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Existence of Nonoscillatory Solution of Second Order Linear Neutral Delay Equation

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Consider the neutral delay differential equation with positive and negative coefficients,

$$\frac{d^2}{dt^2}[x(t) + px(t-\tau)] + Q_1(t)x(t-\sigma_1) - Q_2(t)x(t-\sigma_2) = 0,$$

where $p \in R$ and

$$\tau \in (0,\infty),\ \sigma_1, \sigma_2 \in [0,\infty) \quad \text{and} \quad Q_1, Q_2 \in C([t_0,\infty), R^+).$$

Some sufficient conditions for the existence of a nonoscillatory solution of the above equation expressed in terms of $\int^\infty sQ_i(s)\,ds < \infty$, $i = 1, 2$, and certain technical conditions implying that $Q_1(s)$ dominates $Q_2(s)$ are obtained for values of $p \neq \pm 1$. © 1998 Academic Press

1. INTRODUCTION

Consider the neutral delay differential equation of second order with positive and negative coefficients,

$$\frac{d^2}{dt^2}\big[x(t) + px(t-\tau)\big] + Q_1(t)x(t-\sigma_1) - Q_2(t)x(t-\sigma_2) = 0, \tag{1}$$

where $p \in R$ and

$$\tau \in (0,\infty),\ \sigma_1, \sigma_2 \in [0,\infty) \quad \text{and} \quad Q_1, Q_2 \in C([t_0,\infty), R^+), \tag{2}$$

$$\int^\infty sQ_i(s)\,ds < \infty, \qquad i = 1, 2. \tag{3}$$

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Recently, there has been a lot of activity concerning the oscillation and asymptotic behavior of first order neutral differential equations (see [2, 7–11]), directed mainly at the so-called linearized oscillation theory (see [1] and [4] for a review of this theory and [3] and [4] for some applications). This theory of the corresponding first-order neutral delay equation

$$\frac{d}{dt}\left[x(t) + px(t-\tau)\right] + Q_1(t)x(t-\sigma_1) - Q_2(t)x(t-\sigma_2) = 0 \qquad \text{(E)}$$

was restricted to the case $p \in (0, 1)$, and very recently some of the results have been extended to the case $p > 1$ (see [12] and [13]). The only global results with respect to p (that is, the results that hold for every $p \in R$) can be found in [5] and [12]. The second order neutral equation (1) received much less attention, which is due mainly to the technical difficulties arising in its analysis. See [1, 3, 4] for reviews of this theory. In particular, there is no global result, with respect to p, for (1).

Here we obtain the first global result (with respect to p) in the nonconstant coefficient case, which is a sufficient condition for the existence of a nonoscillatory solution for all values of $p \neq \pm 1$.

Let $m = \max\{\tau, \sigma_1, \sigma_2\}$. By a solution of Eq. (1) we mean a function $y \in C([t_1 - m, \infty), R)$, for some $t_1 \geq t_0$, such that $y(t) + py(t-\tau)$ is twice continuously differentiable on $[t_1, \infty)$ and such that Eq. (1) is satisfied for $t \geq t_1$.

Assume that (2) holds, $t_1 \geq t_0$, and let $\phi \in C([t_1 - m, t_1], R)$ be a given initial function. Then one can easily see by the method of steps that Eq. (1) has a unique solution $y \in C([t_1 - m, \infty), R)$ such that

$$y(t) = \phi(t) \qquad \text{for } t_1 - m \leq t \leq t_1.$$

As is customary, a solution of Eq. (1) is said to oscillate if it has arbitrarily large zeros. Otherwise the solution is called nonoscillatory.

The following result is the special case of our main result.

COROLLARY. *Consider the equation*

$$\frac{d^2}{dt^2}\left[x(t) + px(t-\tau)\right] + Q(t)x(t-\sigma) = 0,$$

where $p \in R$, $p \neq \pm 1$,

$$\tau \in (0, \infty), \quad \sigma \in [0, \infty), \quad \text{and } Q \in C\big([t_0, \infty), R^+\big),$$

and

$$\int^{\infty} sQ(s)\,ds < \infty.$$

Then this equation has a nonoscillatory solution.

