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Idempotency of linear combinations of an idempotent matrix and a t-potent matrix that commute

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Abstract

This paper deals with idempotent matrices (i.e., $A^2 = A$) and t-potent matrices (i.e., $B^t = B$). When both matrices commute, we derive a list of all complex numbers c_1 and c_2 such that $c_1A + c_2B$ is an idempotent matrix. In addition, the real case is also analyzed.

Keywords: Idempotent matrix; t-potent matrix; Linear combination

We consider the following problem: to describe all pairs (c_1, c_2) of nonzero complex numbers for which there exist an idempotent complex matrix A (i.e., $A^2 = A$) and a t-potent complex matrix B (i.e., $B^t = B$) such that their linear combination $c_1A + c_2B$ is an idempotent matrix. This problem was studied in [1] and [2] for $t = 2$ and $t = 3$, respectively. We solve it for all $t > 1$, but only if A and B commute. We suppose that B has at least two distinct nonzero eigenvalues since otherwise $B = \lambda P$, where $P^2 = P$, that is, $c_1A + c_2B = c_1A + c_2\lambda P$ is a linear combination of idempotent matrices studied in [1].

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Theorem 1. *Let A and B be nonzero complex matrices and* $c_1A + c_2B = C$ *satisfy* $A^2 = A$, $B^{k+1} = B$, $AB = BA$, *and* $C^2 = C$. *Assume that A and B are not simultaneously similar to* $A' \oplus 0$ *and* $B' \oplus 0$, *respectively. Also assume that B has at least two distinct nonzero eigenvalues and* $c_1 \neq 0 \neq c_2$. *Then there is a nonsingular matrix S such that* $c_1S^{-1}AS + c_2S^{-1}BS = S^{-1}CS$ *is one of the following linear combinations, in which* $u, v \in \sqrt[k]{1}$, $u \neq v$, *and* $\varepsilon = \frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}$.

$$\frac{v}{v-u}\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \frac{1}{u-v}\begin{bmatrix} vI & 0 \\ 0 & uI \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix},$$

$$-uv^{-1}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + v^{-1}\begin{bmatrix} uI & 0 \\ 0 & vI \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix},$$

$$(1-uv^{-1})\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + v^{-1}\begin{bmatrix} uI & 0 \\ 0 & vI \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} + v^{-1}\begin{bmatrix} -vI & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & vI \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \quad \text{if } 2|k,$$

$$\varepsilon\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} + \varepsilon^{-1}u^{-1}\begin{bmatrix} \varepsilon^{-1}uI & 0 & 0 \\ 0 & uI & 0 \\ 0 & 0 & \varepsilon uI \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \quad \text{if } 6|k.$$

Proof. By simultaneous similarity transformations with A and B, we make $A = I_r \oplus 0$. Then $B = B_1 \oplus B_2$ since $AB = BA$. Both B_1 and B_2 are diagonalizable because $B^{k+1} = B$. By simultaneous similarity transformations with A and B that preserve $A = I_r \oplus 0$, we make $B = \mathrm{diag}(\beta_1, \dots, \beta_n)$, where all $\beta_i^{k+1} = \beta_i$, that is, $\beta_i \in \sqrt[k]{1}$ or $\beta_i = 0$. Since A and B are diagonal, $C = \mathrm{diag}(\gamma_1, \dots, \gamma_n)$ and $C^2 = C$ implies $\gamma_1, \dots, \gamma_n \in \{0, 1\}$.

Therefore,

$$\begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} c_1 + \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \\ \beta_{r+1} \\ \vdots \\ \beta_n \end{bmatrix} c_2 = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_r \\ \gamma_{r+1} \\ \vdots \\ \gamma_n \end{bmatrix}, \quad \begin{matrix} \beta_i \in \{0\} \cup \sqrt[k]{1}, \\ \gamma_i \in \{0, 1\}. \end{matrix} \tag{1}$$

Because $\beta_{r+1}, \dots, \beta_n$ and c_2 are all nonzero, $\gamma_{r+1} = \cdots = \gamma_n = 1$.

We can consider (1) as a system of linear equations with respect to c_1 and c_2. Since this system is solvable, there are at most two linearly independent equations. The number of independent equations equals 2 since $A \neq 0$ and B has at least two nonzero eigenvalues. Let us fix two linearly independent equations.

Case 1: The linearly independent equations have the form

$$c_1 + c_2u = 0, \quad c_1 + c_2v = 0 \quad (u \neq v).$$

Then $c_2 = 0$ and this case is impossible.

Case 2: The linearly independent equations have the form

$$c_1 + c_2 u = 1, \quad c_1 + c_2 v = 0.$$

Then $c_1 = \frac{v}{v-u}$ and $c_2 = \frac{1}{u-v}$. From (1) we get $\frac{v}{v-u} + \frac{\beta_i}{u-v} \in \{0, 1\}$ for all $i = 1, \ldots, r$ and $\frac{\beta_j}{u-v} = 1$ for all $j = r+1, \ldots, n$. Hence $\beta_i = v$ or $\beta_i = u$ for all $i = 1, \ldots, r$ and $\beta_j = u - v$ for all $j = r+1, \ldots, n$.

