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Convergence acceleration of series through a variational approach ☆

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Abstract

By means of a variational approach we find new series representations both for well-known mathematical constants, such as π and the Catalan constant, and for mathematical functions, such as the Riemann zeta function. The series that we have found are all exponentially convergent and provide quite useful analytical approximations. With limited effort our method can be applied to obtain similar exponentially convergent series for a large class of mathematical functions.

Keywords: Acceleration of convergence; Riemann zeta function; Hurwitz zeta function; Linear delta expansion; Variational perturbation theory

1. Introduction

This paper deals with the problem of improving the convergence of a slowly convergent series where a large number of terms is needed to reach the desired accuracy. This is a challenging problem, which has been considered before (see, for example, [1–3]) and which has interesting applications in many physical problems. As a matter of fact it is well known that perturbation theory often gives series which converge very slowly or do not converge at all (one example is the perturbative series for the quantum anharmonic oscillator, which was originally studied in [4]).

In this paper we propose a new method to accelerate some class of mathematical series, which does not rely on a perturbative approach, i.e., on an expansion in some small natural parameter

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(a parameter present in the original expression). The method works by introducing an artificial dependence in the formulas upon an arbitrary parameter, by identifying a new "perturbation" and by then devising an expansion which can be optimized to give faster rates of convergence. The details of how this works will be explained in depth in the next section. This procedure is well known in physics and it has been exploited in the so-called "linear delta expansion" (LDE) and variational perturbation theory (VPT) approaches [5–8].

We will show in the following that such techniques can be used to obtain exponentially convergent series for some mathematical functions. A proof of convergence of the method is also provided. The "flexibility" and simplicity of the method that we propose suggests that application to wider classes of series could be found.

The paper is organized as follows: in Section 2 we first introduce the method and then use it to obtain accelerate series for π and for the Catalan constant; in Sections 3 and 4 we obtain a family of series representations for the Riemann and Hurwitz zeta functions which all converge in a certain domain of the arbitrary parameter; finally in Section 5 we draw our conclusions.

2. Mathematical constants

Many fundamental mathematical constants can be expressed as infinite sums. In many cases such series converge very slowly and a huge number of terms has to be calculated before reaching the desired precision. Several examples of this are discussed, for example, in [1], where the authors consider a particular rearrangement of the series which transforms them into rapidly converging series. In the following we review two of the examples of [1], π and the Catalan constant, and obtain new series representations for these constants which display a fast rate of convergence.

We first consider the series

$$S = 4\sum_{n=1}^{\infty}\left[\frac{1}{4n-3} - \frac{1}{4n-1}\right], \tag{1}$$

which converges very slowly to π and it is known as Gregory's formula. The sum of the to the first 10^3 terms yields an approximate value of π, which only has the first 3 decimals correct.

Flajolet and Vardi [1] have shown that it is possible to convert series such as the one in Eq. (1) into rapidly converging ones. Here we generalize the method of Flajolet and Vardi, introducing an arbitrary parameter in the series. Such parameter is then tuned to accelerate the convergence of the series itself by using the principle of minimal sensitivity (PMS) [9].

Theorem 1. *The series*

$$S = 4\sum_{m=1}^{\infty}\left(\frac{1}{1+\lambda}\right)^{m+1}\sum_{k=1}^{m}\binom{m}{k}\lambda^{m-k}\frac{3^k-1}{4^{k+1}}\zeta(k+1)$$

converges to π for $\lambda > -1/2$, with λ real.

Proof. We write the series of Eq. (1) in the equivalent form

$$S = \sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{1+\lambda}\left(\frac{1}{1-\left(\frac{3}{4n}+\lambda\right)/(1+\lambda)} - \frac{1}{1-\left(\frac{1}{4n}+\lambda\right)/(1+\lambda)}\right), \tag{2}$$

with an arbitrary parameter $\lambda \neq -1$.

Provided that

$$\left|\frac{\frac{3}{4n}+\lambda}{1+\lambda}\right|<1 \quad \text{and} \quad \left|\frac{\frac{1}{4n}+\lambda}{1+\lambda}\right|<1$$

for all $n \geqslant 1$, i.e., $\lambda > -1/2$, we can expand Eq. (2) as

$$S = 4\sum_{n=1}^{\infty}\frac{1}{4n}\sum_{m=1}^{\infty}\left(\frac{1}{1+\lambda}\right)^{m+1}\sum_{k=1}^{m}\binom{m}{k}\lambda^{m-k}\left[\left(\frac{3}{4n}\right)^{k}-\left(\frac{1}{4n}\right)^{k}\right].$$

As the series in m and n contain only positive terms, we can perform the series over n and obtain the result

$$S = \sum_{m=1}^{\infty}\left(\frac{1}{1+\lambda}\right)^{m+1}\sum_{k=1}^{m}\binom{m}{k}\lambda^{m-k}\frac{3^k-1}{4^k}\zeta(k+1). \qquad \square \tag{3}$$

Remark 2. The series of Eq. (3) for $\lambda = 0$ coincides with the result of Flajolet and Vardi [1]:

$$S^{(\mathrm{FV})} = \sum_{m=1}^{\infty}\frac{3^m-1}{4^m}\zeta(m+1) \tag{4}$$

and converges geometrically to π.