This result, which is important for its own sake, will be used in the future to prove the linearized oscillation result for second order neutral equations of the form

$$\frac{d^2}{dt^2}\big[x(t) + px(t-\tau)\big] + q_1 f(x(t-\sigma_1)) - q_2 f(x(t-\sigma_2)) = 0$$

for the values of parameter $p \neq \pm 1$.

The condition (3) seems to be reasonable, since in the particular case $p = 0$, $Q_2(t) \equiv 0$, it becomes the well-known nonoscillatory result for delay equations (see [6]).

2. MAIN RESULT

Our main result is the following:

THEOREM. *Consider Eq.* (1), *subject to conditions* (2) *and* (3). *If*

$$aQ_1(s) - Q_2(s) \geq 0 \quad \text{for every } t \geq T_1 \quad \text{and} \quad a > 0, \tag{4}$$

where $p \neq \pm 1$ *and* T_1 *is large enough, then* (1) *has a nonoscillatory solution.*

Proof. The proof of this theorem will be divided into four claims, depending on the four different ranges of the parameter p.

Claim 1. $p \in [0, 1)$. Choose a $t_1 > t_0$ sufficiently large such that

$$t_1 \geq \max\{T_1, t_0 + \sigma\}, \qquad \sigma = max\{\tau, \sigma_1, \sigma_2\}, \tag{5}$$

$$\int_{t_1}^{\infty} s\big[Q_1(s) + Q_2(s)\big]\, ds < 1 - p, \tag{6}$$

$$0 \leq \int_{t_1}^{\infty} s\big[M_2 Q_1(s) - M_1 Q_2(s)\big]\, ds \leq p - 1 + M_2, \tag{7}$$

$$\int_{t_1}^{\infty} s\big[M_1 Q_1(s) - M_2 Q_2(s)\big]\, ds \geq 0 \tag{8}$$

hold, where M_1 and M_2 are positive constants such that

$$1 - M_2 < p \leq \frac{1 - M_1}{1 + M_2}$$

holds.

Let X be the set of all continuous and bounded functions on $[t_0, \infty)$ with the sup norm. Set

$$A = \big\{x \in X\colon M_1 \leq x(t) \leq M_2, t \geq t_0\big\}.$$

Define a mapping $T: A \to X$ as follows

$$(Tx)(t) = \begin{cases} 1 - p - px(t-\tau) + t\int_t^\infty \left[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\right] ds \\ \qquad + \int_{t_1}^t s\left[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\right] ds, \qquad t \geq t_1 \\ (Tx)(t_1), \qquad t_0 \leq t \leq t_1. \end{cases}$$

Clearly, Tx is continuous. For every $x \in A$ and $t \geq t_1$, using (4) and (7) we get

$$\begin{aligned} (Tx)(t) &= 1 - p - px(t-\tau) + t\int_t^\infty \left[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\right] ds \\ &\quad + \int_{t_1}^t s\left[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\right] ds \\ &\leq 1 - p + t\int_t^\infty \left[M_2Q_1(s) - M_1Q_2(s)\right] ds \\ &\quad + \int_{t_1}^t s\left[M_2Q_1(s) - M_1Q_2(s)\right] ds \\ &\leq 1 - p + \int_t^\infty s\left[M_2Q_1(s) - M_1Q_2(s)\right] ds \\ &\quad + \int_{t_1}^t s\left[M_2Q_1(s) - M_1Q_2(s)\right] ds \\ &= 1 - p + \int_{t_1}^\infty s\left[M_2Q_1(s) - M_1Q_2(s)\right] ds \\ &\leq M_2. \end{aligned}$$

Furthermore, in view of (4) and (8) we have

$$\begin{aligned} (Tx)(t) &= 1 - p - px(t-\tau) + t\int_t^\infty \left[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\right] ds \\ &\quad + \int_{t_1}^t s\left[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\right] ds \\ &\geq 1 - p - pM_2 + t\int_t^\infty \left[M_1Q_1(s) - M_2Q_2(s)\right] ds \\ &\quad + \int_{t_1}^t s\left[M_1Q_1(s) - M_2Q_2(s)\right] ds \\ &\geq 1 - p - pM_2 \\ &\geq M_1. \end{aligned}$$

Thus we proved that $TA \subset A$. Since A is a bounded, closed, and convex subset of X we have to prove that T is a contraction mapping on A to apply the contraction principle.

