbf{Z}} |2u_i - u_{i+1} - u_{i-1}|^2 \leq 16\|u\|^2, \quad \forall u \in \ell^2.$$

Hence $\|A\| \leq 4$, where $\|A\|$ is the norm of the operator in the set of linear operators from ℓ^2 to ℓ^2, i.e., $A \in \mathcal{L}(\ell^2)$. Thus $I+\nu A \in \mathcal{L}(\ell^2)$. At the same time, one can easily check that $I+\nu A : \ell^2 \longmapsto \ell^2$ is one-to-one and onto. By Banach's theorem, there exists $(I+\nu A)^{-1} \in \mathcal{L}(\ell^2)$ such that $(I+\nu A)^{-1}(I+\nu A) = (I+\nu A)(I+\nu A)^{-1} = I$. Now by (2.2), for any $u \in \ell^2$, we have

$$(I+\nu A)^* = I^* + \nu A^* = I+\nu A, \qquad ((I+\nu A)u, u) = \|u\|^2 + \nu\|Bu\|^2 \geq 0, \quad \forall u \in \ell^2.$$

Thus, $I+\nu A$ is a positive and self-adjoint operator on ℓ^2. Clearly, $(I+\nu A)^{-1}$ is also a positive and self-adjoint operator on ℓ^2. The proof is completed. □

By Lemma 3.1 and classical theory of Functional Analysis (see, e.g., [2]), there exists a unique positive and self-adjoint operator $D \in \mathcal{L}(\ell^2)$ such that

$$D^2 = (I+\nu A)^{-1}. \tag{3.1}$$

Moreover, we have

$$\|D\|^2 \geq \|D^2\| = \|(I+\nu A)^{-1}\| = \frac{1}{\|I+\nu A\|} \geq \frac{1}{1+\nu\|A\|} \geq \frac{1}{1+4\nu}, \tag{3.2}$$

$$\|Du\|^2 = (D^2u, u) = ((I+\nu A)^{-1}u, u) \leq \|(I+\nu A)^{-1}\|\|u\|^2 \leq \frac{\|u\|^2}{1-4\nu}, \quad \forall u \in \ell^2. \tag{3.3}$$

(3.2) and (3.3) imply that D is invertible and its inverse operator D^{-1} satisfies $\|D^{-1}\|^2 = \frac{1}{\|D\|^2} \leq 1+4\nu$ and thus we have

$$\|Du\|^2 \geq \frac{\|u\|^2}{\|D^{-1}\|^2} \geq \frac{\|u\|^2}{1+4\nu}, \quad \forall u \in \ell^2. \tag{3.4}$$

Since $\nu \in [0, 1/8]$, we conclude from (3.3) and (3.4) that

$$\frac{2}{3}\|u\|^2 \leq \frac{1}{1+4\nu}\|u\|^2 \leq \|Du\|^2 \leq \frac{1}{1-4\nu}\|u\|^2 \leq 2\|u\|^2, \quad \forall u \in \ell^2. \tag{3.5}$$

We now make another assumption on the function f.

($\mathbf{A_4}$) $(Df(u), Du) \geq 0, \forall u \in \ell^2$, where D is the operator defined by (3.1).

Using Lemma 3.1, we can put problem (1.1) and (1.5) into the following first-order lattice system with initial condition:

$$\dot{u} + (I+\nu A)^{-1}Au + (I+\nu A)^{-1}\lambda u + (I+\nu A)^{-1}f(u) = (I+\nu A)^{-1}g, \quad t > 0, \tag{3.6}$$

$$u(0) = (u_{i,0})_{i\in\mathbf{Z}} = u_0. \tag{3.7}$$

Lemma 3.2. *If* $(\mathbf{A_1})$–$(\mathbf{A_4})$ *hold and* $g \in \ell^2$*, then for any* $\nu \in [0, \frac{1}{8}]$*, problem* (3.6) *and* (3.7) *possesses a unique solution* $u \in C([0, +\infty), \ell^2) \cap C^1((0, +\infty), \ell^2)$ *and the solution operators*

$$S_\nu(t) : u_0 \in \ell^2 \longmapsto S_\nu(t)u_0 = u(t) \in \ell^2, \quad t \geq 0, \tag{3.8}$$

form a continuous semigroup $\{S_\nu(t)\}_{t\geq 0}$ *on* ℓ^2*. Moreover,* $\{S_\nu(t)\}_{t\geq 0}$ *has a uniform (with respect to* ν*) bounded absorbing set* B_0 *and a global attractor* $\mathcal{A}_\nu \subset B_0 \subset \ell^2$*.*

Proof. Here we only verify the existence of a uniform (with respect to ν) bounded absorbing set. The rest of the proof is similar to that in [4].