Notice that Eq. (3) defines a family of series converging to π as long as $\lambda > -1/2$. Clearly the dependence upon λ in Eq. (3) is artificial and shows up only when a finite number of terms is considered. If we set $S_N(\lambda)$ to be the partial series of Eq. (3) the dependence upon λ in $S_N(\lambda)$ then disappears in the limit $N \to \infty$.

For fixed N we evaluate the partial sum at the points where

$$dS_N(\lambda)/d\lambda = 0,$$

since there the expression is less sensitive to changes of the arbitrary parameter λ, a property which shares with the full series (3). This is called principle of minimal sensitivity (PMS) [9] and provides an equation which, once solved at a given order, provides an *optimal* value of λ for a fixed partial sum $S_N(\lambda)$.

For $S_2(\lambda)$ we obtain

$$\lambda^{(1)} = -\frac{3}{\pi^2}\zeta(3) \approx -0.365381 > -1/2. \tag{5}$$

Remark 3. The series of Eq. (3) converges geometrically to π. We can estimate the rate of convergence by approximating the mth term in the series with

$$\begin{aligned} s_m &= \left(\frac{1}{1+\lambda}\right)^{m+1}\sum_{k=1}^{m}\binom{m}{k}\lambda^{m-k}\frac{3^k-1}{4^k} \\ &= \frac{1}{(1+\lambda)^{m+1}}\left[(\lambda+3/4)^m-(\lambda+1/4)^m\right] \approx m\left(\frac{\lambda+3/4}{1+\lambda}\right)^m. \end{aligned} \tag{6}$$

Using the PMS value of Eq. (5) we obtain $s_m \approx 1.65^{-m}$. This improves the rate $s_m \approx 1.33^{-m}$ of the series of Flajolet and Vardi (4).

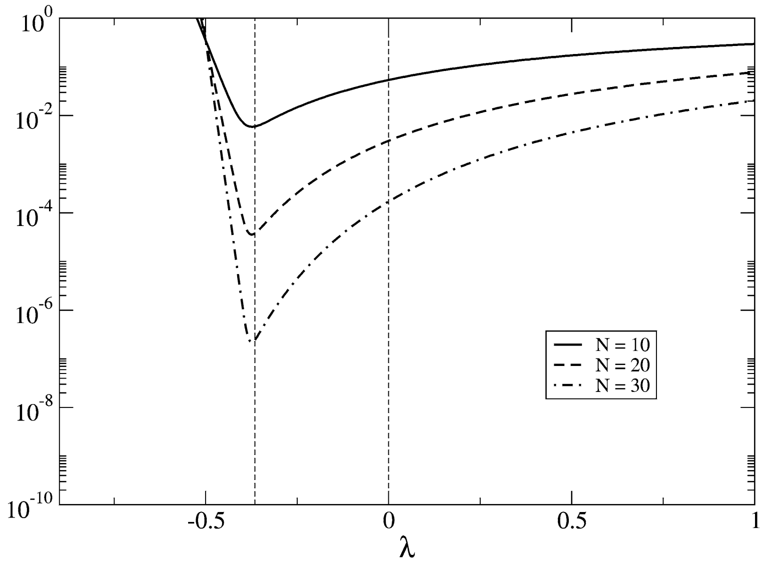

Fig. 1. The error obtained using the partial sum of Eq. (3) over the first 10, 20 and 30 terms respectively as a function of λ.

In Fig. 1 we display the partial sums of Eq. (3) with 10, 20 and 30 terms as a function of λ: the locations of $\lambda^{(1)}$ and $\lambda = 0$ are marked with vertical lines. It turns out that $\lambda^{(1)}$ is an excellent approximation to the exact minimum of the partial sum even for large values of terms.

In Fig. 2 we plot the error obtained by using Eq. (3) with λ given by Eq. (5) and by using the formula of Flajolet and Vardi, Eq. (4). Our series converges exponentially more rapidly than Eq. (4).

We now consider another series, which was also considered in [1]. The series

$$S = \sum_{n=1}^{\infty} \left[\frac{1}{(4n-3)^2} - \frac{1}{(4n-1)^2} \right] \tag{7}$$

is known to slowly converge to the Catalan constant, $G \approx 0.9159656$.

Theorem 4. *The series defined as*

$$S = \sum_{m=1}^{\infty} \left(\frac{1}{1+\lambda} \right)^{m+1} \sum_{k=1}^{m} \binom{m}{k} \lambda^{m-k} k \frac{3^{k-1}-1}{4^{k+1}} \zeta(k+1), \tag{8}$$

converges to the Catalan constant for any $\lambda > -1/2$, with λ real.

Proof. We can rewrite Eq. (7) as

$$S = \lim_{a \to 0} \frac{d}{da} \tilde{S}(a) \tag{9}$$

where

$$\tilde{S}(a) \equiv \sum_{n=1}^{\infty} \left[\frac{1}{(4n-3-a)} - \frac{1}{(4n-1-a)} \right].$$

