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A quick proof that $K_{10} \neq P + P + P$

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There are a number of proofs of the fact that the complete graph on 10 vertices, K_{10}, cannot be factored into three copies of the Petersen Graph, P. These are typically based on certain symmetries of the graph or properties of its eigenvalues. The following short proof of this novelty is based on the existence of strongly independent edges in P (such edges are mutually at distance at least two).

Note in Fig. 1(i) that the edges 12, 34 and 56 of P are strongly independent. Suppose that $K_{10} = P_1 + P_2 + P_3$ where the edges of the copies, P_i, of P are coloured red, blue and green respectively. Let $H = K_{10} - P_3$. Then H must be factorable as a red and a blue copy of P. H contains an induced $K_{2,2,2}$ on the vertices 1, 2, 3, 4, 5 and 6 as shown in Fig. 1(ii). Since the Petersen graph has girth five, the edges of this $K_{2,2,2}$ must be coloured red and blue so as to avoid a monochromatic 3 or 4-cycle. Suppose vertex 1 has at most one incident blue edge say 16, then 13, 14, 15 are red, 35 and 45 are blue but now no two of 23, 24 and 25 can be the same colour without creating a monochromatic 3-cycle or 4-cycle. This implies that a correct colouring consists of a red 6-cycle and a blue 6-cycle. A representative of the six such colourings (some three consecutive edges of the

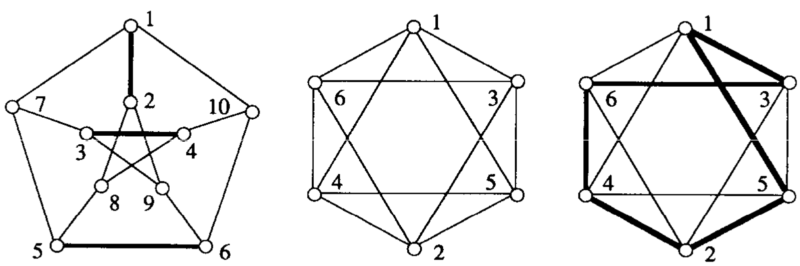

Fig. 1. (i) The Petersen Graph. (ii) Induced $K_{2,2,2}$ of H. (iii) A colouring.

outer 6 cycle must be either red or blue) is shown in Fig. 1(iii) where the dark edges are red. Consider vertex 7 and the triangle on the vertices 2, 4 and 6. We must have that 27 and 47 are blue while 67 is red. Thus 18 and 38 are blue while 68 is red. Now either colouring of the edge 78 results in a monochromatic 3 or 4-cycle and we have a final contradiction.