If $r = n$, rearranging if necessary the eigenvalues of B, we get

$$A = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad B = \begin{bmatrix} vI & 0 \\ 0 & uI \end{bmatrix}.$$

If $r < n$, then $\beta_j = \varepsilon u$, $v = \varepsilon^{-1} u$, and $6|k$ because $u = \beta_j + v$ and $u, \beta_j, v \in \sqrt[k]{1}$. Rearranging if necessary the eigenvalues of B, we get

$$A = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \varepsilon^{-1} uI & 0 & 0 \\ 0 & uI & 0 \\ 0 & 0 & \varepsilon uI \end{bmatrix}.$$

Case 3: The linearly independent equations have the form

$$c_1 + c_2 u = 1, \quad c_1 + c_2 v = 1 \qquad (u \neq v).$$

Then $c_2 = 0$ and this case is impossible.

Case 4: The linearly independent equations have the form

$$c_1 + c_2 u = 0, \quad c_2 v = 1.$$

Then $c_1 = -uv^{-1}$ and $c_2 = v^{-1}$. From (1) we get $-uv^{-1} + v^{-1}\beta_i \in \{0, 1\}$ for all $i = 1, \ldots, r$ and $v^{-1}\beta_j = 1$ for all $j = r+1, \ldots, n$. Hence $\beta_i = u$ or $\beta_i = u + v$ for all $i = 1, \ldots, r$ and $\beta_j = v$ for all $j = r+1, \ldots, n$.

If $\beta_i = u$ for all $i = 1, \ldots, r$, rearranging if necessary the eigenvalues of B, we get

$$A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} uI & 0 \\ 0 & vI \end{bmatrix}.$$

Suppose that there exists $i \in \{1, \ldots, r\}$ such that $\beta_i = u + v$. If $\beta_i \neq 0$, then $u = \varepsilon^{-1}\beta_i$, $v = \varepsilon\beta_i$, and $6|k$ since $\beta_i, u, v \in \sqrt[k]{1}$. Rearranging if necessary the eigenvalues of B, we get

$$A = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \varepsilon^{-1}\beta_i I & 0 & 0 \\ 0 & \beta_i I & 0 \\ 0 & 0 & \varepsilon\beta_i I \end{bmatrix},$$

and $c_1 = -uv^{-1} = \varepsilon$, $c_2 = v^{-1} = \varepsilon^{-1}\beta_i^{-1}$. If $\beta_i = 0$, then $u = -v$, $c_1 = 1$, $c_2 = v^{-1}$, and k is even. Rearranging if necessary the eigenvalues of B, we get

$$A = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -vI & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & vI \end{bmatrix}.$$

Case 5: The linearly independent equations have the form

$$c_1 + c_2 u = 1, \quad c_2 v = 1.$$

Then $c_1 = 1 - uv^{-1}$ and $c_2 = v^{-1}$. From (1) we get $1 - uv^{-1} + v^{-1}\beta_i \in \{0, 1\}$ for all $i = 1, \ldots, r$ and $v^{-1}\beta_j = 1$ for all $j = r + 1, \ldots, n$. Hence $\beta_i = u$ or $\beta_i = u - v$ for all $i = 1, \ldots, r$ and $\beta_j = v$ for all $j = r + 1, \ldots, n$.

If $\beta_i = u$ for all $i = 1, \ldots, r$, rearranging if necessary the eigenvalues of B, we get

$$A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} uI & 0 \\ 0 & vI \end{bmatrix}.$$

If there exists $i \in \{1, \ldots, r\}$ such that $\beta_i = u - v$, then $\beta_i = \varepsilon^{-1}u$, $v = \varepsilon u$, and $6|k$ because $u, \beta_i, v \in \sqrt[k]{1}$. Rearranging if necessary the eigenvalues of B, we get

$$A = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \varepsilon^{-1}uI & 0 & 0 \\ 0 & uI & 0 \\ 0 & 0 & \varepsilon uI \end{bmatrix},$$

and $c_1 = 1 - uv^{-1} = \varepsilon$, $c_2 = v^{-1} = \varepsilon^{-1}u^{-1}$.

Case 6: The linearly independent equations have the form

$$c_2 u = 1, \quad c_2 v = 1 \qquad (u \neq v).$$

This system is unsolvable.

This completes the proof. □

It seems interesting to show that k must be less than 3 when c_1 and c_2 are restricted to be real numbers.

Corollary 1. *Let c_1 and c_2 be nonzero real numbers. Let A and B be nonzero complex matrices and $c_1 A + c_2 B = C$ satisfy $A^2 = A$, $B^{k+1} = B$, $AB = BA$, $A \neq B$, and $C^2 = C$. Then $B^2 = B$ or $B^3 = B$.*

Proof. The case $k = 1$ was studied in [1] and it yields $B^2 = B$. So, we suppose $k > 1$. From Theorem 1, $c_1 = \overline{c_1}$, and $c_2 = \overline{c_2}$ we get $\{u, v\} = \{1, -1\}$. Hence, we deduce that $B^3 = B$ since B is diagonalizable. □

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