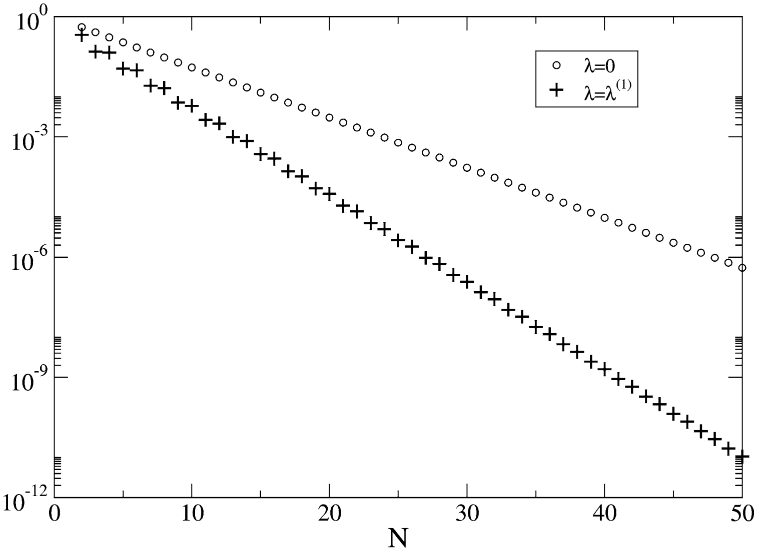

Fig. 2. The error obtained using the partial sum of Eq. (3) with $\lambda = 0$ and $\lambda = \lambda^{(1)}$ as a function of the number of terms in the sum.

Note that this series converges uniformly in a so we can differentiate term by term. We can apply Theorem 1 to $\tilde{S}(a)$ and obtain

$$\tilde{S}(a) = \sum_{m=1}^{\infty} \sum_{k=1}^{m} \binom{m}{k} \frac{\lambda^{m-k}}{(1+\lambda)^{m+1}} \frac{(3+a)^k - (1+a)^k}{4^{k+1}} \zeta(k+1). \tag{10}$$

As the series (10) converges uniformly in a, the proof is complete once the limit (9) is evaluated. □

Notice that the series of Theorem 4 reduces to the formula given in [1] for $\lambda = 0$:

$$S^{(\mathrm{FV})} = \sum_{m=1}^{\infty} m \frac{3^{m-1} - 1}{4^{m+1}} \zeta(m+1). \tag{11}$$

Remark 5. The series defined as

$$S = \frac{1}{3} \sum_{m=1}^{\infty} \sum_{k=1}^{m} \frac{1}{2^{3+2k}} \frac{\lambda_0^{m-k}}{(1+\lambda_0)^{m+2}} \binom{m}{k} \zeta(k+1) \times \left[-\left(\left(3+3^k\right)k(1+\lambda_0)\right) - 3\left(-1+3^k\right)(\lambda_0 - m)\right] \tag{12}$$

converges to the Catalan constant, where $\lambda_0 \equiv -\frac{3}{\pi^2}\zeta(3)$.

This follows from Eq. (10) for $\tilde{S}(a)$ applying the PMS

$$\frac{d\tilde{S}(a)}{d\lambda} = 0.$$

The optimal value of λ taking the first two terms in the series

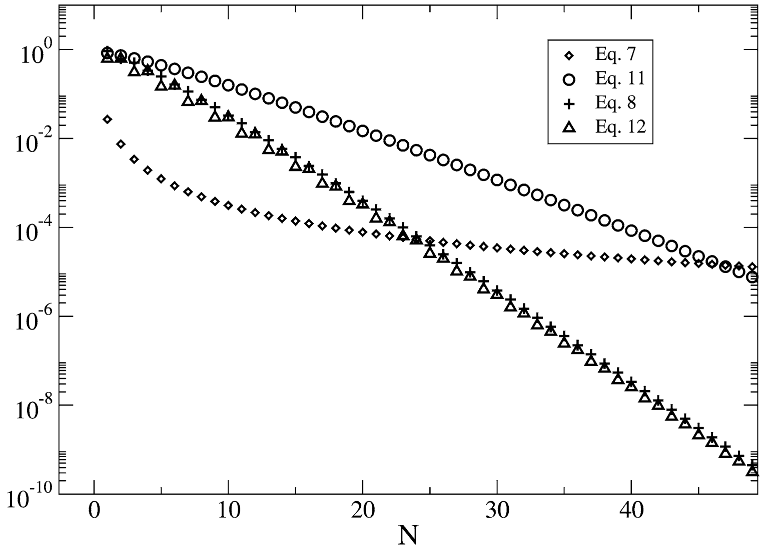

Fig. 3. $|S^{(N)} - G|$ as a function of the number of terms in the sum using Eq. (7) (diamonds), Eq. (11) (circles), Eq. (8) (pluses) and Eq. (12) (triangles).

$$\lambda^{(1)} = -\frac{3}{\pi^2}\zeta(3)\left(1 + \frac{a}{2}\right), \tag{13}$$

and Eq. (12) is obtained.

In Fig. 3 we compare the different approximations, showing that Eq. (8) (with $\lambda = \lambda^{(1)}$) and Eq. (12) have a greater rate of convergence then the corresponding equation in [1]. Equation (12) provides a slightly better approximation.

Remark 6. The accelerated series (8) and (12) converge geometrically to the Catalan constant. For example, the mth term of the series (8) behaves as

$$s_m \approx \frac{m}{16(1+\lambda)^{m+1}}\left[(\lambda + 3/4)^{m-1} - (\lambda + 1/4)^{m-1}\right] \approx m\left(\frac{\lambda + 3/4}{1+\lambda}\right)^m. \tag{14}$$

Taking $\lambda = \lambda_0$ we have $s_m \approx m 1.65^{-m}$.

It is clear that these results can be generalized to sums of the form

$$S_n = \sum_{n=1}^{\infty}\left[\frac{1}{(4n-3)^n} - \frac{1}{(4n-1)^n}\right]. \tag{15}$$

3. The Riemann zeta function

In this section we apply the same strategy outlined above to the calculation of the Riemann zeta function [10], and prove the following theorem.