Let $u(t) \in \ell^2$ be a solution of (3.6) and (3.7). Taking the inner product $(\cdot, \cdot)$ of (3.6) with u, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + ((I + \nu A)^{-1}Au, u) + ((I + \nu A)^{-1}\lambda u, u) + ((I + \nu A)^{-1}f(u), u) = ((I + \nu A)^{-1}g, u). \tag{3.9}$$

By (2.2) and (3.1) and $(\mathbf{A_4})$,

$$((I + \nu A)^{-1}Au, u) = \|DBu\|^2 \geq 0, \tag{3.10}$$

$$((I + \nu A)^{-1}\lambda u, u) = (\lambda Du, Du) = \sum_{i\in\mathbf{Z}} \lambda_i (Du)_i (Du)_i \geq \lambda_0\|Du\|^2, \tag{3.11}$$

$$((I + \nu A)^{-1}f(u), u) = (Df(u), Du) \geq 0, \tag{3.12}$$

$$((I + \nu A)^{-1}g, u) = (Dg, Du) \leq \frac{1}{2\lambda_0}\|Dg\|^2 + \frac{\lambda_0}{2}\|Du\|^2. \tag{3.13}$$

Taking (3.5) and (3.9)–(3.13) into account, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \frac{2\lambda_0}{3}\|u\|^2 \leq \frac{2}{\lambda_0}\|g\|^2. \tag{3.14}$$

Applying Gronwall's inequality to (3.14), we obtain

$$\|u(t)\|^2 \leq \|u_0\|^2 \mathrm{e}^{-\frac{2}{3}\lambda_0 t} + \frac{3}{\lambda_0^2}\|g\|^2, \quad \forall t \geq 0, \tag{3.15}$$

which implies that $\{S_\nu(t)\}_{t\geq 0}$ possesses a uniform (with respect to ν) bounded absorbing set B_0 in ℓ^2, where B_0 is a ball centered at 0 with radius $r_0 = \frac{2}{\lambda_0}\|g\|$. □

4. Upper semicontinuity of the global attractor

To study the limiting behavior of lattice systems (1.1) and (1.5) as $\nu \to 0$, one important step is to consider the continuous dependence of solutions as $\nu \to 0$, which also makes independent sense.

Lemma 4.1. *Let* $(\mathbf{A_1})$–$(\mathbf{A_4})$ *hold and* $g \in \ell^2$*; then for any* $\nu \in [0, 1/8]$ *and any* $R, T > 0$ *given, there exists a positive constant* $C = C(R, T, \lambda_0, \hat{\lambda}_0, \|g\|)$ *such that*

$$\|S_\nu(t)u_0 - S_0(t)u_0\| \leq C\nu, \quad \forall t \in [0, T] \text{ and } \|u_0\| \leq R.$$

Proof. Let $u_0 \in \ell^2$ with $\|u_0\| \leq R$. Set $u = S_\nu(t)u_0$ and $v = S_0(t)u_0$. Then $w = u - v = S_\nu(t)u_0 - S_0(t)u_0$ is a solution of the following problem:

$$\dot{w} + Aw + \nu A\dot{u} + \lambda w + f(u) - f(v) = 0, \quad t > 0, \tag{4.1}$$

$$w(0) = 0. \tag{4.2}$$

Taking the inner product $(\cdot, \cdot)$ of (4.1) with $\dot{w}$, we obtain

$$\|\dot{w}\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Bw\|^2 + \nu(B\dot{u}, B\dot{w}) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i\in\mathbf{Z}} \lambda_i w_i^2 + (f(u) - f(v), \dot{w}) = 0. \tag{4.3}$$

Clearly, we have $\|B\| \leq 2$ and thus

$$\nu(B\dot{u}, B\dot{w}) \leq \nu\|B\dot{u}\|\,\|B\dot{w}\| \leq 4\nu\|\dot{u}\|\,\|\dot{w}\| \leq \frac{1}{2}\|\dot{w}\|^2 + 8\nu^2\|\dot{u}\|^2. \tag{4.4}$$

By (2.3), (3.5), (3.6) and (3.15),

$$\begin{aligned}\|\dot{u}\| &= \|(I+\nu A)^{-1}(Au + \lambda u + f(u) - g)\| \leq 2(\|Au\| + \|\lambda u\| + \|f(u)\| + \|g\|)\\ &\leq 2(4\|u\| + \hat{\lambda}_0\|u\| + \alpha(R)\|u\| + \|g\|) \leq 2(4 + \hat{\lambda}_0 + \alpha(R))\|u\| + 2\|g\|\\ &\leq 2(4 + \hat{\lambda}_0 + \alpha(R))\left(R^2 + \frac{3}{\lambda_0^2}\|g\|^2\right)^{\frac{1}{2}} + 2\|g\|\\ &\doteq C_1(R, \lambda_0, \hat{\lambda}_0, \|g\|), \quad \forall t \geq 0.\end{aligned} \tag{4.5}$$

Also by (2.3),

$$\begin{aligned}(f(u) - f(v), \dot{w}) \leq \|f(u) - f(v)\|\,\|\dot{w}\| &\leq \frac{1}{2}\|\dot{w}\|^2 + \frac{1}{2}\|f(u) - f(v)\|^2\\ &\leq \frac{1}{2}\|\dot{w}\|^2 + \frac{\alpha^2(R)}{2}\|w\|^2.\end{aligned} \tag{4.6}$$