Now, for $x_1, x_2 \in A$ and $t \geq t_1$ we have

$$\begin{aligned}
&\left|(Tx_1)(t) - (Tx_2)(t)\right| \\
&\quad \leq p\left|x_1(t-\tau) - x_2(t-\tau)\right| \\
&\qquad + t\int_t^{\infty} Q_1(s)\left|x_1(s-\sigma_1) - x_2(s-\sigma_1)\right| ds \\
&\qquad + t\int_t^{\infty} Q_2(s)\left|x_1(s-\sigma_2) - x_2(s-\sigma_2)\right| ds \\
&\qquad + \int_{t_1}^{t} sQ_1(s)\left|x_1(s-\sigma_1) - x_2(s-\sigma_1)\right| ds \\
&\qquad + \int_{t_1}^{t} sQ_2(s)\left|x_1(s-\sigma_2) - x_2(s-\sigma_2)\right| ds \\
&\quad \leq p\|x_1 - x_2\| + \|x_1 - x_2\| \\
&\qquad \times \left\{\int_t^{\infty} s\left[Q_1(s) + Q_2(s)\right] ds + \int_{t_1}^{t} s\left[Q_1(s) + Q_2(s)\right] ds\right\} \\
&\quad = \|x_1 - x_2\|\left\{p + \int_{t_1}^{\infty} s\left[Q_1(s) + Q_2(s)\right] ds\right\} \\
&\quad = q_1\|x_1 - x_2\|
\end{aligned}$$

where we used sup norm. This immediately implies that

$$\|Tx_1 - Tx_2\| \leq q_1\|x_1 - x_2\|,$$

where in view of (6), $q_1 < 1$, which proves that T is a contraction mapping. Consequently T has the unique fixed point x, which is obviously a positive solution of Eq. (1). This completes the proof of Claim 1.

Claim 2. $p \in (1, +\infty)$. Choose a $t_1 > T_1 > t_0$ sufficiently large such that

$$t_1 + \tau \geq t_0 + \max\{\sigma_1, \sigma_2\}, \tag{9}$$

$$\int_{t_1}^{\infty} s\left[Q_1(s) + Q_2(s)\right] ds < p - 1, \tag{10}$$

$$0 \leq \int_{t_1}^{\infty} s\left[N_2 Q_1(s) - N_1 Q_2(s)\right] ds \leq 1 - p + pN_2, \tag{11}$$

and

$$\int_{t_1}^{\infty} s\left[N_1 Q_1(s) - N_2 Q_2(s)\right] ds \geq 0, \tag{12}$$

where N_1 and N_2 are positive constants such that

$$(1 - N_1)p \geq 1 + N_2 \quad \text{and} \quad p(1 - N_2) < 1.$$

Let X be the set as in Claim 1. Set

$$A = \{x \in X \colon N_1 \leq x(t) \leq N_2, t \geq t_0\}.$$

Define a mapping $T\colon A \to X$ as follows:

$$(Tx)(t) = \begin{cases} 1 - \frac{1}{p} - \frac{1}{p}x(t+\tau) \\ \quad + \frac{t+\tau}{p}\int_{t+\tau}^{\infty}\left[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\right] ds \\ \quad + \frac{1}{p}\int_{t_1}^{t+\tau} s\left[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\right] ds, & t \geq t_1 \\ (Tx)(t_1), \qquad t_0 \leq t \leq t_1 \end{cases}$$