Theorem 7. *A convergent series for the Riemann zeta function, which is valid for* $\Re(s) > 0$*, with the exclusion of* $s = 1$*, is*

$$\zeta(s)=\frac{1}{1-2^{1-s}}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\frac{\lambda^{k-j}}{(1+\lambda)^{k+1}}\frac{(-1)^j}{(1+j)^s} \tag{16}$$

for $\lambda>0$.

Proof. We use the integral representation

$$\zeta(s)=\frac{1}{1-2^{1-s}}\frac{1}{\Gamma(s)}\int_0^1\frac{\log^{s-1}\frac{1}{x}}{1+x}\,dx, \tag{17}$$

valid for $\Re(s)>0$, and write it as

$$\zeta(s)=\frac{1}{1-2^{1-s}}\frac{1}{\Gamma(s)}\int_0^1\frac{1}{1+\lambda}\frac{\log^{s-1}\frac{1}{x}}{1+\frac{x-\lambda}{1+\lambda}}\,dx, \tag{18}$$

where λ is an arbitrary parameter introduced by hand. The condition $\left|\frac{x-\lambda}{1+\lambda}\right|<1$ is fulfilled uniformly for all $x\in[0,1]$ provided that $\lambda>0$; in this case one can expand the denominator in powers of $\left(\frac{x-\lambda}{1+\lambda}\right)$ and obtain

$$\begin{aligned}\zeta(s)&=\frac{1}{1-2^{1-s}}\frac{1}{\Gamma(s)}\sum_{k=0}^{\infty}\frac{1}{(1+\lambda)^{k+1}}\int_0^1(-x+\lambda)^k\log^{s-1}\frac{1}{x}\,dx\\&=\frac{1}{1-2^{1-s}}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\frac{\lambda^{k-j}}{(1+\lambda)^{k+1}}\frac{(-1)^j}{(1+j)^s},\end{aligned} \tag{19}$$

which completes our proof. □

Remark 8. The series (16) for $\lambda=1$,

$$\zeta(s)=\frac{1}{1-2^{1-s}}\sum_{k=0}^{\infty}\frac{1}{2^{k+1}}\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^j}{(1+j)^s}, \tag{20}$$

has been first conjectured by Knopp around 1930 [11], and later proved by Hasse [12] and rediscovered more recently by Sondow [13].

Although λ appears explicitly in the series (16), the series itself does not depend upon λ, as long as $\lambda>0$. However the partial sums $\zeta^{(K)}(\lambda,s)$, defined as

$$\zeta^{(K)}(\lambda,s)\equiv\frac{1}{1-2^{1-s}}\sum_{k=0}^{K}\sum_{j=0}^{k}\binom{k}{j}\frac{\lambda^{k-j}}{(1+\lambda)^{k+1}}\frac{(-1)^j}{(1+j)^s}, \tag{21}$$

must show a dependence upon λ. However such dependence may be minimized by applying the PMS, i.e.,

$$\frac{d}{d\lambda}\zeta^{(K)}(\lambda,s)=0. \tag{22}$$

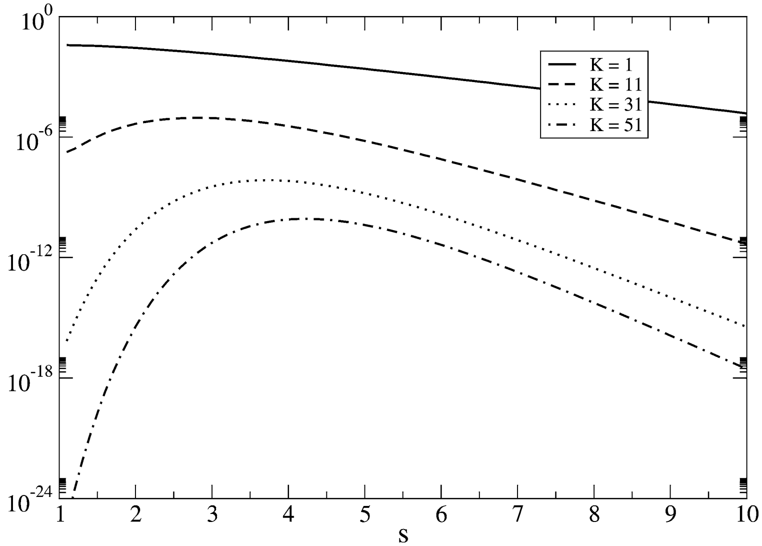

Fig. 4. The error $|(\zeta^{(K)}(s) - \zeta(s))/\zeta(s)|$ obtained by using Eq. (23) to different orders.

To lowest order, which corresponds to sum up to $K = 1$, one has that $\lambda_{\mathrm{PMS}}^{(1)} = 2^{-s}$ and the corresponding formula is found to be

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{k=0}^{\infty} \frac{1}{(1 + 2^{-s})^{k+1}} \sum_{j=0}^{k} \binom{k}{j} 2^{-s(k-j)} \frac{(-1)^j}{(1 + j)^s}. \tag{23}$$

Equation (23) is an *exact* series representation of the Riemann zeta function: this simple formula yields an excellent approximation to the zeta function as it can be appreciated by looking at Fig. 4, where we plot the error $|(\zeta^{(K)}(s) - \zeta(s))/\zeta(s)|$ obtained by using Eq. (23) to different orders, for $1 < s \leqslant 10$. It is remarkable that this simple analytical formula works quite well even in the proximity of $s = 1$, where the ζ function diverges.