It then follows from (4.3)–(4.6) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|Bw\|^2 + \sum_{i\in\mathbf{Z}}\lambda_i w_i^2\right) \leq \frac{\alpha^2(R)}{2\lambda_0}\left(\sum_{i\in\mathbf{Z}}\lambda_i w_i^2 + \|Bw\|^2\right) + 8C_1^2(R, \lambda_0, \hat{\lambda}_0, \|g\|)\nu^2. \tag{4.7}$$

Applying Gronwall's inequality to (4.7), we obtain

$$\begin{aligned}\|Bw\|^2 + \sum_{i\in\mathbf{Z}}\lambda_i w_i^2 &\leq \left(\|Bw(0)\|^2 + \sum_{i\in\mathbf{Z}}\lambda_i w_i^2(0)\right)\mathrm{e}^{\frac{\alpha^2(R)}{2\lambda_0}t} + 8C_1^2\nu^2\int_0^t \mathrm{e}^{\frac{\alpha^2(R)}{2\lambda_0}(t-s)}\mathrm{d}s\\ &= 8C_1^2(R, \lambda_0, \hat{\lambda}_0, \|g\|)\nu^2\int_0^t \mathrm{e}^{\frac{\alpha^2(R)}{2\lambda_0}(t-s)}\mathrm{d}s\\ &\doteq C_2(R, T, \lambda_0, \hat{\lambda}_0, \|g\|)\nu^2, \quad \forall t \in [0, T],\end{aligned}$$

from which we get $\|w\| \leq \sqrt{\frac{C_2(R,T,\lambda_0,\hat{\lambda}_0,\|g\|)}{\lambda_0}}\,\nu \doteq C(R, T, \lambda_0, \hat{\lambda}_0, \|g\|)\nu$. □

Theorem 4.1. *Let* $(\mathbf{A_1})$–$(\mathbf{A_4})$ *hold and* $g \in \ell^2$*; then the global attractor* $\mathcal{A}_\nu$ *of* $\{S_\nu(t)\}_{t\geq 0}$ *corresponding to lattice systems (1.1) and (1.5) is upper semicontinuous at* $\nu = 0$ *in the following sense:*

$$d_{\ell^2}(\mathcal{A}_\nu, \mathcal{A}_0) = 0 \quad \textit{as } \nu \to 0, \tag{4.8}$$

where $d_{\ell^2}(X, Y) = \sup_{x\in X}\inf_{y\in Y}\|x - y\|$ *is the Hausdorff semidistance from* $X \subset \ell^2$ *to* $Y \subset \ell^2$.

Proof. On the one hand, for any $\nu \in [0, 1/8]$, $\mathcal{A}_\nu$ is uniform (w.r.t. ν) bounded in ℓ^2. On the other hand, $\mathcal{A}_0$ attracts any bounded set of ℓ^2. Thus for any $\varepsilon > 0$, there exists some $T > 0$ such that

$$d_{\ell^2}(S_0(T)\mathcal{A}_\nu, \mathcal{A}_0) < \frac{\varepsilon}{2}, \quad \forall \nu \in [0, 1/8]. \tag{4.9}$$

At the same time, for above $T > 0$, Lemma 3.2 and Lemma 4.1 show that there exists some $C = C(r_0, T, \lambda_0, \hat{\lambda}_0, \|g\|)$ such that

$$d_{\ell^2}(\mathcal{A}_\nu, S_0(T)\mathcal{A}_\nu) = d_{\ell^2}(S_\nu(T)\mathcal{A}_\nu, S_0(T)\mathcal{A}_\nu) < C\nu, \quad \forall \nu \in [0, 1/8]. \tag{4.10}$$

It then follows from (4.9) and (4.10) that

$$d_{\ell^2}(\mathcal{A}_\nu, \mathcal{A}_0) \leq d_{\ell^2}(\mathcal{A}_\nu, S_0(T)\mathcal{A}_\nu) + d_{\ell^2}(S_0(T)\mathcal{A}_\nu, \mathcal{A}_0)$$

$$< d_{\ell^2}(S_\nu(T)\mathcal{A}_\nu, S_0(T)\mathcal{A}_\nu) + \frac{\varepsilon}{2}$$
$$< C\nu + \frac{\varepsilon}{2}, \quad \forall \nu \in [0, 1/8].$$

Choose $\nu_0 = \min\{\varepsilon/2C, 1/8\}$ and we get

$$d_{\ell^2}(\mathcal{A}_\nu, \mathcal{A}_0) < \varepsilon, \quad \forall \nu \in [0, \nu_0].$$

The proof is completed. □

References

[1] P.W. Bates, K. Lu, B. Wang, Attractors for lattice dynamical systems, Internat. J. Bifur. Choas 11 (1) (2001) 143–153.
[2] W. Rudin, Functional Analysis, 2nd ed., McGraw-Hill Company, 1991.
[3] E.V. Vlecka, B. Wang, Attractors for lattice FitzHugh–Nagumo systems, Physica D 212 (2005) 317–336.
[4] S. Zhou, Attractors for first order dissipative lattice dynamical systems, Physica D 178 (2003) 51–61.