Clearly, Tx is continuous. For every $x \in A$ and $t \geq t_1$, using (4) and (11) we get

$$\begin{aligned} (Tx)(t) &= 1 - \frac{1}{p} - \frac{1}{p}x(t+\tau) \\ &\quad + \frac{t+\tau}{p}\int_{t+\tau}^{\infty}\left[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\right] ds \\ &\quad + \frac{1}{p}\int_{t_1}^{t+\tau} s\left[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\right] ds \\ &\leq 1 - \frac{1}{p} + \frac{t+\tau}{p}\int_{t+\tau}^{\infty}\left[N_2Q_1(s) - N_1Q_2(s)\right] ds \\ &\quad + \frac{1}{p}\int_{t_1}^{t+\tau} s\left[N_2Q_1(s) - N_1Q_2(s)\right] ds \\ &\leq 1 - \frac{1}{p} + \frac{1}{p}\left\{\int_{t+\tau}^{\infty} s\left[N_2Q_1(s) - N_1Q_2(s)\right] ds \right. \\ &\qquad\qquad \left. + \int_{t_1}^{t+\tau} s\left[N_2Q_1(s) - N_1Q_2(s)\right] ds\right\} \\ &= 1 - \frac{1}{p} + \frac{1}{p}\int_{t_1}^{\infty} s\left[N_2Q_1(s) - N_1Q_2(s)\right] ds \\ &\leq N_2. \end{aligned}$$

Furthermore, in view of (12) we have

$$\begin{aligned}(Tx)(t) &= 1 - \frac{1}{p} - \frac{1}{p}x(t+\tau) \\ &\quad + \frac{t+\tau}{p}\int_{t+\tau}^{\infty}\big[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\big]\,ds \\ &\quad + \frac{1}{p}\int_{t_1}^{t+\tau} s\big[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\big]\,ds \\ &\geq 1 - \frac{1}{p} - \frac{N_2}{p} + \frac{t+\tau}{p}\int_{t+\tau}^{\infty}\big[N_1Q_1(s) - N_2Q_2(s)\big]\,ds \\ &\quad + \frac{1}{p}\int_{t_1}^{t+\tau} s\big[N_1Q_1(s) - N_2Q_2(s)\big]\,ds \\ &\geq 1 - \frac{1}{p} - \frac{N_2}{p} \\ &\geq N_1.\end{aligned}$$

Thus we proved that $TA \subset A$. Since A is a bounded, closed, and convex subset of X we have to prove that T is a contraction mapping on A to apply the contraction principle.

Now for $x_1, x_2 \in A$ and $t \geq t_1$ we have

$$\begin{aligned}&\big|(Tx_1)(t) - (Tx_2)(t)\big| \\ &\quad\leq \frac{1}{p}\big|x_1(t+\tau) - x_2(t+\tau)\big| \\ &\quad + \frac{t+\tau}{p}\bigg[\int_{t+\tau}^{\infty} Q_1(s)\big|x_1(s-\sigma_1) - x_2(s-\sigma_1)\big|\,ds \\ &\qquad\qquad + \int_{t+\tau}^{\infty} Q_2(s)\big|x_1(s-\sigma_2) - x_2(s-\sigma_2)\big|\,ds\bigg] \\ &\quad + \frac{1}{p}\bigg[\int_{t_1}^{t+\tau} sQ_1(s)\big|x_1(s-\sigma_1) - x_2(s-\sigma_1)\big|\,ds \\ &\qquad\qquad + \int_{t_1}^{t+\tau} sQ_2(s)\big|x_1(s-\sigma_2) - x_2(s-\sigma_2)\big|\,ds\bigg] \\ &\quad\leq \frac{1}{p}\|x_1 - x_2\| + \frac{1}{p}\|x_1 - x_2\| \\ &\qquad \times \bigg[\int_{t+\tau}^{\infty} s\big[Q_1(s) + Q_2(s)\big]\,ds + \int_{t_1}^{t+\tau} s\big[Q_1(s) + Q_2(s)\big]\,ds\bigg] \\ &\quad = \frac{1}{p}\|x_1 - x_2\|\bigg[1 + \int_{t_1}^{\infty} s\big[Q_1(s) + Q_2(s)\big]\,ds\bigg] \\ &\quad = q_2\|x_1 - x_2\|,\end{aligned}$$

where we used sup norm. This immediately implies that

$$\left\| Tx_1 - Tx_2 \right\| \le q_2 \left\| x_1 - x_2 \right\|,$$

where in view of (10), $q_2 < 1$, which proves that T is a contraction mapping. Consequently T has the unique fixed point x, which is obviously a positive solution of Eq. (1). This completes the proof of Claim 2.