The rate of convergence of the series is greatly improved by applying the PMS to higher orders.[1] Although it is possible to find the analytical solution to the PMS equation only to low orders, we have calculated λ numerically in Fig. 5 for $s = 2, 3, 4, 5$. For example, to order $K = 101$ we find that $\lambda_{\mathrm{PMS}}^{(101)}(2) = 0.482$, $\lambda_{\mathrm{PMS}}^{(101)}(3) = 0.467$, $\lambda_{\mathrm{PMS}}^{(101)}(4) = 0.452$ and $\lambda_{\mathrm{PMS}}^{(101)}(5) = 0.439$.

Remark 9. The thin solid lines correspond to the best fit of λ with the function

$$\lambda_{\mathrm{FIT}}(K) = \kappa_1 + \frac{\kappa_2 K \log K}{\kappa_3 K + (\kappa_4 + K) \log K}$$

for K up to 101. The coefficients κ_i for $s = 2, 3, 4, 5$ can be found in Table 1.

In Fig. 6 we display the dependence upon λ of the partial sums over the first $K = 11, 31, 51$ terms in the case of $\zeta(3)$. The vertical line corresponds to the location of $\lambda_{\mathrm{PMS}}^{(101)}$. In Fig. 7 we plot

[1] A real solution is found only for odd values of K.

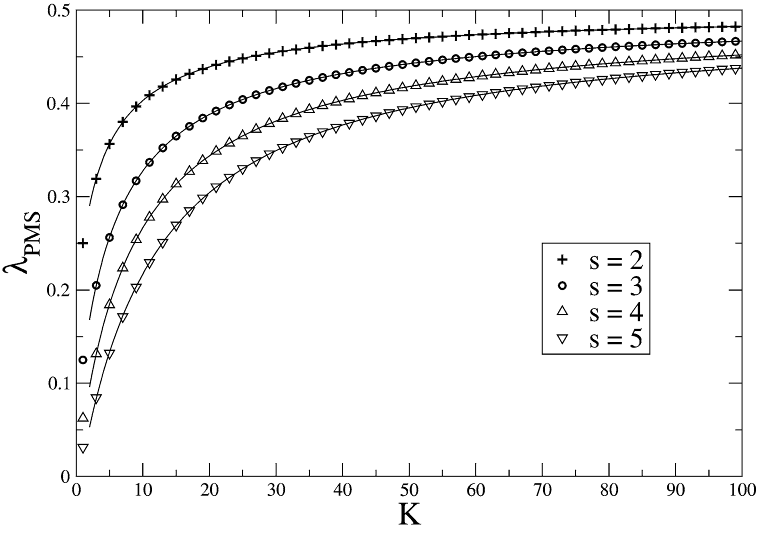

Fig. 5. The optimal parameter λ_{PMS} for $s = 2, 3, 4, 5$ calculated to different orders.

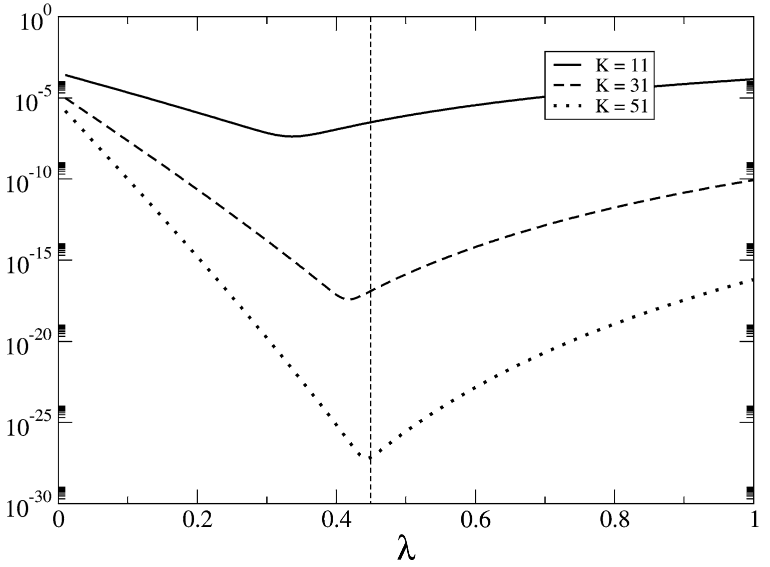

Fig. 6. Dependence upon the variational parameter of $|(\zeta^{(K)}(s) - \zeta(s))/\zeta(s)|$ with $K = 11, 31, 51$.

Table 1
Coefficients of the best fit of the λ_{PMS} as a function of K

s	κ_1	κ_2	κ_3	κ_4
2	0.201	0.318	0.416	3.9396
3	0.0655	0.461	0.415	5.791
4	0.0035	0.514	0.288	8.283
5	−0.0245	0.515	0.0031	11.31

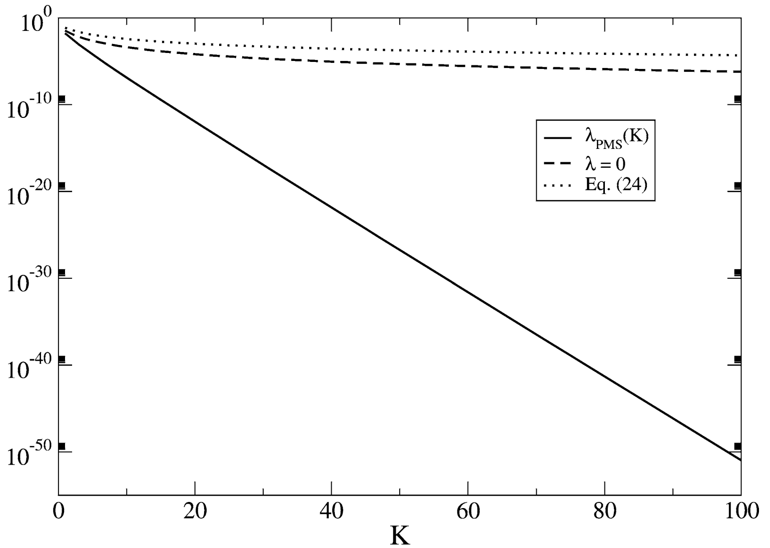

Fig. 7. Difference $|\zeta^{(K)}(3) - \zeta(3)|$ as a function of the number of terms in the sum.

the difference $|\zeta^{(K)}(3) - \zeta(3)|$ using Eq. (16) with $\lambda = \lambda_{\mathrm{PMS}}^{(K)}$ (solid line), $\lambda = 0$ (dashed line) and the series representation

$$\zeta(3) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^3} \tag{24}$$

which corresponds to the dotted line in the plot. This last series converges quite slowly and a huge number of terms (of the order of 10^{25}) is needed to obtain the same accuracy that our series with $\lambda_{\mathrm{PMS}}^{(K)}$ reaches with just 10^2 terms.

Remark 10. We now define

$$c_K(\lambda, s) \equiv \sum_{j=0}^{K} \binom{K}{j} \frac{\lambda^{K-j}}{(1+\lambda)^{K+1}} \frac{(-1)^j}{(1+j)^s}, \tag{25}$$

which we can write as

$$\begin{aligned} c_K(\lambda, s) &= \sum_{j=0}^{K} \binom{K}{j} \frac{\lambda^{K-j}}{(1+\lambda)^{K+1}} \frac{(-1)^j}{\Gamma(s)} \int_0^\infty e^{-(1+j)t}\, t^{s-1}\, dt \\ &= \frac{1}{\Gamma(s)} \int_0^\infty e^{-t} t^{s-1} \frac{(\lambda - e^{-t})^K}{(1+\lambda)^{K+1}}\, dt. \end{aligned} \tag{26}$$

For $\lambda > 1$ we have the inequality

$$\left|c_K(\lambda, s)\right| \leqslant \frac{1}{\Gamma(s)} \int_0^\infty e^{-t} t^{s-1} \frac{\lambda^K}{(1+\lambda)^{K+1}}\, dt = \frac{\lambda^K}{(1+\lambda)^{K+1}}. \tag{27}$$

For $0<\lambda<1$ we can split the integral in the two regions $0<t<\log(1/\lambda)$ and $t>\log(1/\lambda)$ and obtain the inequality

$$\begin{aligned}\left|c_K(\lambda,s)\right| &\leqslant \frac{1}{\Gamma(s)}\int_0^{\log(1/\lambda)} e^{-t}t^{s-1}\frac{(\lambda-e^{-t})^K}{(1+\lambda)^{K+1}}\,dt+\frac{1}{\Gamma(s)}\int_{\log(1/\lambda)}^{\infty}\frac{e^{-t(K+1)}t^{s-1}}{(1+\lambda)^{K+1}}\,dt\\ &\leqslant \frac{\lambda^K}{(1+\lambda)^{K+1}}\left(1-\frac{\Gamma(s,\log(1/\lambda))}{\Gamma(s)}\right)\\ &\quad+\frac{1}{(1+\lambda)^{K+1}}\frac{\Gamma(s,(K+1)\log(1/\lambda))}{\Gamma(s)}\frac{1}{(K+1)^s},\end{aligned} \tag{28}$$

where $\Gamma(a,x)$ is the incomplete gamma function. Using the asymptotic behavior of $\Gamma(a,x)$ (see [14, 6.5.32]),

$$\Gamma(a,x)\approx x^{a-1}e^{-x},$$

we can estimate the rate of convergence to be

$$\left|c_K(\lambda,s)\right| \lessapprox \frac{\lambda^K}{(1+\lambda)^{K+1}}\left[1-\frac{\lambda}{\Gamma(s)}\left(\log\frac{1}{\lambda}\right)^{s-1}\left(1-\frac{1}{1+K}\right)\right], \tag{29}$$

which is essentially geometric.

Notice that the Kth term of series of Eq. (20) decays with a much slower rate, given by

$$c_K^{(\mathrm{KHS})}\approx 2^{-K}. \tag{30}$$

If s is on the critical line, i.e., $s=1/2+i\tau$, we have

$$\zeta\left(\frac{1}{2}+i\tau\right)=\frac{(2)^{-1/2+i\tau}}{2^{-1/2+i\tau}-1}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\frac{\lambda^{k-j}}{(1+\lambda)^{k+1}}\frac{(-1)^j}{(1+j)^{1/2+i\tau}}. \tag{31}$$

In Fig. 8 we have plotted the error (in percent) over the real part of the zeta function, i.e.,

$$\Xi\equiv\Re\left[\frac{\zeta^{(K)}\left(\frac{1}{2}+i\tau\right)-\zeta\left(\frac{1}{2}+i\tau\right)}{\zeta\left(\frac{1}{2}+i\tau\right)}\right]\times 100,$$

as a function of the number of terms considered in the sum of Eq. (31). We use $\tau=50$. The solid line corresponds to Eq. (20), whereas the dashed line corresponds to using our formula, Eq. (31), with $\lambda=0.3$.