Claim 3. $p \in (-1, 0)$. Choose a $t_1 > T_1 > t_0$ sufficiently large so that (5) and the inequalities

$$\int_{t_1}^{\infty} s\left[Q_1(s) + Q_2(s)\right] ds < p + 1 \tag{13}$$

$$0 \le \int_{t_1}^{\infty} s\left[M_4 Q_1(s) - M_3 Q_2(s)\right] ds \le (p+1)(M_4 - 1) \tag{14}$$

hold, where the constants M_3 and M_4 satisfy

$$0 < M_3 \le 1 < M_4.$$

Let X be the set as in Claim 1. Set

$$A = \left\{x \in X\colon M_3 \le x(t) \le M_4, t \ge t_0\right\}.$$

Define a mapping $T\colon A \to X$ as follows:

$$(Tx)(t) = \begin{cases} 1 + p - px(t-\tau) \\ \quad + t\int_t^{\infty}\left[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\right] ds \\ \quad + \int_{t_1}^{t} s\left[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\right] ds, & t \ge t_1 \\ (Tx)(t_1), \qquad t_0 \le t \le t_1. \end{cases}$$

Clearly, Tx is continuous. For every $x \in A$ and $t \ge t_1$, using (14) we get

$$\begin{aligned} (Tx)(t) &= 1 + p - px(t-\tau) \\ &\quad + t\int_t^{\infty}\left[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\right] ds \\ &\quad + \int_{t_1}^{t} s\left[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\right] ds \le 1 + p - pM_4 \\ &\quad + t\int_t^{\infty}\left[M_4 Q_1(s) - M_3 Q_2(s)\right] ds + \int_{t_1}^{t} s\left[M_4 Q_1(s) - M_3 Q_2(s)\right] ds \\ &\le 1 + p - pM_4 + \int_{t_1}^{\infty} s\left[M_4 Q_1(s) - M_3 Q_2(s)\right] ds \\ &\le 1 + p - pM_4 + (p+1)(M_4 - 1) \\ &= M_4. \end{aligned}$$

Furthermore, in view of (4) we have

$$
\begin{aligned}
(Tx)(t) = 1 + p - px(t-\tau) & \\
&+ t\int_t^\infty \left[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\right] ds \\
&+ \int_{t_1}^t \left[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\right] ds \\
\geq 1 + p - pM_3 &+ t\int_t^\infty \left[M_3Q_1(s) - M_4Q_2(s)\right] ds \\
&+ \int_{t_1}^t s[M_4Q_1(s) - M_4Q_2(s)]\, ds \\
\geq 1 + p - pM_3 & \\
\geq M_3. &
\end{aligned}
$$

Thus, we proved that $TA \subset A$. Since A is a bounded, closed, and convex subset of X, we have to prove that T is contraction mapping on A to apply the contraction principle.

Now, for $x_1, x_2 \in A$ and $t \geq t_1$ we have

$$
\begin{aligned}
&\left|(Tx_1)(t) - (Tx_2)(t)\right| \\
&\quad \leq -p\left|x_1(t-\tau) - x_2(t-\tau)\right| \\
&\qquad + t\int_t^\infty Q_1(s)\left|x_1(s-\sigma_1) - x_2(s-\sigma_1)\right| ds \\
&\qquad + t\int_t^\infty Q_2(s)\left|x_1(s-\sigma_2) - x_2(s-\sigma_2)\right| ds \\
&\qquad + \int_{t_1}^t sQ_1(s)\left|x_1(s-\sigma_1) - x_2(s-\sigma_1)\right| ds \\
&\qquad + \int_{t_1}^t sQ_2(s)\left|x_1(s-\sigma_2) - x_2(s-\sigma_2)\right| ds \\
&\quad \leq -p\left\|x_1 - x_2\right\| + \left\|x_1 - x_2\right\|\left(\int_t^\infty s\left[Q_1(s) + Q_2(s)\right] ds\right) \\
&\quad = \left\|x_1 + x_2\right\|\left\{-p + \int_{t_1}^\infty s\left[Q_1(s) + Q_2(s)\right] ds\right\} \\
&\quad = q_4\left\|x_1 - x_1\right\|,
\end{aligned}
$$

where we used sup norm. This immediately implies that

$$
\left\|Tx_1 - Tx_2\right\| \leq q_3\left\|x_1 - x_2\right\|,
$$

where in view of (13), $q_3 < 1$. This proves that T is a contraction mapping. Consequently, T has the unique fixed point x, which is obviously a positive solution of Eq. (1). This completes the proof of Claim 3.