4. The generalized Hurwitz zeta function

We now turn our attention to the generalized Hurwitz zeta function given by

$$\bar{\zeta}(s,u,\xi)=\sum_{n=0}^{\infty}\frac{1}{(n^u+\xi)^s}, \tag{32}$$

which includes as special cases both the Riemann ($\bar{\zeta}(s,1,1)$) and the Hurwitz ($\bar{\zeta}(s,1,\xi)$) zeta functions.

We prove the following theorem:

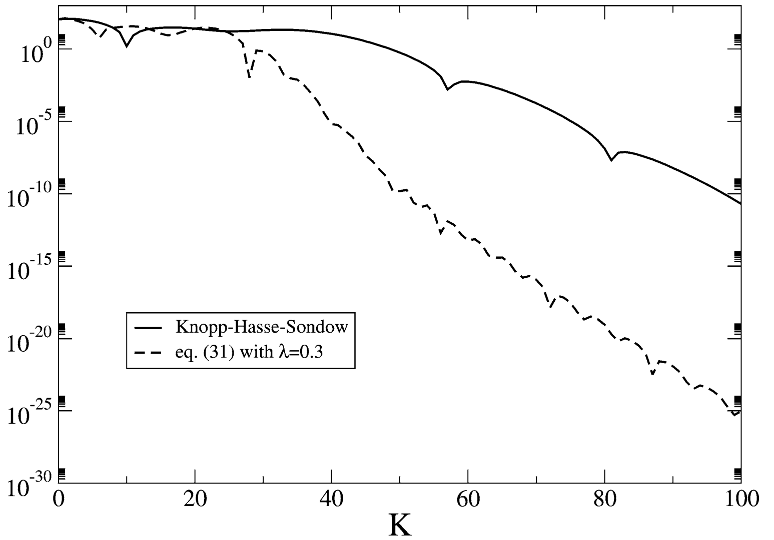

Fig. 8. $\Xi \equiv \Re[(\zeta^{(K)}(\frac{1}{2}+i\tau)-\zeta(\frac{1}{2}+i\tau))/\zeta(\frac{1}{2}+i\tau)] \times 100$, as a function of the number of terms considered in the sum of Eq. (31). The dashed curve is obtained using $\lambda = 0.3$.

Theorem 11. *Let s and u be real numbers such that $su > 1$; then*

$$\bar{\zeta}(u,s,\xi) = \frac{1}{\xi^s} + \sum_{k=0}^{\infty} \frac{\Gamma(k+s)}{\Gamma(s)} \Psi_k(\lambda,u,s,\xi) \tag{33}$$

where

$$\Psi_k(\lambda,u,s,\xi) \equiv \sum_{j=0}^{k} \frac{(-\xi)^j}{j!(k-j)!} \frac{\lambda^{2(k-j)}}{(1+\lambda^2)^{s+k}} \zeta(us+uj) \tag{34}$$

and $\lambda^2 > (\xi-1)/2$ $(\xi > 0)$.

Proof. For $us > 1$ the series $\bar{\zeta}(u,s,\xi)$ converges; we use the identity

$$\bar{\zeta}(u,s,\xi) = \frac{1}{\xi^s} + \sum_{n=1}^{\infty} \frac{1}{n^{su}} \frac{1}{(1+\lambda^2)^s} \frac{1}{(1+\Delta(n))^s}, \tag{35}$$

where

$$\Delta(n) \equiv \frac{\xi/n^u - \lambda^2}{1+\lambda^2}. \tag{36}$$

Provided that $\lambda^2 > (\xi-1)/2$ and $\xi > 0$, $|\Delta(n)| < 1$ and therefore by the binomial theorem

$$\frac{1}{(1+\Delta(n))^s} = \sum_{k=0}^{\infty} \frac{\Gamma(k+s)}{\Gamma(s)k!} \left[-\Delta(n)\right]^k.$$

Using this result in Eq. (35) we have

$$\bar{\zeta}(u,s,\xi)=\frac{1}{\xi^s}+\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{\Gamma(k+s)}{\Gamma(s)k!}\sum_{j=0}^{k}\binom{k}{j}\frac{\lambda^{2(k-j)}}{(1+\lambda^2)^{s+k}}\frac{(-\xi)^j}{n^{u(s+j)}}. \tag{37}$$

As the sum over n and k converges absolutely, we can sum over n and obtain

$$\bar{\zeta}(u,s,\xi)=\frac{1}{\xi^s}+\sum_{k=0}^{\infty}\frac{\Gamma(k+s)}{\Gamma(s)}\sum_{j=0}^{k}\frac{(-\xi)^j}{j!(k-j)!}\frac{\lambda^{2(k-j)}}{(1+\lambda^2)^{s+k}}\zeta\big(u(s+j)\big),$$

which completes our proof. □

Having proved our fundamental result, Eq. (33), we now discuss some of the properties of the new series. We first stress that (33) is independent of the arbitrary parameter λ, although λ appears explicitly in the expression. This happens because Eq. (32) is independent of λ and it has just been proved that our new series, Eq. (33) converges to Eq. (32) provided that $\lambda^2 > (\xi-1)/2$ and $\xi > 0$. In other words we can say that Eq. (33) describes a family of series, each corresponding to a different value of λ and each converging to the same series, Eq. (32).