Claim 4. $p \in (-\infty, -1)$. Choose a $t_1 > T_1 > t_0$ sufficiently large such that (9) and the inequalities

$$\int_{t_1}^{\infty} s\big[Q_1(s) + Q_2(s)\big]\,ds < -p - 1 \tag{15}$$

$$0 \le \int_{t_1}^{\infty} s\big[N_4 Q_1(s) - N_3 Q_2(s)\big]\,ds \le (p+1)(N_3 - 1) \tag{16}$$

hold, where the positive constants N_3 and N_4 satisfy

$$0 < N_3 < 1 \le N_4.$$

Let X be the set as in Claim 1. Set

$$A = \big\{x \in X\colon N_3 \le x(t) \le N_4,\, t \ge t_0\big\}.$$

Define a mapping $T\colon A \to X$ as follows

$$(Tx)(t) = \begin{cases} 1 + \dfrac{1}{p} - \dfrac{1}{p}x(t+\tau) \\ \quad + \dfrac{t+\tau}{p}\displaystyle\int_{t+\tau}^{\infty}\big[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\big]\,ds \\ \quad + \dfrac{1}{p}\displaystyle\int_{t_1}^{t+\tau} s\big[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\big]\,ds, \qquad t \ge t_1 \\ (Tx)(t_1), \qquad t_0 \le t \le t_1. \end{cases}$$

Clearly, Tx is continuous. For every $x \in A$ and $t \ge t_1$, using (4), we get

$$\begin{aligned} (Tx)(t) &= 1 + \frac{1}{p} - \frac{1}{p}x(t+\tau) \\ &\quad + \frac{t+\tau}{p}\int_{t+\tau}^{\infty}\big[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\big]\,ds \\ &\quad + \frac{1}{p}\int_{t_1}^{t+\tau} s\big[Q_1(s)x(s-\sigma_1) - Q_2(s)x(s-\sigma_2)\big]\,ds \\ &\le 1 + \frac{1}{p} - \frac{N_4}{p} + \frac{t+\tau}{p}\int_{t+\tau}^{\infty}\big[N_3 Q_1(s) - N_4 Q_2(s)\big]\,ds \\ &\quad + \frac{1}{p}\int_{t_1}^{t+\tau} s[N_3 Q_1(s) - N_4 Q_2(s)]\,ds \\ &\le 1 + \frac{1}{p} - \frac{N_4}{p} \\ &\le N_4. \end{aligned}$$

Furthermore, in view of (16) we have

$$\begin{aligned}(Tx)(t) &= 1+\frac{1}{p}-\frac{1}{p}x(t+\tau)\\ &\quad+\frac{t+\tau}{p}\int_{t+\tau}^{\infty}\big[Q_1(s)x(s-\sigma_1)-Q_2(s)x(s-\sigma_2)\big]\,ds\\ &\quad+\frac{1}{p}\int_{t_1}^{t+\tau}s[Q_1(s)x(s-\sigma_1)-Q_2(s)x(s-\sigma_2)]\,ds\\ &\ge 1+\frac{1}{p}-\frac{N_3}{p}+\frac{t+\tau}{p}\int_{t+\tau}^{\infty}\big[N_4Q_1(s)-N_3Q_2(s)\big]\,ds\\ &\quad+\frac{1}{p}\int_{t_1}^{t+\tau}s\big[N_4Q_1(s)-N_3Q_2(s)\big]\,ds\\ &\ge 1+\frac{1}{p}-\frac{N_3}{p}+\frac{1}{p}\int_{t_1}^{\infty}s\big[N_4Q_1(s)-N_3Q_2(s)\big]\,ds\\ &\ge 1+\frac{1}{p}-\frac{N_3}{p}+\frac{1}{p}(p+1)(N_3-1)\\ &= N_3.\end{aligned}$$

Thus, we proved that $TA\subset A$. Since A is a bounded, closed, and convex subset of X, we have to prove that T is a contraction mapping on A to apply the contraction principle.

Now, for $x_1, x_2\in A$ and $t\ge t_1$ we have

$$\begin{aligned}&\big|(Tx_1)(t)-(Tx_2)(t)\big|\\ &\quad\le -\frac{1}{p}\big|x_1(t+\tau)-x_2(t+\tau)\big|\\ &\qquad-\frac{t+\tau}{p}\bigg[\int_{t+\tau}^{\infty}Q_1(s)\big|x_1(s-\sigma_1)-x_2(s-\sigma_1)\big|\,ds\\ &\qquad\qquad+\int_{t+\tau}^{\infty}Q_2(s)\big|x_1(s-\sigma_2)-x_2(s-\sigma_2)\big|\,ds\bigg]\\ &\qquad-\frac{1}{p}\bigg[\int_{t_1}^{t+\tau}sQ_1(s)\big|x_1(s-\sigma_1)-x_2(s-\sigma_1)\big|\,ds\\ &\qquad\qquad+\int_{t_1}^{t+\tau}sQ_2(s)\big|x_1(s-\sigma_2)-x_2(s-\sigma_2)\big|\,ds\bigg]\\ &\quad\le -\frac{1}{p}\|x_1-x_2\|-\frac{1}{p}\|x_1-x_2\|\\ &\qquad\times\left\{\int_{t+\tau}^{\infty}s\big[Q_1(s)+Q_2(s)\big]\,ds+\int_{t_1}^{t+\tau}s\big[Q_1(s)+Q_2(s)\big]\,ds\right\}\end{aligned}$$

$$= -\frac{1}{p}\|x_1 - x_2\|\left\{1 + \int_{t_1}^{\infty} s\big[Q_1(s) + Q_2(s)\big]\,ds\right\}$$
$$= q_4\|x_1 - x_2\|,$$

where we used sup norm. This immediately implies that

$$\|Tx_1 - Tx_2\| \le q_4\|x_1 - x_2\|.$$

In view of (15), $q_4 < 1$, which proves that T is a contraction mapping. Consequently, T has the unique fixed point x, which is obviously a positive solution of Eq. (1). This completes the proof of Claim 4.

The proof of the theorem is complete.

Remark. Condition (4), which implies that $Q_1(t)$ dominates $Q_2(t)$, may look too restrictive. This condition is actually affected by the choice of the constants M_i and N_i, $i = 1, 2, 3, 4$. Choosing those constants in an appropriate way, we can specify that this condition hold for a single value of a; in this case this condition becomes very easy to check and use. For instance, if $M_{2k} = \alpha M_{2k-1}$, $N_{2k} = \alpha N_{2k-1}$, $k = 1, 2$, then $a = \alpha$ in (4), where $\alpha > 1$ is a given number. Choosing α to be as close to 1 as we please, we get very precise asymptotic behavior for the nonoscillatory solution we constructed, since in all cases we have

$$M_{2k-1} \le x(t) \le \alpha M_{2k-1}, \qquad k = 1, 2$$

or

$$N_{2k-1} \le x(t) \le \alpha N_{2k-1}, \qquad k = 1, 2.$$

We can also specify our choice of constants by choosing $M_1 = M_3 = N_1 = N_3$ and $M_2 = M_4 = N_2 = N_4$, which can be achieved by taking M_1 and M_2 to satisfy $0 < M_1 < M_2$ and $M_2{}^2 > M_1$. In this case in all four cases we will have the same asymptotic behavior of nonoscillatory solution as $M_1 \le x(t) \le M_2$ with the same value of $a = M_2/M_1$.

Combining the last two choices of constants, we get $M_1 \le x(t) \le \alpha M_1$ and $a = \alpha$.

Finally, in the special case where $Q_2(t) \equiv 0$, condition (4) is redundant and the theorem holds under condition (3) only. This result is stated as the Corollary.

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