Consider the partial sum

$$\bar{\zeta}^{(N)}(\lambda,u,s,\xi)=\frac{1}{\xi^s}+\sum_{k=0}^{N}\frac{\Gamma(k+s)}{\Gamma(s)}\Psi_k(\lambda,u,s,\xi)$$

obtained by restricting the infinite sum to the first $N+1$ terms. Obviously $\bar{\zeta}^{(N)}(\lambda,u,s,\xi)$ depends upon λ as a result of having neglected an infinite number of terms. We can use this feature to our advantage and fix λ so that the convergence rate of the series is maximal.

The proper value of λ is chosen using the PMS, which amounts to find λ fulfilling the equation

$$\frac{d}{d\lambda}\bar{\zeta}^{(N)}(s,\lambda)=0. \tag{38}$$

A straightforward mathematical interpretation of this condition is that the value of λ complying with this equation also minimizes the difference [15]

$$\Xi\equiv\big[\bar{\zeta}(u,s,\xi)-\bar{\zeta}^{(N)}(\lambda,u,s,\xi)\big]^2. \tag{39}$$

To lowest order, which corresponds to choosing $N=1$, one obtains the optimal value

$$\lambda_{\mathrm{PMS}}^{(1)}=\sqrt{\xi\,\frac{\zeta(u(1+s))}{\zeta(su)}}, \tag{40}$$

which can be used as long as $\lambda_{\mathrm{PMS}}^2 > (\xi-1)/2$. Notice that, since λ_{PMS} depends upon ξ then $\bar{\zeta}^{(N)}(\lambda_{\mathrm{PMS}},u,s,\xi)$ will not be a polynomial in ξ. On the other hand, if we had chosen $\lambda=0$, then $\bar{\zeta}^{(N)}(0,u,s,\xi)$ would be a polynomial of Nth order in ξ. In that case however the convergence of the series would be strictly limited to the region $\xi<1$. For this reason we will refer to our accelerated series corresponding to λ_{PMS} and to $\lambda=0$ as being "nonperturbative" and "perturbative," respectively.

In Fig. 9 we plot the difference $|\bar{\zeta}^{(K)}(2,\frac{3}{5},1)-\bar{\zeta}(2,\frac{3}{5},1)|$ as a function of the number of terms in the sum of Eq. (32). In this case $su=6/5$ is close to 1 and (32) converges very slowly. The horizontal lines are the values obtained by using Eq. (33) with the optimal value given in Eq. (40).

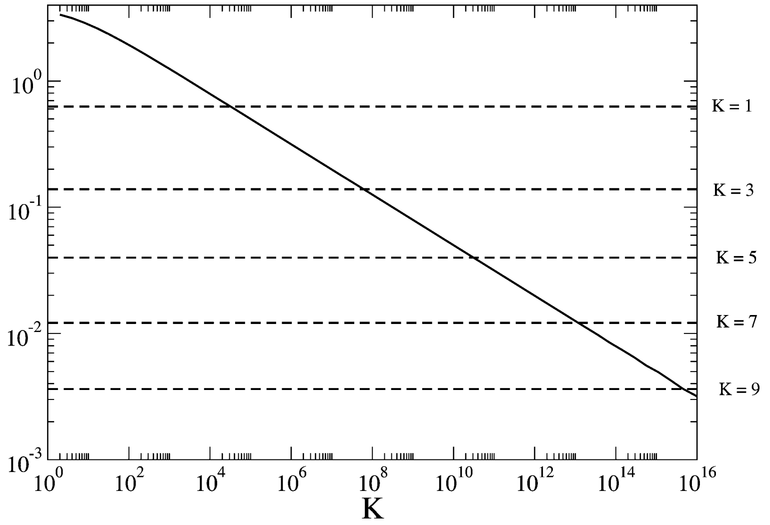

Fig. 9. Difference $|\bar{\zeta}^{(K)}(2, \frac{3}{5}, 1) - \bar{\zeta}(2, \frac{3}{5}, 1)|$ as a function of the number of terms in the sum.

5. Conclusions

The variational approach that we have described in this paper allows in many cases to convert a slowly converging series into a series which converges exponentially. In most cases the results that are obtained by following this approach are explicit. Although we have considered only a few examples, we believe that it should be possible to apply this method to a larger class of series. Indeed the results of [16,17], which focus on the calculation of the period of a classical oscillator, and of [18], which focuses on the calculation of the spectrum a quantum oscillator in the WKB approximation, provide a similar series representation for elliptic functions. In [19] the author has also applied the accelerated series for the generalized Hurwitz zeta function obtained in the present paper to the calculation of bosonic one loop integrals at finite temperature in quantum field theory. The author and collaborators are presently applying the results of this paper to the calculation of the Casimir energy for massive scalar fields between parallel plates [20]. In our opinion, the strongest results of the present paper are Eqs. (16) and (23); it remains to study the possible uses of such equations both in mathematical and physical problems (for example, these equations could be used to further improve the rate of convergence of series such as the one in Eq. (4)).

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